A NEW DERIVATION OF UP-ALGEBRAS BY MEANS OF UP-ENDOMORPHISMS

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Abstract. The notions of left (resp. right)-f-derivations of type I and of left (resp. right)-f-derivations of type II of UP-algebras are introduced, some useful examples are discussed, and related properties are investigated. Moreover, we show that the kernel of right-f-derivations of type I and of right-f-derivations of type II of UP-algebras is a UP-subalgebra, and also give examples to show that the kernel of left (resp. right)-f-derivations of type I and of left (resp. right)-f-derivations of type II of UP-algebras is not a UP-ideal, the fixed set of right-f-derivations of type I and of left (resp. right)-f-derivations of type II of UP-algebras is not a UP-subalgebra, and the fixed set of left-f-derivations of type I of UP-algebras is not a UP-ideal in general.

Keywords: UP-algebra; UP-subalgebra; UP-ideal; left-f-derivation; right-f-derivation.

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1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [14], BCI-algebras [15], BCH-algebras [11], KU-algebras

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SU-algebras [19] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [15] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [14, 15] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

In the theory of rings and near rings, the properties of derivations is an important topic to study [32, 22]. In 2004, Jun and Xin [18] applied the notions of rings and near rings theory to BCI-algebras and obtained some properties. Zhan and Liu [38] introduced the notion of left-right (resp. right-left) $f$-derivations of BCI-algebras, investigated some related properties by using the idea of regular $f$-derivations and they gave characterizations of $p$-semisimple BCI-algebras. Several researches were conducted on the generalizations of the notion of derivations and application to many logical algebras such as: In 2005, Zhan and Liu [38] introduced the notion of left-right (resp. right-left) $f$-derivations of BCI-algebras. In 2006, Abujabal and Al-shehri [1] studied derivations of $p$-semisimple BCI-algebras and proved that for any derivations $d_1, d_2$ of a $p$-semisimple BCI-algebra $X$, $d_1 \circ d_2$ is also a derivation of $X$, $d_1 \circ d_2 = d_2 \circ d_1$ and $d_1 \ast d_2 = d_2 \ast d_1$. In 2007, Abujabal and Al-shehri [2] introduced the notion of left derivations of BCI-algebras, investigated regular left derivations and studied left derivations on $p$-semisimple BCI-algebras. In 2009, Javed and Aslam [17] studied derivations of $p$-semisimple BCI-algebras and proved that for any $f$-derivations $d_f, d'_f$ of a $p$-semisimple BCI-algebra $X$, $d_f \circ d'_f$ is also an $f$-derivation of $X$ and $d_f \circ d'_f = d'_f \circ d_f$. Nisar [31] introduced the notions of right $F$-derivations and left $F$-derivations of BCI-algebras. Nisar [30] characterized $f$-derivations of BCI-algebras. Prabpayak and Leerawat [33] studied left-right derivations and right-left derivations of BCC-algebras and also considered regular derivations of BCC-algebras. In 2010, Al-shehri [4] applied the notion of derivations in ring and near-ring theory to MV-algebras and investigated some of its properties. They introduced additive derivations of MV-algebras, investigated several properties and proved that an additive derivation of a linearly ordered MV-algebra is an isotope by used the notion of an isotone derivation and characterized derivations of MV-algebras.
Kim [20] introduced the notion of $f$-derivations which is a generalization of derivations in subtraction algebras, and some related properties are investigated. In 2011, Thomys [36] described $f$-derivations of weak BCC-algebras in which the condition $((xy)z = (xz)y$ for any $x, y, z$ in a BCI-algebra $X$ when $x, y$ belong to the same branch. In 2012, Al-shehri and Bawazeer [5] studied the notion of left-right (resp. right-left) $t$-derivations of BCC-algebras, investigated some properties on $t$-derivations of BCC-algebras and considered $t$-regular, $t$-derivations and the $d_t$-invariant on ideals of BCC-algebras. Lee and Kim [23] considered