A GENERALIZATION OF COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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Abstract. We say that a module $M$ is a closed cofinitely weak generalized supplemented module or briefly ccwgs-module if for every $N \leq_{cc} M$, $N$ has a weak Rad-supplement in $M$. In this article, the various properties of ccwgs-modules are given as a proper generalization of cofinitely weak Rad-supplemented modules. We prove that every cofinite direct sum of a ccwgs-module is a ccwgs-module. In particular, we also prove that every ccwgs-module over a left Bass ring is a ccws-module. Finally, we show that the notion of cofinitely weak Rad-supplemented modules and the notion of ccwgs-modules are equivalent under some special conditions.

Keywords: supplements; cofinite submodule; cofinitely weak Rad-supplemented module; closed weak Rad-supplemented modules.

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1. Introduction

Throughout this paper, it is assumed that $R$ is an associative ring with identity and all modules are unital left $R$-modules. Let $R$ be such a ring and let $M$ be an $R$-module. The notation $K \leq M$
(K < M) means that K is a (proper) submodule of M. A submodule N of M is called cofinite in M if the factor module \( \frac{M}{N} \) is finitely generated. A module M is called extending if every submodule is essential in a direct summand of M [4]. Here a submodule K \( \leq M \) is said to be essential in M, denoted as K \( \leq \text{essential} M \), if K \( \cap N \neq 0 \) for every non-zero submodule N \( \leq M \). A closed submodule N of M, denoted by N \( \leq \text{c} M \), is a submodule which has no proper essential extension in M. Every direct summand of a module M is a closed submodule of M. If L \( \leq \text{c} N \) and N \( \leq \text{c} M \), then L \( \leq \text{c} M \) by [8, Proposition 1.5]. If N is closed and cofinite submodule of M, we denote as N \( \leq \text{cc} M \). As a dual notion of an essential submodule, a proper submodule S of M is called small (in M), denoted as S \( \ll M \), if M \( \neq S + L \) for every proper submodule L of M [20, 19.1].

The Jacobson radical of M will be denoted by \( \text{Rad}_M \). It is known that \( \text{Rad}_M \) is the sum of all small submodules of M [20, 21.5]

A non-zero module M is said to be hollow if every proper submodule of M is small in M, and it is said to be local if it is hollow and is finitely generated. A module M is local if and only if it is finitely generated and \( \text{Rad}_M \) is maximal (see [4, 2.12 §2.15]). A ring R is said to be local if J is maximal, where J is the Jacobson radical of R.

An R-module M is called supplemented if every submodule of M has a supplement in M. Here a submodule K \( \leq M \) is said to be a supplement of N in M if K is minimal with respect to N + K = M, or equivalently, if N + K = M and N \( \cap K \ll K \) [20, page 349]. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a proper generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. As a generalization of supplemented modules, a module M is called weakly supplemented if any submodule N of M has a weak supplement K, i.e. there exists a submodule K of M such that M = N + K and N \( \cap K \ll M \) as in [11].

Alizade et al. [1] have defined cofinitely supplemented modules as a proper generalization of supplemented modules. They call a module M cofinitely supplemented if every cofinite submodule N of M has a supplement in M, and give characterizations of these modules over any rings and commutative domains (see [1]). In particular, it is shown in [1, Theorem 2.8] that a module M is cofinitely supplemented if and only if every maximal submodule of M.
has a supplement in $M$. A module $M$ is called *cofinitely weak supplemented* if every cofinite submodule has a weak supplement in $M$ [2].

A module $M$ is called *lifting* (or $D_1$-module) if, for every submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$ [4, 22.2]. Mohamed and Müller have generalized the concept of lifting modules to $\oplus$-supplemented modules. $M$ is called $\oplus$-*supplemented* if every submodule $N$ of $M$ has a supplement that is a direct summand of $M$ [12]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [12, Lemma A.4 (2)]). It is shown in [12, Proposition A.7 and Proposition A.8] that if $R$ is a dedekind domain, every supplemented $R$-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

In [5], Çalışıcı and Pancar call a module $M$ $\oplus$-*cofinitely supplemented* if every cofinite submodule of $M$ has a supplement that is a direct summand of $M$. They gave in the same paper some properties of these modules. In addition, it is proven in [5, Theorem 2.9] that a ring $R$ is semiperfect (that is, $_RR$ is supplemented) if and only if every free left $R$-module is $\oplus$-cofinitely supplemented.

