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2-PRIMAL WEAK (σ,δ)-RIGID RINGS

M. ABROL, V. K. BHAT*

School of Mathematics, SMVD University, P/o SMVD University, Katra, J and K, India

Abstract. For a ring R, an endomorphism σ of R and δ a σ-derivation of R, we introduce a weak (σ,δ)-rigid ring, which generalizes the notion of (σ,δ)-rigid rings and investigate its properties. Moreover, we state and prove a necessary and sufficient condition for a weak (σ,δ)-rigid ring to be a (σ,δ)-rigid ring. We prove that a (σ,δ)-ring is a weak (σ,δ)-rigid ring and conversely that the prime radical of a weak(σ,δ)-rigid ring is a (σ,δ)-ring. We also find a relation between minimal prime ideals and completely prime ideals of a ring R, where R is a (σ,δ)-ring and R is a 2-primal weak (σ,δ)-rigid ring.

Keywords: minimal prime ideals, completely prime ideals, (σ,δ)-rings, weak (σ,δ)-rigid rings, 2-primal rings.

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1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$, unless otherwise stated. The prime radical and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$ respectively. The ring of integers is denoted by $\mathbb{Z}$, the field of real numbers is denoted by $\mathbb{R}$, the field of rational numbers is denoted by $\mathbb{Q}$ and the field of complex numbers is denoted by $\mathbb{C}$, unless otherwise stated. The set of minimal prime ideals of R is denoted by Min.Spec(R).

* Corresponding author

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Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R, which is defined as an additive map from $R \to R$ such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R.$$

Example 1.1. *Let $R = \mathbb{Z}[\sqrt{2}]$. Then $\sigma : R \to R$ defined as*

$$\sigma(a + b\sqrt{2}) = (a - b\sqrt{2}) \textit{ for } a + b\sqrt{2} \in R.$$

is an endomorphism of R. For any $s \in \mathbb{R}$. Define $\delta_s : R \to R$ by

$$\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}) \textit{ for } a + b\sqrt{2} \in R.$$

Then δ_s is a σ-derivation of R.

According to Krempa [14], an endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R. We recall that σ-rigid rings are reduced rings by Hong et. al. [12]. Recall that ring R is reduced if R has no non-zero nilpotent elements. Observe that reduced rings are abelian. Properties of σ-rigid rings have been studied in Krempa [14], Hong [12] and Hirano [10]. Also Kwak [13] defined $\sigma(*)$-ring. Let R be a ring and σ an endomorphism of R. Then R is said to be $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$. Note that we say a ring R with an endomorphism σ is weak σ-rigid if $a\sigma(a) \in N(R)$ implies and is implied by $a \in N(R)$ for $a \in R$. Clearly this notion of a weak σ-rigid ring generalizes that of a σ-rigid ring. For further details on weak σ-rings refer to [12, 14, 17, 19].

Completely prime ideals:

Completely prime ideals are a special type of prime ideals that play a key role in the notions introduced in this paper. Recall that an ideal P of a ring R is said to be completely prime if $ab \in P$ implies that $a \in P$ or $b \in P$ for $a, b \in R$ (Chapter 3 of [9]). In commutative sense completely prime and prime have the same meaning. We also note that a completely prime ideal of a ring R is a prime ideal, but the converse need not be true. The following example shows that a prime ideal need not be a completely prime ideal.

Example 1.2. *(Example 1.1 of Bhat* [3]*): Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$*. If p is a prime number, then the ideal* $P = M_2(p\mathbb{Z})$ *is a prime ideal of R. But is not completely prime.*
There are examples of rings (non-commutative) in which prime ideals are completely prime.

Example 1.3. *(Example 1.2 of Bhat* [3]*): Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$*. Then* $P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$*,* $P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ *and* $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ *are prime ideals of R. Now all these are completely prime also.*

Minimal prime ideals:

A minimal prime ideal in a ring R is any prime ideal of R that does not properly contain any other prime ideal. In example 1.2 of Bhat [3] (discussed above), $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is a minimal prime ideal. Further more there are examples of rings in which minimal prime ideals are completely prime. For example a reduced ring. Regarding minimal prime ideals we have the following:

Proposition 3.3 of [9]: Any prime ideal U in a ring R contains a minimal prime ideal.

Theorem 3.4 of [9]: In a right Noetherian ring R, there exists only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetition allowed) that equals zero.

Note that a considerable work has been done in the investigation of prime ideals in particular minimal prime ideals refer to [1, 2, 8, 9, 16, 18].

