ON FORMAL LOCAL COHOMOLOGY MODULES

C.H. TOGNON

Department of Mathematics, University of Sao Paulo, Sao Carlos - Sao Paulo, Brazil

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Abstract. Let $I$, $a$ be two ideals of a Noetherian ring $R$. Let $M$ be an $R$-module. There exists a systematic study of the formal cohomology modules $\lim_{\leftarrow n \in \mathbb{N}} H^i_I(M/a^nM)$, $0 \leq i \in \mathbb{Z}$. It is what will be done in this paper.

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1. Introduction

Throughout this paper, $R$ is a commutative ring with non-zero identity. The theory of local cohomology has developed for six decades after its introduction by Grothendieck. There exists a relation between local cohomology and formal local cohomology. We study here this latter module.

2. Preliminaries

E-mail address: carlostognon@gmail.com

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Let $I$ be an ideal of $R$, and let $M$ be an $R$-module. In [1], the $i$th local cohomology module $H^i_I(M)$ of $M$ with respect to $I$ is defined by

$$H^i_I(M) = \varprojlim_{t \in \mathbb{N}} \text{Ext}^i_R(R/I^t, M),$$

for all $0 \leq i \in \mathbb{Z}$. Now, for a other ideal of $R$, consider the family of local cohomology modules given by $\{H^i_I(M/a^nM)\}_{n \in \mathbb{N}}$. According to [4], for every $n \in \mathbb{N}$, we have that there exists a natural homomorphism

$$\phi_{n+1,n} : H^i_I(M/a^{n+1}M) \to H^i_I(M/a^nM).$$

These families form an inverse system. Their inverse limit that is given by $\varprojlim_{n \in \mathbb{N}} H^i_I(M/a^nM)$ is called, according to [4], the $i$th formal local cohomology module of $M$ with respect to $a$, and will be denoted by $\mathcal{H}^i_{a,I}(M)$. Moreover, for a Noetherian local ring $(R, m)$ and $M$ an $R$-module we have the Matlis dual module $D(M) = \text{Hom}_R(M, E)$ of $M$, where $E = E(R/m)$ is the injective envelope of the residue field $R/m$.

The next definition will be used in the sequence of the paper.

**Definition 2.1.** Let $(R, m, k)$ and $(S, n, l)$ be two local rings. A ring homomorphism $(R, m, k) \to (S, n, l)$ is a local homomorphism if $mS \subset n$.

In the next section, the following remark will be used.

**Remark 2.2.**([4, Remark 4.6]) Note that, the short exact sequence

$$0 \to a^nM/a^{n+1}M \to M/a^{n+1}M \to M/a^nM \to 0$$

induces an epimorphism $H^i_I(M/a^{n+1}M) \to H^i_I(M/a^nM) \to 0$, of non-zero $R$-modules for all $n \in \mathbb{N}$. Hence, the inverse limit $\varprojlim_{n \in \mathbb{N}} H^i_I(M/a^nM)$ is not zero.

The following definition will be used in the next section.

**Definition 2.3.**([2, Definition 3.1]) Let $R$ be a Noetherian ring. Let $I$ be an ideal of $R$ and let $M$ be an $R$-module. The $i$th local homology module $H^i_I(M)$ of $M$ with respect to $I$ is defined by,

$$H^i_I(M) := \lim_{\leftarrow t \in \mathbb{N}} \text{Tor}^R_i(R/I^t, M).$$
3. Main Results

In this section, we have a result on formal local cohomology modules.

**Theorem 3.1.** Let \((R, m, k)\) be a Noetherian local ring. Let \((R, m, k) \to (S, n, l)\) be a local homomorphism of local rings, with \(S\) a Noetherian ring, and let \(a\) be an ideal of \(R\). Suppose that \(M\) is a finitely generated \(S\)-module. If \(\bigcap a^i_m(M) = 0\), for each \(i \geq 1\), then \(D(M/a^nM)\) is a flat \(R\)-module, for some \(n \in \mathbb{N}\).