Let $M$ be a module and $U, V$ be submodules of $M$. A submodule $V$ of $M$ is called *Rad-supplement* (according to [19], *generalized supplement*) of $U$ in $M$ if $U + V = M$ and $U \cap V \subseteq \text{Rad}V$ (see [4, Theorem 10.14]). A module $M$ is called *Rad-supplemented* (according to [19], *generalized supplemented*) if every submodule $U$ of $M$ has a Rad-supplement in $M$. Since Jacobson radical of a module $M$ is the sum of all small submodules of $M$, every supplement is a Rad-supplement. Then, clearly every supplemented module is Rad-supplemented but a Rad-supplemented module need not to be supplemented. Note that radical modules are Rad-supplemented. Let $R$ be a non-local dedekind domain with quotient field $K$. Then $K$ is Rad-supplemented.

In [3], a module $M$ is called *cofinitely Rad-supplemented* if every cofinite submodule has a Rad-supplement in $M$, and the closure properties of cofinitely Rad-supplemented modules is given. Lomp [11] calls a module $M$ *semilocal* if $\frac{M}{\text{Rad}M}$ is semisimple. Equivalently, every submodule $N$ of $M$ has a weak Rad-supplement $K$ in $M$, that is, $M = N + K$ and $N \cap K \subseteq \text{Rad}M$. A ring $R$ is called semilocal if the left (or right) $R$-module $R$ is semilocal. He show [11, Theorem
that $R$ is semilocal if and only if every left $R$-module is semilocal. A submodule $V$ of $M$ is called weak Rad-supplement of $U$ in $M$ if $U + V = M$ and $U \cap V \subseteq \text{Rad} M$ ([7]). A module $M$ is called cofinitely weak Rad-supplemented if every cofinite submodule $U$ of $M$, there exists a submodule $V$ of $M$ such that $U + V = M$ and $U \cap V \subseteq \text{Rad} M$.

Let $M$ be an $R$-module. $M$ is called Rad-$\oplus$-supplemented, or generalized $\oplus$-supplemented, if every submodule of $M$ has a Rad-supplement that is a direct summand of $M$ ([6]). Clearly, Rad-$\oplus$-supplemented modules are Rad-supplemented. A module $M$ is called $\oplus$-cofinitely Rad-supplemented (according to [9], generalized $\oplus$-cofinitely supplemented) if every cofinite submodule of $M$ has a Rad-supplement that is a direct summand of $M$. Instead of a $\oplus$-cofinitely radical supplemented module, we will use a cgs$^{\oplus}$-module like for [13].

In [14], the notion of closed weak supplemented modules is studied as a generalization of weak supplemented modules. A module $M$ is called a closed weak supplemented module if every closed submodule has a weak supplement in $M$. Then, Türkmen et al. call a module $M$ closed cofinitely weak supplemented module (or briefly, ccws-module) if for $N \leq_{cc} M$, $N$ has a weak supplement in $M$ ([18]). The various properties of ccws-modules are given in the same paper. A module $M$ is called closed weak generalized supplemented (or, closed weak Rad-supplemented) if every closed submodule has a weak generalized supplement (weak Rad-supplement) in $M$.

In this paper, we introduce the notion of closed cofinitely weak Rad-supplemented modules, denoted by ccwgs, as a proper generalization of ccws-modules. We provide some properties of these modules. An example is given to separate ccwgs-modules and cofinitely weak Rad-supplemented modules. We prove that every cofinite direct summand of a ccwgs-module is a ccwgs-module. We obtain that every ccwgs-module over a left Bass ring is a ccws-module. We also prove that a cofinitely strong refinable module $M$ is cgs$^{\oplus}$-module if and only if $M$ is a cofinitely weak supplemented modules.