2-Primal rings:

Another area of interest since recent past has been the study of 2-primal rings. This involves the notion of prime radicals and the set of nilpotent elements of a ring. Further the concept of completely prime ideals and completely semi-prime ideals are also studied in this area. Due to Birkenmeier et al. [4], a ring R is called 2-primal if $P(R) = N(R)$. Note that every reduced ring is a 2-primal ring and a commutative ring is also 2-primal. Part of the attraction of 2-primal rings in addition to their being a common generalization of commutative rings and rings without nilpotent elements lies in the structure of their prime ideals. Shin showed in ([20], proposition

1.11) that a ring R is 2-primal if and only if every minimal prime ideal $P \subset R$ is completely prime (i.e. R/P is a domain). Other properties and examples of 2-primal can be found in [5, 6, 7, 11, 13, 15, 20].

δ-Rings:

Bhat [2] has defined a δ-ring as : Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a δ-ring if $a\delta(a) \in P(R)$ implies that $a \in P(R)$. Note that a δ-ring is without identity, $1 \neq 0$ as $1.\delta(1) = 0$, but $1 \neq 0$.

Example 1.4. *Let S be a ring without identity and $R = S \times S$ with $P(R) = \{0\}$ (for example we take $S = 2\mathbb{Z}$). Then $\sigma : R \to R$ is an endomorphism defined by*

$$\sigma((a,b)) = (b,a).$$

$\delta_r : R \to R$, for any $r \in \mathbb{R}$ defined by

$$\delta_r((a,b)) = (a,b)r - r\sigma((a,b)) \text{ for } (a,b) \in R.$$

is a σ-derivation. Let $(a,b)\delta_r((a,b)) \in P(R)$.

Then $(a,b)\{(a,b)r - r\sigma((a,b))\} \in P(R)$

or $(a,b)\{(a,b)r - r(b,a)\} \in P(R)$

i.e. $(a,b)(ar - rb, br - ra) \in P(R)$.

Therefore, $(a(ar - rb), b(br - ra)) \in P(R) = \{0\}$ which implies that $a = 0, b = 0$

i.e. $(a,b) = (0,0) \in P(R)$.

Thus R is a δ-ring.

2. Preliminaries

We begin with the following:

Definition 2.1. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ,δ)-ring if $a(\sigma(a) + \delta(a)) \in P(R)$ implies that $a \in P(R)$ for $a \in R$.

Example 2.2. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. *Then* $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

Let $\sigma : R \to R$ be defined by

$$\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R.

For any $s \in R$. Define $\delta_s : R \to R$ by

$$\delta_s(a) = as - s\sigma(a) \text{ for } a \in R.$$

Clearly, δ_s is a σ-derivation of R.

Now let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, s = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$.

Further $A[\sigma(A) + \delta(A)] \in P(R)$ implies that

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big\{ \sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big) + As - s\sigma(A) \Big\} \in P(R)$

or $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} - \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Big) \Big\} \in P(R)$

or $\begin{pmatrix} a^2 & a^2q + abr + bc - acq \\ 0 & c^2 \end{pmatrix} \in P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ *which implies that*

$a^2 = 0, c^2 = 0$ *i.e.* $a = 0, c = 0$.

Therefore, $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P(R)$.

Hence R is a (σ,δ)-ring.

Remark 2.3. :

(1) If $\delta(a) = 0$, then (σ,δ)-ring is a $\sigma(*)$-ring.

(2) If $\sigma(a) = 0$, then (σ,δ)-ring is a δ-ring.

Definition 2.4. Let R be a ring. Let σ be an endomorphism of R and δ a σ-derivation of R. Then R is said to be a (σ,δ)-rigid ring if

$$a(\sigma(a) + \delta(a)) = 0 \text{ implies that } a = 0 \text{ for } a \in R.$$

Example 2.5. *Let $R=\mathbb{C}$ and $\sigma : R \to R$ be defined by*

$$\sigma(a+ib)=a-ib, \text{ for all } a,b\in\mathbb{R}.$$

Then σ is an endomorphism of R.

Define $\delta : R \to R$ by

$$\delta(\alpha)=\alpha-\sigma(\alpha) \text{ for } \alpha\in R.$$

Then δ is a σ-derivation of R. Now it can be easily seen that R is a (σ,δ)-rigid ring.

Ouyang [19] introduced weak σ-rigid rings, where σ is an endomorphism of R. This ring is related to 2-primal rings. In this note we generalize the (σ,δ)-rigid ring by introducing weak (σ,δ)-rigid ring and prove relations involving the concepts discussed above. We begin with the following definition:

Definition 2.6. Let R be a ring. Let σ an endomorphism of R and δ a σ-derivation of R. Then R is said to be a weak (σ,δ)-rigid ring if $a(\sigma(a)+\delta(a))\in N(R)$ implies and is implied by $a\in N(R)$ for $a\in R$.