**Proof.** By the hypothesis, for all \(i \geq 1\), we have that:

\[
\bigcap a^i_m(M) := \lim_{n \in \mathbb{N}} H^i_m(M/a^nM) = 0.
\]

By the Remark , we have that there exists \(n \in \mathbb{N}\) such that the local cohomology module \(H^i_m(M/a^nM) = 0\). Therefore, it follows, as given in prerequisites, that we have:

\[
\lim_{t \in \mathbb{N}} \text{Ext}_R^t(R/m^t, M/a^nM) = 0. \quad (*)
\]

Thus, applying the Matlis dual module \(D(\bullet)\) (see prerequisites) to \((*)\) we obtain that \(D \left( \lim_{t \in \mathbb{N}} \text{Ext}_R^t(R/m^t, M/a^nM) \right) = 0\). Now, by [3, Theorem 2.27], it follows that \(D \left( \lim_{t \in \mathbb{N}} \text{Ext}_R^t(R/m^t, M/a^nM) \right)\), which is equal to

\[
\text{Hom}_R \left( \lim_{t \in \mathbb{N}} \text{Ext}_R^t(R/m^t, M/a^nM), E(R/m) \right),
\]

is isomorphic to \(\lim_{t \in \mathbb{N}} \text{Hom}_R \left( \text{Ext}_R^t(R/m^t, M/a^nM), E(R/m) \right)\), which in turn is equal to \(\lim_{t \in \mathbb{N}} D \left( \text{Ext}_R^t(R/m^t, M/a^nM) \right)\).

By [5, Proposition 3.4.14 (ii)], we have that:

\[
D \left( \text{Ext}_R^t(R/m^t, M/a^nM) \right) \cong \text{Tor}_i^R(R/m^t, D(M/a^nM)).
\]

Therefore, by the Definition , we have that

\[
H^m_i(D(M/a^nM)) = \lim_{t \in \mathbb{N}} \text{Tor}_i^R(R/m^t, D(M/a^nM)) = 0.
\]

By Remark , it follows that there exists \(t \in \mathbb{N}\) such that

\[
\text{Tor}_i^R(R/m^t, D(M/a^nM)) = 0, \text{ for all } i \geq 1.
\]

Thus, also we have \(\text{Tor}_i^R(R/m, D(M/a^nM)) = 0, \text{ for all } i \geq 1 \quad (**).\)
To end the theorem, it suffices to prove that $\text{Tor}^R_i(N, D(M/\mathfrak{a}^nM)) = 0$ for each finitely generated $R$-module $N$, and $i \geq 1$. This we achieve by an induction on $\dim(N)$.

When $\dim(N) = 0$, let’s induce on the length of $N$. If $l_R(N) = 1$, then $N \cong R/\mathfrak{m}$, so the desired result is the mentioned in $(\ast\ast)$. When $l_R(N) \geq 2$, one can get an exact sequence of $R$-modules $0 \to R/\mathfrak{m} \to N \to N' \to 0$. Applying $\bullet \otimes_R D(M/\mathfrak{a}^nM)$ yields an exact sequence

$$\text{Tor}^R_i(R/\mathfrak{m}, D(M/\mathfrak{a}^nM)) \to \text{Tor}^R_i(N, D(M/\mathfrak{a}^nM)) \to \text{Tor}^R_i(N', D(M/\mathfrak{a}^nM)).$$

Since $l_R(N') = l_R(N) - 1$, the induction hypothesis yields the vanishing.

Let $d \geq 1$ be an integer such that for $i \geq 1$ we have that the functor $\text{Tor}^R_i(\bullet, D(M/\mathfrak{a}^nM))$ vanishes on finitely generated $R$-modules of dimension up to $d - 1$. Let $N$ be a finitely generated $R$-module of dimension $d$. Consider the exact sequence of $R$-modules

$$0 \to \Gamma_m(N) \to N \to N' \to 0,$$

and the induced exact sequence on $\text{Tor}^R_i(\bullet, D(M/\mathfrak{a}^nM))$. Since $l_R(\Gamma_m(N))$ is finite, it suffices to verify the vanishing for $N'$. Thus, replacing $N$ by $N'$, one may assume that $\text{depth}(N) \geq 1$. Let $x$ in $R$ be an $N$-regular element; then $\dim(N/(x)N) = \dim(N) - 1$. In view of the induction hypothesis, the exact sequence $0 \to N \xrightarrow{x} N \to N/(x)N \to 0$ induces an exact sequence

$$\text{Tor}^R_i(N, D(M/\mathfrak{a}^nM)) \xrightarrow{x} \text{Tor}^R_i(N, D(M/\mathfrak{a}^nM)) \to \text{Tor}^R_i(N/(x)N, D(M/\mathfrak{a}^nM)) = 0$$

for $i \geq 1$, where $\tilde{M} = M/\mathfrak{a}^nM$. As an $S$-module $\text{Tor}^R_i(N, D(M/\mathfrak{a}^nM))$ is finitely generated: compute it using a resolution of $N$ by finitely generated free $R$-modules. Since, by Definition, $xS$ is in the maximal ideal of $S$, the exact sequence above implies $\text{Tor}^R_i(N, D(M/\mathfrak{a}^nM)) = 0$ by Nakayama’s lemma, for all $i \geq 1$. This completes the induction step.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