the properties of $f$-derivations of BCC-algebras and also characterized $\text{Kerd}$ by $f$-derivations. Muhiuddin and Al-roqi [27] introduced the notion of $t$-derivations of BCI-algebras and proved that for any $t$-derivation $d_t$ of a BCI-algebra $X$, $d_t \circ d'_t$ is also a $t$-derivation of $X$ and $d_t \circ d'_t = d'_t \circ d_t$, and for any $t$-derivation $d_t$ of a $p$-semisimple BCI-algebra $X$, $d_t \star d'_t = d'_t \star d_t$. Muhiuddin and Al-roqi [26] introduced the notion of (regular) $(\alpha, \beta)$-derivations of BCI-algebras. In 2013, Bawazeer, Al-shehri and Babusal [9] introduced the notion of generalized derivations of BCC-algebras. Lee [21] introduced a new kind of derivations of BCI-algebras. Ardekani and Davvaz [6] extend the notion of derivations of MV-algebras, introduced the notion of $f$-derivations and $(f, g)$-derivations of MV-algebras and investigated some properties of them. Muhiuddin, Al-roqi, Jun and Ceven [29] introduced the notion of symmetric left bi-derivations of BCI-algebras. Ghorbani, Torkzadeh and Motamed [10] introduced the notion of $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations for MV-algebras and studied the connection between these derivations on MV-algebras. They characterized the isotone $(\odot, \oplus)$-derivations and proved that $(\ominus, \odot)$-derivations are isotone. And they determined the relationship between $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations for MV-algebras. Leerawat and Bunphan [24] introduced the notion of $f$-derivations of Boolean algebras, namely Boolean $f$-derivations, investigated some related properties and proved that the fixed set and the kernel of Boolean $f$-derivations are ideals in Boolean algebras. Torkzadeh and Abbasian [37] defined the concept of studied $(\odot, \vee)$-derivations for BL-algebras and discussed some related results, studied $(\odot, \vee)$-derivations on boolean center $B(A)$ of a BL-algebras $A$, investigated some properties of isotone $(\odot, \vee)$-derivations on a BL-algebras $A$ and characterized the $(\odot, \vee)$-derivations on the Gödel structure $[0,1]$. In 2014, Al-roqi [3] introduced the notion of generalized (regular) $(\alpha, \beta)$-derivations of BCI-algebras. Muhiuddin and Al-roqi [28] introduced the notion of
generalized left derivations of BCI-algebras. Ardekani and Davvaz [7] introduced the notion of \((f,g)\)-derivations of BCI-algebras. Min, Xiaoao-long and Yi-jun [25] introduced the notion of \(f\)-derivations and \(g\)-derivations of MV-algebras. In 2015, Asawasamrit [8] introduced the notion of \(f\)-derivations of KK-algebras and investigated some related properties. Jana, Senapati and Pal [16] introduced the notion of left-right (resp. right-left) derivations, \(f\)-derivations, generalized derivations of KUS-algebras and proved that \(\text{(Der}(X),\wedge)\) refers to a semigroup for any \(p\)-semisimple KUS-algebra \(X\) and defined the relationship between left-right derivations, right-left derivations and generalized derivations of KUS-algebras. In 2016, Sawika, Intasan, Kaewwasri and Iampan [35] introduced the notions of \((l,r)\)-derivations, \((r,l)\)-derivations and derivations of UP-algebras and investigated some related properties. Iampan [12] introduced the notion of \(f\)-derivations of UP-algebras which is the generalization of the notion of derivations [35].

The notion of derivations play an important role in studying the many logical algebras. In this paper, we introduce the notions of left (resp. right)-\(f\)-derivations of type I and of left (resp. right)-\(f\)-derivations of type II of UP-algebras, some useful examples are discussed, and related properties are investigated.

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 1.1.** [13] An algebra \(A = (A;\cdot,0)\) of type \((2,0)\) is called a *UP-algebra* if it satisfies the following axioms: for any \(x,y,z \in A\),

- \((UP-1)\): \((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0\),
- \((UP-2)\): \(0 \cdot x = x\),
- \((UP-3)\): \(x \cdot 0 = 0\), and
- \((UP-4)\): \(x \cdot y = y \cdot x = 0\) implies \(x = y\).

From [13], we know that the notion of UP-algebras is a generalization of KU-algebras.

