ON SOME PROPERTIES OF GENERALIZED $\delta$-SUPPLEMENTED MODULES AND (GENERALIZED) $f$-$\delta$-SUPPLEMENTED MODULES

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Abstract. In this paper, we give some properties of generalized $\delta$-supplemented modules together with $\delta$-radical and $\delta$-reduced module concepts. Moreover we define (generalized) $f$-$\delta$-supplemented modules and investigate some characterizations of these modules.

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1. Introduction

Throughout this paper, we use $R$ to denote an associative commutative ring with identity and all modules are unitary left $R$-modules. Let $M$ be an $R$-module. By $N \leq M$ we mean that $N$ is a submodule of $M$. A submodule $L$ of a module $M$ is called small in $M$ (denoted by $L << M$) if for every proper submodule $K$ of $M$, $L + K \neq M$. The Jacobson radical of $M$ is denoted by $Rad(M)$. Equivalently, $Rad(M)$ is the sum of all small submodules of $M$. Recall that a submodule $L \leq M$ is called essential, denoted by $L \triangleright M$, if $L \cap K \neq 0$ for each

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nonzero submodule $K \leq M$. The singular submodule of a module $M$ (denoted by $Z(M)$) is $Z(M) = \{ x \in M \mid Ix = 0 \text{ for some ideal } I \not\supseteq M \}$. A module $M$ is called singular if $Z(M) = M$. For further properties of singular modules we refer to [3].

An $R$-module $M$ is called supplemented if every submodule $N$ of $M$ has a supplement that is a submodule $K$ minimal with respect to $N + K = M$. $K$ is a supplement of $N$ in $M$ if and only if $N + K = M$ and $N \cap K \ll K$ [10]. Let $M$ be an $R$-module and let $N$ and $K$ be any submodules of $M$ with $M = N + K$. If $N \cap K \leq \text{Rad}(K)$ then $K$ is called a generalized supplement of $N$ in $M$. In [8] $M$ is called generalized supplemented module (or briefly GS-module) if every submodule $N$ of $M$ has a generalized supplement $K$ in $M$.

In [12], Zhou defined the concept of $\delta$-small submodules as a generalization of small submodules. Let $N$ be a submodule of $M$. $N$ is said to be $\delta$-small in $M$ if $N + K \neq M$ for any proper submodule $K$ of $M$ with $\frac{M}{K}$ singular. $\delta(M) = \sum \{ N \leq M \mid N \ll \delta M \} = \text{Rej}_M(\varphi) = \cap \{ N \leq M \mid \frac{M}{N} \in \varphi \}$, where $\varphi$ be the class of all singular simple modules. A submodule $L$ of $M$ is called a $\delta$-supplement of $N$ in $M$ if $M = N + L$ and $N \cap L$ is $\delta$-small in $L$ and $M$ is called $\delta$-supplemented in case every submodule of $M$ has a $\delta$-supplement in $M$ [4]. Let $M$ be an $R$-module and, $N, K \leq M$ with $M = N + K$. If $N \cap K \leq \delta(N)$ then $N$ is called a generalized $\delta$-supplement of $K$ in $M$. Following [6], $M$ is called a generalized $\delta$-supplemented module (or briefly $\delta$-GS module) if every submodule $N$ of $M$ has a generalized $\delta$-supplement $K$ in $M$.

In this paper, we give some specialized properties of generalized $\delta$-supplemented modules and (generalized) $f$-$\delta$-supplemented modules.

\section{2. Preliminaries}

We begin by stating the following lemmas for the completeness.

\textbf{Lemma 1.} Let $N$ be a submodule of $M$. The following are equivalent:

1. $N \ll \delta M$;
2. If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule $Y$ with $Y \subseteq N$;
3. If $X + N = M$ with $\frac{M}{X}$ Goldie torsion, then $X = M$ "(see [12])".

\textbf{Lemma 2.} Let $M$ be a module.
(1) For submodules $N, K, L$ of $M$ with $K \subseteq N$, we have
\[
(a) \ N \lessdot \delta M \text{ if and only if } K \lessdot \delta M \text{ and } \frac{N}{K} \lessdot \delta \frac{M}{K}.
\]
\[
(b) \ N + L \lessdot \delta M \text{ if and only if } N \lessdot \delta M \text{ and } L \lessdot \delta M.
\]
(2) If $K \lessdot \delta M$ and $f : M \rightarrow N$ is a homomorphism, then $f(K) \lessdot \delta N$.
\[
\text{In particular, if } K \lessdot \delta M \subseteq N, \text{ then } K \lessdot \delta N.
\]
(3) Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \lessdot \delta M_1 \oplus M_2$ if and only if $K_1 \lessdot \delta M_1$ and $K_2 \lessdot \delta M_2$ ”(see [12])”.