Example 2.7. *Let σ be an endomorphism of R and δ a σ-derivation of R. Let R be a (σ,δ)-rigid ring. Let*

$$R_3=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} : a,b,c,d\in R\right\}$$

be a subring of $T_3(R)$. The endomorphism σ of R can be extended to the endomorphism $\overline{\sigma}$: $R_3\to R_3$ defined by

$$\overline{\sigma}((a_{ij}))=(\sigma(a_{ij}))$$

and δ a σ-derivation of R can be extended to $\overline{\delta} : R_3\to R_3$ by

$$\overline{\delta}((a_{ij}))=(\delta(a_{ij})).$$

Let

$$\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\left\{\overline{\sigma}\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}+\overline{\delta}\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\right\}\in N(R).$$

Then there is some positive integer n such that

$$\Big[\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\Big\{\begin{pmatrix} \sigma(a) & \sigma(b) & \sigma(c) \\ 0 & \sigma(a) & \sigma(d) \\ 0 & 0 & \sigma(a) \end{pmatrix}+\begin{pmatrix} \delta(a) & \delta(b) & \delta(c) \\ 0 & \delta(a) & \delta(d) \\ 0 & 0 & \delta(a) \end{pmatrix}\Big\}\Big]^n=0,$$

which implies that

$$\Big[\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} \sigma(a)+\delta(a) & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & \sigma(a)+\delta(a) & \sigma(d)+\delta(d) \\ 0 & 0 & \sigma(a)+\delta(a) \end{pmatrix}\Big]^n=0$$

$$\begin{pmatrix} a(\sigma(a)+\delta(a)) & a(\sigma(b)+\delta(b))+b(\sigma(a)+\delta(a)) & a(\sigma(c)+\delta(c))+b(\sigma(d)+\delta(d))+c(\sigma(a)+\delta(a)) \\ 0 & a(\sigma(a)+\delta(a)) & a(\sigma(d)+\delta(d))+d(\sigma(a)+\delta(a)) \\ 0 & 0 & a(\sigma(a)+\delta(a)) \end{pmatrix}$$

$=0$, *which gives*

$$a(\sigma(a)+\delta(a))\in N(R).$$

Since R is reduced, we have

$$a(\sigma(a)+\delta(a))=0$$

which implies that $a=0$*, since R is a* (σ,δ)*-rigid ring.*

Hence

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}=\begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}\in N(R).$$

Conversely, assume that

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\in N(R).$$

Then there is some positive integer n such that

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^n=\begin{pmatrix} a^n & * & * \\ 0 & a^n & * \\ 0 & 0 & a^n \end{pmatrix}=0$$

which implies that $a=0$*, because R is reduced. (Here* $*$ *are non-zero terms involving summation of powers of some or all of a, b, c, d.)*

So

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \left\{ \overline{\sigma} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} + \overline{\delta} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 0 & \sigma(b) & \sigma(c) \\ 0 & 0 & \sigma(d) \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \delta(b) & \delta(c) \\ 0 & 0 & \delta(d) \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 0 & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & 0 & \sigma(d)+\delta(d) \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 0 & 0 & b(\sigma(d)+\delta(d)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(R).$$

Therefore, R_3 is a weak $(\overline{\sigma}, \overline{\delta})$-rigid ring.

With this we prove the following result.

Theorem A. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a (σ, δ)-rigid ring if and only if R is weak (σ, δ)-rigid ring and reduced. (This has been proved in Theorem (3.7)).*

Theorem B. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ, δ)-ring. Then R is a weak (σ, δ)-rigid ring. Conversely a 2-primal weak (σ, δ)-rigid ring is a (σ, δ)-ring. (This has been proved in Theorem (3.8)).*

Theorem C. *Let R be a Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(U) = U$ and $\delta(U) \subseteq U$ where $U \in$ Min.Spec(R). Then R is a (σ, δ) -ring if and only if for each $U \in$ Min.Spec(R), $\sigma(U) + \delta(U) = U$ and U is a completely prime ideal of R. (This has been proved in Theorem (3.9)).*

Theorem D. *Let R be a Noetherian ring which is $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(U) = U$ and $\delta(U) \subseteq U$ where $U \in$ Min.Spec(R).*

Then R is a 2-primal weak (σ,δ)-rigid ring if and only if for each minimal prime U of R, $\sigma(U)+\delta(U)=U$ and U is a completely prime ideal of R. (This has been proved in Theorem (3.10)).