**Example 1.2.** [13] Let \(X\) be a universal set. Define a binary operation \(\cdot\) on the power set of \(X\) by putting \(A \cdot B = B \cap A = A' \cap B = B - A\) for all \(A, B \in \mathcal{P}(X)\). Then \((\mathcal{P}(X);\cdot,\emptyset)\) is a UP-algebra and we shall call it the *power UP-algebra of type 1*. 
**Example 1.3.** [13] Let $X$ be a universal set. Define a binary operation $*$ on the power set of $X$ by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

**Example 1.4.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th>$\cdot$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let $A$ and $B$ denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

**Proposition 1.5.** [13] In a UP-algebra $A$, the following properties hold: for any $x, y, z \in A$,

1. $x \cdot x = 0$,
2. $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,
3. $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,
4. $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,
5. $x \cdot (y \cdot x) = 0$,
6. $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and
7. $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on $A$ [13] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

**Proposition 1.6.** [13] In a UP-algebra $A$, the following properties hold: for any $x, y, z \in A$,

1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ imply $x = y$, and
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$,
(4) $x \leq y$ implies $z \cdot x \leq z \cdot y$,
(5) $x \leq y$ implies $y \cdot z \leq x \cdot z$,
(6) $x \leq y \cdot x$, and
(7) $x \leq y \cdot y$.

**Definition 1.7.** [13] A nonempty subset $B$ of $A$ is called a *UP-ideal* of $A$ if it satisfies the following properties:

1. the constant 0 of $A$ is in $B$, and
2. for any $x, y, z \in A$, $x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, $A$ and $\{0\}$ are UP-ideals of $A$.

**Definition 1.8.** [13] A subset $S$ of $A$ is called a *UP-subalgebra* of $A$ if the constant 0 of $A$ is in $S$, and $(S; \cdot, 0)$ is a UP-ideal of $A$. Clearly, $A$ and $\{0\}$ are UP-subalgebras of $A$.

**Proposition 1.9.** [13] A nonempty subset $S$ of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the $\cdot$ multiplication on $A$.

**Definition 1.10.** [13] Let $(A; \cdot, 0)$ and $(A'; \cdot', 0')$ be UP-algebras. A mapping $f$ from $A$ to $A'$ is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) \cdot' f(y)$$

for all $x, y \in A$.

A UP-homomorphism $f : A \to A'$ is called a *UP-endomorphism* of $A$ if $A' = A$.

**Definition 1.11.** [35] For any $x, y \in A$, we define a binary operation $\wedge$ on $A$ by $x \wedge y = (y \cdot x) \cdot x$.

**Definition 1.12.** [35] A UP-algebra $A$ is called *meet-commutative* if $x \wedge y = y \wedge x$ for all $x, y \in A$, that is, $(y \cdot x) \cdot x = (x \cdot y) \cdot y$ for all $x, y \in A$.

**Proposition 1.13.** [35] In a UP-algebra $A$, the following properties hold:

1. $0 \wedge x = 0$,
2. $x \wedge 0 = 0$, and
3. $x \wedge x = x$. 
2. **Left-\(f\)-derivations and right-\(f\)-derivations of type I**

In this section, we first introduce the notion of left (resp. right)-\(f\)-derivations of type I of \(A\) and study some of their basic properties. Finally, two subsets \(\text{Ker}_d(A)\) and \(\text{Fix}_d(A)\) for left (resp. right)-\(f\)-derivation \(d\) of type I of \(A\) are studied.

**Definition 2.1.** Let \(A\) be a UP-algebra and let \(f\) be a UP-endomorphism of \(A\). A self-map \(d: A \rightarrow A\) is called a **left-\(f\)-derivation of type I** (in short, an **l-\(f\)-derivation of type I**) of \(A\) if it satisfies the identity \(d(x \cdot y) = (d(x) \cdot f(y)) \land (x \cdot y)\) for all \(x, y \in A\). Similarly, a self-map \(d: A \rightarrow A\) is called a **right-\(f\)-derivation of type I** (in short, an **r-\(f\)-derivation of type I**) of \(A\) if it satisfies the identity \(d(x \cdot y) = (f(x) \cdot d(y)) \land (x \cdot y)\) for all \(x, y \in A\). Moreover, if \(d\) is both a left-\(f\)-derivation and a right-\(f\)-derivation of type I of \(A\), it is called an **\(f\)-derivation of type I of \(A\)**.