**Proposition 3.** Let $U$ and $V$ be submodules of a module $M$. Assume that $V$ is a $\delta$-supplement of $U$ in $M$. Then the following statements hold

(1) If $W + V = M$ for some $W \subseteq U$, then $V$ is a $\delta$-supplement of $W$ in $M$,

(2) If $K \lessdot \delta M$, then $V$ is a $\delta$-supplement of $U + K$ in $M$,

(3) For $K \lessdot \delta M$ we have $K \cap V \lessdot \delta V$ and so $\delta(V) = V \cap \delta(M)$,

(4) For $L \subseteq U$, $\frac{V + L}{L}$ is a $\delta$-supplement of $\frac{U}{L}$ in $\frac{M}{L}$

(5) If $\delta(M) \lessdot \delta M$, or $\delta(M) \subseteq U$ and if $p : M \rightarrow \frac{M}{\delta(M)}$ is the canonical projection, then
\[
\frac{M}{\delta(M)} = p(U) \oplus p(V) \text{ ”(see [5])”}.
\]

**Proposition 4.** Let $A, B$ be submodules of $M$ such that $B$ is a generalized $\delta$-supplement submodule of $A$ in $M$. Then:

(1) If $W + B = M$ for some $W \subseteq A$ then $B$ is a generalized $\delta$-supplement of $W$.

(2) If $K \lessdot \delta M$ then $B$ is generalized $\delta$-supplement of $A + K$.

(3) For $K \lessdot \delta M$ then $K \cap B \lessdot \delta B$ and so $\delta(B) = B \cap \delta(M)$.

(4) For $L \subseteq A$, $\frac{B + L}{L}$ is a generalized $\delta$-supplement of $\frac{A}{L}$ in $\frac{M}{L}$ ”(see [11])”.

3. $f - \delta$ Supplemented Modules

**Definition 1.** Let $M$ be an $R$-module. If every finitely generated submodule of $M$ has a $\delta$-supplement in $M$, then $M$ is called finitely $\delta$-supplemented module or briefly $f - \delta$-supplemented module.
Proposition 5. Let $M$ be an $f$-$\delta$-supplemented module and $L << \delta M$. Then, $\frac{M}{L}$ is also $f$-$\delta$-supplemented.

Proof. Let $\frac{K}{L}$ be a finitely generated submodule of $\frac{M}{L}$. It follows that $\frac{K}{L} = \langle k_1 + L, k_2 + L, \ldots, k_n + L \rangle$ for some $k_1, k_2, \ldots, k_n \in K$. If we say $S = \langle k_1, k_2, \ldots, k_n \rangle$ then it can be seen easily $K = S + L$. Because $S$ is finitely generated in $M$, there exists a submodule $V$ in $M$ which is a $\delta$-supplement for $S$. By hypothesis, $V$ is also a $\delta$-supplement for $K$. Therefore, $\frac{V + L}{L}$ is a $\delta$-supplement of $\frac{K}{L}$ in $\frac{M}{L}$.

Proposition 6. Let $M$ be an $f$-$\delta$-supplemented module and $\delta(M) << \delta M$. Then every finitely generated submodule of $\frac{M}{\delta(M)}$ is a direct summand.

Proof. It is clear that $\frac{M}{\delta(M)}$ is $f$-$\delta$-supplemented from the previous proposition. Let $\frac{K}{\delta(M)} \leq \frac{M}{\delta(M)}$ be a finitely generated submodule. By hypothesis, there is a $\delta$-supplement $\frac{V}{\delta(M)}$ of $\frac{K}{\delta(M)}$ in $\frac{M}{\delta(M)}$. That means, $\frac{K}{\delta(M)} + \frac{V}{\delta(M)} = \frac{M}{\delta(M)}$ and $\frac{K}{\delta(M)} \cap \frac{V}{\delta(M)} = \frac{K \cap V}{\delta(M)} << \delta \frac{V}{\delta(M)}$. Since $K \cap V$ is $\delta$-small in $M$, we can write $K \cap V \leq \delta(M)$. Hence $\frac{K}{\delta(M)} \cap \frac{V}{\delta(M)} = 0 \frac{M}{\delta(M)}$ and this completes the proof.

Proposition 7. Let $M$ be an $f$-$\delta$-supplemented module and $L$ be a generated submodule of $M$. Then $\frac{M}{L}$ is also $f$-$\delta$-supplemented.