3. Proof of Main Results

For the proof of the main result, we need the following:

Proposition 3.1. *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. Then for $u\neq 0$, $\sigma(u)+\delta(u)\neq 0$.*

Proof. Let $0\neq u\in R$, we show that $\sigma(u)+\delta(u)\neq 0$. Let for $0\neq u$, $\sigma(u)+\delta(u)=0$ which implies that

$$\delta(u)=-\sigma(u). \tag{3.1}$$

We know that for $0\neq a, 0\neq b\in R$, $\delta(ab)=\delta(a)\sigma(b)+a\delta(b)$. By using (3.1), this implies that $\delta(ab)=-\sigma(a)\sigma(b)+a(-\sigma(b))$ or $-\sigma(ab)=-[\sigma(a)+a]\sigma(b)$. Since σ is an endomorphism of R, this gives $-\sigma(a)\sigma(b)=-[\sigma(a)+a]\sigma(b)$ i.e. $\sigma(a)=\sigma(a)+a$. Therefore, $a=0$, which is not possible. Hence the result. □

Theorem 3.2. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ,δ)-ring, then R is 2-primal.*

Proof. R is a (σ,δ)-ring. We know that a reduced ring is 2-primal. We use the principle of Mathematical induction to prove that R is a reduced ring.

Let for $x\in R$, $x^n=0$. We use induction on n and show that $x=0$. Result is trivially true for $n=1$, as $x^n=x^1=a(\sigma(a)+\delta(a))=0$. Now Proposition (3.1), implies that $a=0$. Hence $x=0$. Therefore, the result is true for $n=1$. Let us assume that the result is true for $n=k$, i.e. $x^k=0$ which implies that $x=0$. Let $n=k+1$. Then $x^{k+1}=0$ which implies that

$$a^{k+1}(\sigma(a)+\delta(a))^{k+1}=0.$$

Again by Proposition (3.1), we get $a=0$. Hence $x=0$. Therefore, the result is true for $n=k+1$ also. Thus the result is true for all n. □

The converse of the above is not true.

Example 3.3. *Let $R = F(x)$, the field of rational polynomials in one variable, x. Then R is 2-primal with $P(R) = \{0\}$.*
Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For $r \in R$, $\delta_r : R \to R$ is a σ-derivation defined as

$$\delta_r(a) = ar - r\sigma(a) \text{ for } a \in R.$$

Then R is not a (σ,δ)-ring. For take $f(x) = xa + b, r = \frac{-b}{xa}$.

Towards the proof of the next Theorem, we require the following:

J. Krempa [14] has investigated the relation between minimal prime ideals and completely prime ideals of a ring R. With this he proved the following:

Theorem 3.4. *For a ring R the following conditions are equivalent:*
(1) R is reduced.
(2) R is semiprime and all minimal prime ideals of R are completely prime.
(3) R is a subdirect product of domains.

Theorem 3.5. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. If R is a (σ,δ)-ring, then $P(R)$ is completely semi-prime.*

Proof. As proved in Theorem (3.2), R is a reduced ring and by using Theorem (3.4), the result follows. □

The converse of the above is not true.

Example 3.6. *Let F be a field, $R = F \times F$. Let $\sigma : R \to R$ be an automorphism defined as*

$$\sigma((a,b)) = (b,a), \text{ for all } a,b \in F.$$

Here $P(R)$ is a completely semi-prime ring, as R is a reduced ring.
For $r \in F$. Define $\delta_r : R \to R$ by

$$\delta_r((a,b)) = (a,b)r - r\sigma((a,b)) \text{ for } a,b \in F.$$

Then δ_r is a σ-derivation of R. Also R is not a (σ,δ)-ring. For take $A=(1,-1), r=\frac{1}{2}$.

We now state and prove the main results of this paper in the form of the following Theorems:

Theorem 3.7. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then R is a (σ,δ)-rigid ring if and only if R is weak (σ,δ)-rigid ring and reduced.*

Proof. Let R be a (σ,δ)-rigid ring, then R is reduced as in the proof of Theorem (3.2). We will show that R is a weak (σ,δ)-rigid ring. Suppose $a\in N(R)$ then $a^n=0$, for some positive integer n, which implies that $a=0$. Hence $a(\sigma(a)+\delta(a))=0\in N(R)$, by Proposition (3.1), since $(\sigma(a)+\delta(a))\neq 0$.