2. ccwgs-Modules

In this section, we define the concept of ccwgs-modules as a generalization of cofinitely weak Rad-supplemented modules, and give various properties of them.
Definition 2.1. Let $M$ be a module. $M$ is called a closed cofinitely weak generalized supplemented module (or briefly a ccwgs-module) if, for every $N \leq_{cc} M$, there exists a submodule $K$ of $M$ such that $M = K + N$ and $K \cap N \subseteq \text{Rad} M$.

Under given definitions, we clearly have the following implications on modules:

$$
\text{cofinitely Rad-supplemented modules} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Lemma 2.4. (See [15, Lemma 3.5]) Let $N$ and $L$ be cofinite submodules of a module $M$ such that $N + L$ has a weak Rad-supplement $H$ in $M$ and $N \cap (H + L)$ has a weak Rad-supplement $G$ in $N$. Then $H + G$ is a weak Rad-supplement of $L$ in $M$.

Proposition 2.5. Let $M = M_1 \oplus M_2$ such that each $M_i$ ($i = 1, 2$) is a ccwgs-module. Suppose that $M_i \cap (M_j + L) \leq_{cc} M_i$ and $M_j \cap (L + K) \leq_{cc} M_j$, where $K$ is a weak Rad-supplement of $M_i \cap (M_j + L)$ in $M_i$, $i \neq j$, for any $L \leq_{cc} M$. Then $M$ is a ccwgs-module.

Proof. Let $L \leq_{cc} M$, then $M = M_1 + M_2 + L$ has a weak Rad-supplement 0 in $M$. Since $M_1 \cap (M_2 + L) \leq_{cc} M_1$ and $M_1$ is a ccwgs-module, then there exists a submodule $K$ of $M_1$ such that $M_1 = M_1 \cap (M_2 + L) + K$ and $M_1 \cap (M_2 + L) \cap K = K \cap (M_2 + L) \subseteq \text{Rad} M_1$. By Lemma 2.4, $K$ is a weak Rad-supplement of $M_2 + L$ in $M$, i.e. $M = K + (M_2 + L)$. Since $M_2 \cap (K + L) \leq_{cc} M_2$ and $M_2$ is a ccwgs-module, then $M_2 \cap (K + L)$ has a weak Rad-supplement $J$ in $M_2$. Again by Lemma 2.4, $K + J$ is a weak Rad-supplement of $L$ in $M$. Hence $M$ is a ccwgs-module.

Proposition 2.6. Let $M = M_1 + M_2$, where $M_1$ is a ccwgs-module and $M_2$ is any $R$-module. Suppose that for any $N \leq_{cc} M$, $N \cap M_1 \leq_{cc} M_1$. Then $M$ is a ccwgs-module if and only if every $N \leq_{cc} M$ with $M_2$ not contained in $N$ has a weak Rad-supplement.

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Let $N \leq_{cc} M$ with $M_2 \leq N$. Then $M = M_1 + M_2 = M_1 + N$ and $M_1 + N$ has a weak Rad-supplement 0 in $M$. Since $N \cap M_1 \leq_{cc} M_1$ and $M_1$ is a ccwgs-module, $N \cap M_1$ has a weak Rad-supplement $H$ in $M_1$. By Lemma 2.4, $H$ is a weak Rad-supplement of $N$ in $M$. By the hypothesis, $M$ is a ccwgs-module.

Recall from [10, page 185] that a left $R$-module $M$ is said to be singular (respectively, non-singular) if $Z(M) = M$ (respectively, $Z(M) = 0$), where $Z(M) = \{ m \in M | \text{Ann}(m) \trianglelefteq R \}$.

Let $M$ be a non-singular module and $N \leq_{cc} M$, then $N \cap L \leq_{cc} L$ for any submodule $L$ of $M$ and $M = N + L$.

Corollary 2.7. Let $M = M_1 + M_2$ be a non-singular module with $M_1$ ccwgs and $M_2$ any $R$-module. Then $M$ is a ccwgs-module if and only if every $N \leq_{cc} M$ with $M_2$ not contained in $N$ has a weak Rad-supplement.
Recall from [16, 1.11] that a module $M$ is said to be *distributive* if $(X+Y) \cap Z = (X \cap Z) + (Y \cap Z)$ for any submodules $X, Y$, and $Z$ of $M$. This means that the submodule lattice $\text{Lat}(M)$ is distributive.