Proof. Let $\frac{K}{L}$ be a finitely generated submodule of $\frac{M}{L}$. Since $\frac{K}{L}$ and $L$ finitely generated so is $K$. From hypothesis there exists a $\delta$-supplement $V$ of $K$ in $M$. Then $\frac{V + L}{L} \leq \frac{M}{L}$ is also a $\delta$-supplement for $\frac{K}{L}$.

4. Generalized $\delta$—Supplemented Modules

Proposition 8. Let $M$ be an $R$-module and $U, V \leq M$. $V$ is a generalized $\delta$-supplement of $U$ if and only if $U + V = M$ and $Rm << \delta V$ for every $m \in U \cap V$.

Proof. Let $V$ be a generalized $\delta$-supplement of $U$. Then $U + V = M$ and $U \cap V \leq \delta(V)$. Since, $\delta(V)$ is the sum of all $\delta$-small submodules of $V$ we can write $m = m_1 + m_2 + \ldots + m_n$ for every $m \in U \cap V$ such that $m_i \in V_i << \delta V, \forall i = 1, 2, \ldots, n$. Since $V_i << \delta V$, also $Rm_i << \delta V$. And so, $Rm << \delta V$ since $Rm \subseteq Rm_1 + Rm_2 + \ldots + Rm_n$.

Conversely, let $U + V = M$ and $Rm << \delta V$ for every $m \in U \cap V$. Since $\delta(V) = \sum_{L << \delta V} L$ for every element $m$ in $U \cap V, m \in Rm \leq \delta(V)$. Hence, $U \cap V \leq \delta(V)$.
A module $M$ is called radical if $\text{Rad} M = M$, and $M$ is called reduced if it has no nonzero radical submodule. See [13] for details for the notion of reduced and radical modules. By using these concepts we can give following definition.

**Definition 2.** A module $M$ is called $\delta$-radical if $\delta(M) = M$, and $M$ is called $\delta$-reduced if it has no nonzero $\delta$-radical submodule.

**Corollary 9.** Let $V$ be a $\delta$-radical submodule of $M$. Then, $V$ is a generalized $\delta$-supplement of every submodule in $M$ including $V$.

**Proposition 10.** Let $U, V \leq M$ and $V$ be a generalized $\delta$-supplement of $U$ that is not $\delta$-radical. Then there is a maximal essential submodule of $M$ including $U$.

**Proof.** By hypothesis, there is a maximal submodule $K$ of $M$ such that $\frac{V}{K}$ is simple singular. Here, $K$ is a maximal submodule of $V$ so $V = K + Rv$ for $v \in V \setminus K$. Assume that $M = U + K$. Then $v = u + k$, $u \in U$, $k \in K$ and so $u \in U \cap V$. Because $V$ is a generalized $\delta$-supplement of $U$ we can say $U \cap V \leq \delta(V) \leq K$. And $v \in K$ is obtained but this contradicts with $M \neq U + K$. Since $\frac{M}{U + K} \cong \frac{V}{K}$, $U + K$ is maximal and essential in $M$.

**Corollary 11.** Let $M$ be a $\delta$-reduced module. If $U$ has a generalized $\delta$-supplement in $M$, $U$ is included by a maximal essential submodule in $M$.

**Proposition 12.** Let $M$ be an $R$-module. If every proper submodule of $M$ is contained in a maximal submodule then $\delta(M) \ll M$.

**Proof.** Let $L \leq M$ with $\delta(M) + L = M$, and let $\frac{M}{L}$ be singular. If $L \neq M$, then there exists a maximal submodule $K$ in $M$ containing $L$. Since $\frac{M}{K}$ is simple and singular, $\delta(M) \leq K$. So $M = K$ is obtained but this contradicts with the maximality of $K$. Hence $L = M$ is obtained.

**Corollary 13.** If $M$ is a finitely generated $R$-module then $\delta(M) \ll M$

**Corollary 14.** Let $M$ be an $R$-module and $V$ be a generalized $\delta$-supplement of $U$. If $V$ is finitely generated then $V$ is also $\delta$-supplement of $U$ in $M$.

**Proposition 15.** Let $M$ be an $R$-module and $V$ be a generalized $\delta$-supplement of $U$ in $M$. If $U$ is a maximal and singular submodule of $M$ then $U \cap V = \delta(V)$.
Proof.  $U \cap V \leq \delta(V)$ since $V$ is a generalized $\delta$-supplement of $U$. Additively, $\delta(V) \leq \delta(M)$
and $\delta(M) \leq U$ since $U$ is maximal and singular and so $\delta(V) \leq U \cap V$.

Proposition 16. Let $M$ be an $R$-module and $V$ be a generalized $\delta$-supplement of $U$. If $K \leq \delta(M)$
then $K \cap V \leq \delta(V)$.