If $a(\sigma(a)+\delta(a))\in N(R)$ for $a\in R$, then $a(\sigma(a)+\delta(a))=0$ and so $a\in N(R)$, because R is a (σ,δ)-rigid ring. Thus R is a weak (σ,δ)-rigid ring.

Conversely, suppose that R is a weak (σ,δ)-rigid ring and reduced.

Let $a(\sigma(a)+\delta(a))=0$ for $a\in R$, then $a\in N(R)$, since R is a weak (σ,δ)-rigid ring. Thus $a=0$. Hence R is a (σ,δ)-rigid ring. □

Theorem 3.8. *Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a (σ,δ)-ring. Then R is a weak (σ,δ)-rigid ring. Conversely a 2-primal weak (σ,δ)-rigid ring is a (σ,δ)-ring.*

Proof. Let R be a (σ,δ)-ring, then by Theorem (3.5), R is completely semi-prime and therefore 2-primal, which implies that $N(R)=P(R)$ and therefore $a(\sigma(a)+\delta(a))\in N(R)=P(R)$. Since R is a (σ,δ)-ring, therefore $a\in P(R)=N(R)$. Hence R is a weak (σ,δ)-rigid ring.

Conversely, let R be 2-primal and weak (σ,δ)-rigid ring. Then $N(R)=P(R)$ and so $a(\sigma(a)+\delta(a))\in P(R)$ which implies that $a(\sigma(a)+\delta(a))\in N(R)$. Since R is a weak (σ,δ)-rigid ring, therefore $a\in N(R)$ which implies that $a\in P(R)$. Hence R is a (σ,δ)-ring.

□

Theorem 3.9. *Let R be a Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(U)=U$ and $\delta(U)\subseteq U$ where $U\in Min.Spec(R)$. Then R is a (σ,δ)-ring if and only if for each $U\in Min.Spec(R)$, $\sigma(U)+\delta(U)=U$ and U is a completely prime ideal of R.*

Proof. Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U)+\delta(U)=U$ and U is a completely prime ideal of R. Let $a\in R$ be such that $a(\sigma(a)+\delta(a))\in P(R)=\bigcap_{i=1}^{n}U_i$, where U_i are the minimal primes of R. Now for each i, $a\in U_i$ or $\sigma(a)+\delta(a)\in U_i$ and U_i is completely prime. Now $\sigma(a)+\delta(a)\in U_i=\sigma(U_i)+\delta(U_i)$ which implies that $a\in U_i$ and hence $a\in P(R)$. Thus R is a (σ,δ)-ring.

Conversely, suppose that R is a (σ,δ)-ring and let $U=U_1$ be a minimal prime ideal of R. Then by Theorem (3.5), $P(R)$ is completely semi-prime. Let $U_2,U_3,..,U_n$ be the other minimal primes of R. Suppose that $\sigma(U)+\delta(U)\neq U$. Then $\sigma(U)+\delta(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U)+\delta(U)=U_n$. Let $a\in\bigcap_{i=1}^{n-1}U_i$. Then $\sigma(a)+\delta(a)\in U_n$, and so $a(\sigma(a)+\delta(a))\in\bigcap_{i=1}^{n}U_i=P(R)$ and therefore $a\in P(R)$ and thus $\bigcap_{i=1}^{n-1}U_i\subseteq U_n$ which implies that $U_i\subseteq U_n$ for some $i\neq n$, which is impossible. Hence $\sigma(U)+\delta(U)=U$.

Now suppose that $U=U_1$ is not completely prime. Then there exists $a,b\in R/U$ with $ab\in U$. Let c be any element of $b(U_2\bigcap U_3\bigcap\ldots\bigcap U_n)a$. Then $c^2\in\bigcap_{i=1}^{n}U_i=P(R)$. So $c\in P(R)$ and thus $b(U_2\bigcap U_3\bigcap\ldots\bigcap U_n)a\subseteq U$. Therefore $bR(U_2\bigcap U_3\bigcap\ldots\bigcap U_n)Ra\subseteq U$ and as U is prime, $a\in U$, $U_i\subseteq U$ for some $i\neq 1$ or $b\in U$. None of these can occur, so U is completely prime. □

Theorem 3.10. *Let R be a Noetherian ring which is $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(U)=U$ and $\delta(U)\subseteq U$ where $U\in$ Min.Spec(R). Then R is a 2-primal weak (σ,δ)-rigid ring if and only if for each minimal prime U of R, $\sigma(U)+\delta(U)=U$ and U is completely prime ideal of R.*

Proof. It follows by Theorems (3.8) and (3.9). □

Conflict of Interests

The authors declare that there is no conflict of interests.

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