**Theorem 2.8.** Let $M = M_1 \oplus M_2$ be a distributive module. Then $M$ is a ccwgs-module if and only if, for each $M_i$ of $M$, $i \in \{1, 2\}$, $M_i$ is a ccwgs-module.

**Proof.** Let $L \leq_{cc} M$. By the isomorphisms $M/L \cong M_1/M_1 \cap L + M_2/M_2 \cap L$ and $M/L \cong M_i/M_i \cap L$ for each $M_i$, $i \in \{1, 2\}$ and $i \neq j$, we have $M_i \cap L$ is a cofinite submodule of $M_i$. In addition, since $L$ is a closed submodule of $M$, then for each $i, i \in \{1, 2\}$, $M_i \cap L$ is closed in $M_i$. So $M_i \cap L \leq_{cc} M_i$. In fact, suppose that $M_i \cap L$ is essential in $K$. Since $M_2 \cap L$ is essential in $M_2 \cap L$ and $M$ is distributive, we have that $L = (M_1 \cap L) \oplus (M_2 \cap L) = K \oplus (M_2 \cap L)$, because $L$ is closed in $M$. Since for each $i, i \in \{1, 2\}, M_i$ is a ccwgs-module, there exists a submodule $K_i$ of $M_i$ such that $M_i = (L \cap M_i) + K_i, (L \cap M_1) \cap K_i = L \cap K_i \subseteq \text{Rad}M_1$. Hence $M = M_1 \oplus M_2 = [(L \cap M_1) \oplus (L \cap M_2)] + (K_1 + K_2) = L + (K_1 \oplus K_2)$ and $L \cap (K_1 \oplus K_2) = (L \cap K_1) \oplus (L \cap K_2) \subseteq \text{Rad}M_1 \oplus \text{Rad}M_2 = \text{Rad}(M_1 \oplus M_2) = \text{Rad}M$. Thus $M$ is a ccwgs-module. The converse holds by Proposition 2.3.

**Corollary 2.9.** Let $M = \oplus_{i=1}^n M_i$ be a duo module. Then $M$ is a ccwgs-module if and only if for each cofinite direct summand $M_i, i \in \{1, 2, \ldots, n\}, M_i$ is a ccwgs-module.

Recall from [17] that a module $M$ is called *cofinitely strong refinable* if, for every cofinite submodule $U$ of $M$ and any submodule $V$ of $M$ with $U + V = M$, there exists submodules $U'$ and $V'$ of $M$ with $U' \subseteq U, V' \subseteq V, M = U' + V$ and $M = U' \oplus V'$.

**Proposition 2.10.** Let $M$ be a cofinitely strong refinable module. Then the following statements are equivalent.

(1) $M$ is a cgs$^\oplus$-module.

(2) $M$ is a cofinitely $\text{Rad}$-supplemented module.

(3) $M$ is a cofinitely weak $\text{Rad}$-supplemented module.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Suppose that $M$ is a cofinitely weak $\text{Rad}$-supplemented module. Let $N$ be any cofinite submodule of $M$. Then there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \subseteq \text{Rad}M$. Since $M$ is a cofinitely strong refinable module, there exist submodules $N'$
and $K'$ with $N' \subseteq N, K' \subseteq K, M = N + K'$ and $M = N' \oplus K'. \text{ It follows that } M = N + K'$ and $N \cap K' \subseteq \text{Rad}K'. \text{ Therefore } M \text{ is a } cgs^{\ominus}-\text{module.}

\textbf{Proposition 2.11.} Let $M$ be an $R$-module with $\text{Rad}(M) = 0$. Then, $M$ is a ccwgs-module if and only if every closed cofinite submodule is a direct summand of $M$.

\textbf{Proof.} ($\Rightarrow$) Let $N \leq_{cc} M$. By the hypothesis, there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \subseteq \text{Rad}M$. So $N \cap K = 0$. Thus $M = N \oplus K$. Therefore $N$ is a direct summand of $M$.