Proof. Assume that $K \cap V \not\subseteq \delta(V) = \cap \{N \leq V \mid \frac{V}{N} \in \phi\}$ where $\phi$ be the class of all singular simple modules. Then, there is an element $N$ in $\delta(V)$ such that $K \cap V \not\subseteq V$. So, $N + Rm = V$.
Following this $M = U + V = U + N + Rm$ is obtained. Note that $\frac{M}{U+N}$ is singular and $Rm \ll \delta M$ since $K \leq \delta(M)$. Hence, $U + N = M$. By using modular law, $V = (U \cap V) + N$ and $\delta(V) + N = V$,
since $U \cap V \leq \delta(V)$. It is easy to see that $\delta(V) \leq N$. At the end we have $N = V$ is obtained.
This is a contradiction. So, $K \cap V \leq \delta(V)$.

Corollary 17. Let $M$ be an $R$-module and $V$ be a generalized $\delta$-supplement of $U$. Then $\delta(V) = V \cap \delta(M)$.

Proof. Trivially, $\delta(V) = V \cap \delta(M)$. By the previous proposition, it is easy to see that $V \cap \delta(M) \leq \delta(V)$.

Proposition 18. Let $M$ be an $R$-module and $V$ be a generalized $\delta$-supplement of $U$. If $K$ is a maximal submodule of $V$ with $\frac{V}{K}$ singular, then $U + K$ is maximal in $M$ and $\frac{M}{U+K}$ is singular.

Proof. By hypothesis $U + V = M$ and $U \cap V \leq \delta(V)$ . First we will show $U + K \neq M$. Assume
that $U + K = M$. Since $K$ is maximal in $V$, $K + Rx = V$ for $x \in V - K$. We have $M = U + V =
U + K + Rx$ so $Rx \subseteq U + K$ hence $x \in U + K$. Therefore, $x = y + k$ such that $y \in U$, $k \in K$.
It is easy to see that $y \in U \cap V \leq \delta(V) \leq K$. This contradicts with $x \notin K$. Then, there is an element
$m \in M \setminus (U + K)$. Since $m \in M = U + V$ we can write $m = u + v$ such that $u \in U$, $v \in V$.
Here $v \notin U + K$ and so $v \notin K$. Following this, $U + K + Rm = U + K + Rv$ can be shown simply.
$K + Rv = V$ since $K$ is maximal in $V$. This means that $U + K + Rm = M$. Namely $U + K$ is a maximal submodule of $M$. Additively, $\frac{M}{U+K}$ is singular since, $\frac{M}{U+K} \cong \frac{V}{K+(U\cap V)} \leq \frac{V}{K}$.

Proposition 19. Let $M$ be an $R$-module, $\delta(M) \subseteq U$, $V$ be a generalized $\delta$-supplement of $U$ and
$p : M \rightarrow \frac{M}{\delta(M)}$ be natural epimorphism. Then, $\frac{M}{\delta(M)} = p(U) \oplus p(V)$.
Proof. By hypothesis, \( U + V = M \) and \( U \cap V \leq \delta(V) \). It is clear that \( p(U) = \frac{U}{\delta(M)} \) and \( p(V) = \frac{V + \delta(M)}{\delta(M)} \). Following this, we have \( p(U) + p(V) = \frac{U + V + \delta(M)}{\delta(M)} = \frac{M}{\delta(M)} \) and \( p(U) \cap p(V) = \frac{U}{\delta(M)} \cap \frac{V + \delta(M)}{\delta(M)} = \frac{(U \cap V) + \delta(M)}{\delta(M)} = 0 \frac{M}{\delta(M)} = \{ \delta(M) \} \).

Proposition 20. Let \( M \) be an \( R \)-module and \( V \) be a generalized \( \delta \)-supplement of \( U \). Then, \( \frac{V + L}{L} \) is a generalized \( \delta \)-supplement of \( \frac{U}{L} \) in \( \frac{M}{L} \) where \( L \leq U \).

Proof. Clearly, \( U + V = M \) and \( U \cap V \leq \delta(V) \leq \delta(V + L) \). Following this \( \frac{U}{L} + \frac{V + L}{L} = \frac{M}{L} \) and \( \frac{U}{L} \cap \frac{V + L}{L} = \frac{(U \cap V) + L}{L} = p(U \cap V) \) such that \( p : V + L \rightarrow \frac{V + L}{L} \) is natural epimorphism where \( p(\delta(V + L)) \leq \delta(\frac{V + L}{L}) \). So, \( \frac{(U \cap V) + L}{L} \leq \delta(\frac{V + L}{L}) \) since \( p(U \cap V) \leq p(\delta(V + L)) \).