By using Microsoft Excel, we have all examples.

**Example 2.2.** Let \(A = \{0, 1, 2, 3\}\) be a set with a binary operation \(\cdot\) defined by the following Cayley table:

<table>
<thead>
<tr>
<th>(\cdot)</th>
<th>0</th>
<th>1</th>
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</table>

Then \((A; \cdot, 0)\) is a UP-algebra. We define a self-map \(f: A \rightarrow A\) as follows:

\[
f(0) = 0, f(1) = 1, f(2) = 3, \text{ and } f(3) = 2.
\]

Then \(f\) is a UP-endomorphism. We define a self-map \(d: A \rightarrow A\) as follows:

\[
d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 0.
\]

Then \(d\) is an l-\(f\)-derivation of type I of \(A\).
Example 2.3. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

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Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \to A$ as follows:

$f(0) = 0, f(1) = 0, \text{ and } f(2) = 1$.

Then $f$ is a UP-endomorphism. We define a self-map $d : A \to A$ as follows:

$d(0) = 0, d(1) = 1, \text{ and } d(2) = 1$.

Then $d$ is an $r$-$f$-derivation of type I of $A$.

Example 2.4. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
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</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \to A$ as follows:

$f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3, \text{ and } f(4) = 0$.

Then $f$ is a UP-endomorphism. We define a self-map $d : A \to A$ as follows:

$d(0) = 0, d(1) = 1, d(2) = 2, d(3) = 3, \text{ and } d(4) = 0$.

Then $d$ is an $f$-derivation of type I of $A$.

Definition 2.5. An $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) $d$ of type I of $A$ is called regular if $d(0) = 0$.

Theorem 2.6. In a UP-algebra $A$, the following statements hold:

1. every $l$-$f$-derivation of type I of $A$ is regular, and
2. every $r$-$f$-derivation of type I of $A$ is regular.
**Proof.** (1) Assume that \( d \) is an \( l \)-\( f \)-derivation of type I of \( A \). Then

\[
d(0) = d(x \cdot 0)
\]

(By UP-3)

\[
= (d(x) \cdot f(0)) \land (x \cdot 0)
\]

(By Definition 2.1)

\[
= (d(x) \cdot 0) \land 0
\]

(By UP-3)

\[
= 0 \land 0
\]

(By Proposition 1.13 (3))

\[
= 0.
\]

Hence, \( d \) is regular.

(2) Assume that \( d \) is an \( r \)-\( f \)-derivation of type I of \( A \). Then

\[
d(0) = d(0 \cdot 0)
\]

(By UP-3)

\[
= (f(0) \cdot d(0)) \land (0 \cdot 0)
\]

(By Definition 2.1)

\[
= (0 \cdot d(0)) \land 0
\]

(By UP-2)

\[
= d(0) \land 0
\]

(By Proposition 1.13 (2))

\[
= 0.
\]

Hence, \( d \) is regular.

\[\square\]

**Corollary 2.7.** Every \( f \)-derivation of type I of \( A \) is regular.

**Theorem 2.8.** In a UP-algebra \( A \), the following statements hold:

(1) if \( d \) is an \( l \)-\( f \)-derivation of type I of \( A \), then \( d(x) = f(x) \land x \) for all \( x \in A \), and

(2) if \( d \) is an \( r \)-\( f \)-derivation of type I of \( A \), then \( d(x) = d(x) \land x \) for all \( x \in A \).

**Proof.** (1) Assume that \( d \) is an \( l \)-\( f \)-derivation of I of \( A \). Then, for all \( x \in A \),

\[
d(x) = d(0 \cdot x)
\]

(By UP-2)

\[
= (d(0) \cdot f(x)) \land (0 \cdot x)
\]

(By Definition 2.1)

\[
= (0 \cdot f(x)) \land (0 \cdot x)
\]

(By Theorem 2.6 (1))

\[
= f(x) \land x.
\]
(2) Assume that \( d \) is an \( r-f \)-derivation of type I of \( A \). Then, for all \( x \in A \),

\[
\begin{align*}
  (\text{By UP-2}) & \quad d(x) = d(0 \cdot x) \\
  (\text{By Definition 2.1}) & \quad = (f(0) \cdot d(x)) \land (0 \cdot x) \\
  (\text{By UP-2}) & \quad = (0 \cdot d(x)) \land x \\
  (\text{By UP-2}) & \quad = d(x) \land x.
\end{align*}
\]

\( \square \)

**Corollary 2.9.** If \( d \) is an \( f \)-derivation of type I of \( A \), then \( d(x) = f(x) \land x = d(x) \land x \) for all \( x \in A \).