($\Leftarrow$) The converse is clear.

Using Proposition 2.10 and [4, 1.16], we obtain the following fact.

\textbf{Corollary 2.12.} Let $M$ be a finitely generated $R$-module with $\text{Rad}(M) = 0$. Then the following are equivalent.

(1) $M$ is a ccwgs-module.

(2) $M$ is extending.

Recall [20, page 192] that a ring $R$ is called a \textit{left V-ring} if every simple left $R$-module is injective. Equivalently, a ring $R$ is a left V-ring if and only if $\text{Rad}(M) = 0$ for all left $R$-modules $M$.

\textbf{Theorem 2.13.} Let $R$ be a left nonsingular V-ring. Then the following statements are equivalent.

(1) Every nonsingular left $R$-module $M$ is a ccwgs-module.

(2) For any closed cofinite submodule $N$ of every nonsingular left $R$-module $M$, $N$ is a direct summand of $M$.

\textbf{Proof.} Clear by Proposition 2.11.

Any finite sum of ccwgs-modules need not to be a ccwgs-module, in general. The following Example shows this.

\textbf{Example 2.14.} Let $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of all integers. It can be seen that the left $R$-module $R$ is a ccwgs-module and $M = R \oplus R$ is not an extending $R$-module. As $\text{Rad}(M) = 0$, by Corollary 2.12, $M$ is not a ccwgs-module.
Lemma 2.15. (See [15, Lemma 4.10]) Let $U$ and $K$ be submodules of $M$ such that $K$ is a weak generalized supplement of a maximal submodule $N$ of $M$. If $K + U$ has a weak Rad-supplement $X$ in $M$, then $U$ has a weak Rad-supplement in $M$.

Theorem 2.16. Suppose that for any cofinite submodule $U$ of $M$, there exists a submodule $K$ of $M$, which is a weak Rad-supplement of some maximal submodule $N$ of $M$, such that $K + U$ is closed in $M$. Then $M$ is a ccwgs-module if and only if $M$ is a cofinitely weak Rad-supplemented module.

Proof. ($\Rightarrow$) Let $U \leq_{cc} M$. By the hypothesis, there exists a submodule $K$ of $M$ such that $M = N + K$, $N \cap K \subseteq \text{Rad}M$ and $K + U \leq_{c} M$ for a maximal submodule $N$ of $M$. It follows from $\frac{M}{K + U} \cong \frac{M}{K + U}$ and $U$ is a cofinite submodule of $M$ that $K + U$ is a cofinite submodule of $M$. Since $M$ is a ccwgs-module, there exists a submodule $X$ of $M$ such that $X$ is a weak Rad-supplement of $K + U$. By Lemma 2.15, $U$ has a weak Rad-supplement in $M$. So $M$ is a cofinitely weak Rad-supplemented module.

($\Leftarrow$) Clear.

Lemma 2.17. Let $M$ be a ccwgs-module. Suppose that $\text{Rad}M$ is small in $M$. Then $M$ is a ccwgs-module if and only if $M$ is a ccws-module.

Proof. Let $N \leq_{cc} M$. Since $M$ is a ccwgs-module, there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \subseteq \text{Rad}M$. Since $\text{Rad}M \ll M$, $N \cap K \ll M$. Thus $M$ is a ccws-module. The converse is clear.

A module $M$ is called coatomic if every proper submodule of $M$ is contained in a maximal submodule of $M$. Note that coatomic modules have a small radical.

Corollary 2.18. Let $M$ be a coatomic module. Then $M$ is a ccwgs-module if and only if it is a ccws-module.

Recall from [4] that a ring $R$ is a left Bass ring if every non-zero left $R$-module has a maximal submodule. It is known that the ring $R$ is left Bass if and only if $\text{Rad}M$ is small in $M$ for every non-zero left $R$-module $M$. By using Lemma 2.17, we obtain the following corollary.

Corollary 2.19. Every ccwgs-module over a left Bass ring is a ccws-module.
Conflict of Interests
The authors declare that there is no conflict of interests.

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