Definition 3. A module \( M \) is said to be \( \delta \)-local if \( \delta(M) \bowtie \delta M \) and \( \delta(M) \) is a maximal submodule of \( M \) [2].

Proposition 21. Any \( \delta \)-local module is generalized \( \delta \)-supplemented.

Proof. Let \( N \leq M \) be a proper submodule of \( M \). Since, \( \delta(M) \) is a maximal submodule of \( M \), we have either \( N \leq \delta(M) \) or \( \delta(M) + N = M \). If \( N \leq \delta(M) \) then trivially \( M \) is a generalized \( \delta \)-supplement of \( N \) in \( M \). Now suppose that \( \delta(M) + N = M \). Since \( \delta(M) \bowtie \delta M \), we have by lemma 1, \( N \oplus Y = M \) for some projective semisimple submodule \( Y \leq \delta(M) \). Clearly \( Y \) is a generalized \( \delta \)-supplement of \( N \) in \( M \). Therefore, \( M \) is generalized \( \delta \)-supplemented.

Proposition 22. Let \( M \) be an \( R \)-module. Then \( M \) is a generalized \( \delta \)-supplement of \( \delta(M) \).

Proposition 23. Let \( M \) be an \( R \)-module and \( V \) be a generalized \( \delta \)-supplement of \( U \) in \( M \) such that \( \delta(V) \bowtie \delta V \). then \( V \) is also a \( \delta \)-supplement of \( U \).

Proposition 24. \( U, V \leq M \) and \( V \) is a generalized \( \delta \)-supplement of \( U \) in \( M \). If \( U \cap V \) is a \( \delta \)-supplement of \( U \) then \( V \) is a \( \delta \)-supplement in \( M \).

Proof. By hypothesis, \( U + V = M \) and \( U \cap V \leq \delta(V) \). Let \( U \cap V \) be a \( \delta \)-supplement of \( X \leq U \). Then \( X + U \cap V = U \) and \( X \cap (U \cap V) = X \cap V \bowtie \delta U \cap V \). So we have \( M = U + V = (X + U \cap V) + V = X + V \). Additionally \( X \cap V \bowtie \delta V \) since \( X \cap V \bowtie \delta U \cap V \leq V \).
Corollary 25. Generalized $\delta$-supplement of a semisimple submodule is also a $\delta$-supplement.

Proof. Let $M$ be an $R$-module, $U$ be a semisimple submodule and $V$ be a generalized $\delta$-supplement of $U$. Since $U \cap V$ is a direct sum it is a $\delta$-supplement. And so, $V$ is a $\delta$-supplement by the previous proposition.

5. Generalized $f – \delta$ Supplemented Modules

Definition 4. Let $M$ be an $R$-module. If every finitely generated submodule has a generalized $\delta$-supplement then $M$ is called finitely generated $\delta$-supplement module (denoted briefly $f$-$\delta$-GS).

Proposition 26. Let $M$ be an $f$-$\delta$-GS module and $L$ be a finitely generated submodule. Then, $\frac{M}{L}$ is $f$-$\delta$-GS.

Proof. $\frac{K}{L} \leq \frac{M}{L}$ be any finitely generated submodule. It can be shown that $K$ is finitely generated. From hypothesis, $K$ has a generalized $\delta$-supplement $N \leq M$. Then $\frac{N+L}{L}$ is a generalized $\delta$-supplement of $\frac{K}{L}$ in $\frac{M}{L}$.

Proposition 27. Let $M$ be an $f$-$\delta$-GS module. If $\delta(M)$ is finitely generated then every finitely generated submodule of $\frac{M}{\delta(M)}$ is a direct sum.

Proof. $\frac{M}{\delta(M)}$ is an $f$-$\delta$-GS module by the previous proposition. Let $\frac{K}{\delta(M)} \leq \frac{M}{\delta(M)}$ be a finitely generated submodule. Then $\frac{K}{\delta(M)}$ has a generalized $\delta$-supplement $\frac{X}{\delta(M)}$ such that $\frac{K}{\delta(M)} + \frac{X}{\delta(M)} = \frac{M}{\delta(M)}$ and $\frac{K}{\delta(M)} \cap \frac{X}{\delta(M)} \leq \delta(\frac{X}{\delta(M)})$. And so, $\frac{K}{\delta(M)} \cap \frac{X}{\delta(M)} = \frac{K \cap X}{\delta(M)} \leq \delta(\frac{X}{\delta(M)}) \leq \delta(\frac{M}{\delta(M)}) = \{\delta(M)\}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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