**Proposition 2.10.** Let \( d \) be an \( l-f \)-derivation of type I of \( A \). Then the following properties hold:

for any \( x, y \in A \),

(1) \( f(x) \leq d(x) \),
(2) \( d(x) \cdot f(y) \leq d(x \cdot y) \),
(3) \( d(x \cdot f(x)) \cdot f(f(x)) \leq d(d(x)) \),
(4) \( d(x) \cdot f(d(x)) \leq d(x \cdot d(x)) \),
(5) \( d(d(x)) \cdot f(x) \leq d(d(x) \cdot x) \),
(6) \( d(y \cdot x) \cdot f(x) \leq d(x \land y) \), and
(7) \( d(x) = d(x) \land f(x) \).

**Proof.** (1) For all \( x \in A \),

\[
\begin{align*}
  (\text{By Theorem 2.8 (1)}) & \quad f(x) \cdot d(x) = f(x) \cdot (f(x) \land x) \\
  (\text{By Definition 1.11}) & \quad = f(x) \cdot ((x \cdot f(x)) \cdot f(x)) \\
  (\text{By Proposition 1.5 (5)}) & \quad = 0.
\end{align*}
\]

Hence, \( f(x) \leq d(x) \) for all \( x \in A \).
(2) For all $x, y \in A$,

\[(d(x) \cdot f(y)) \cdot d(x \cdot y) = (d(x) \cdot f(y)) \cdot ((d(x) \cdot f(y)) \land (x \cdot y))\]

\[= (d(x) \cdot f(y)) \cdot (((x \cdot y) \cdot (d(x) \cdot f(y)))\land (x \cdot y)).\]

(By Definition 1.11)

\[(d(x) \cdot f(y))) = 0.\]

(By Proposition 1.5 (5))

Hence, $d(x) \cdot f(y) \leq d(x \cdot y)$ for all $x, y \in A$.

(3) For all $x \in A$,

\[(d(x \cdot f(x)) \cdot f(f(x))) \cdot d(d(x)) = (d(x \cdot f(x)) \cdot f(f(x))).\]

(By Definition 2.1)

\[= (d(f(x) \land x))\]

\[= (d(x \cdot f(x)) \cdot f(f(x))).\]

(By Definition 1.11)

\[= (d((x \cdot f(x)) \cdot f(f(x)))).\]

\[= (d(x \cdot f(x)) \cdot f(f(x))) \land ((d((x \cdot f(x)) \cdot f(f(x)))))\land (x \cdot f(x)).\]

(By Definition 1.11)

\[= (d(x \cdot f(x)) \cdot f(f(x))) \land ((d((x \cdot f(x)) \cdot f(f(x))))\land (x \cdot f(x))).\]

(By Definition 1.11)

\[= (d(x \cdot f(x)) \cdot f(f(x))) \land ((d((x \cdot f(x)) \cdot f(f(x))))\land (x \cdot f(x))).\]

(By Definition 1.11)

\[= (d(x \cdot f(x)) \cdot f(f(x))) \land ((d((x \cdot f(x)) \cdot f(f(x))))\land (x \cdot f(x))).\]

(By Definition 1.11)

\[= (d(x \cdot f(x)) \cdot f(f(x))).\]

\[= (d(x \cdot f(x)) \cdot f(f(x))) \land ((d((x \cdot f(x)) \cdot f(f(x))))\land (x \cdot f(x))).\]

(By Proposition 1.5 (5))

\[= 0.\]

Hence, $d(x \cdot f(x)) \cdot f(f(x)) \leq d(d(x))$ for all $x \in A$. 
(4) For all $x \in A$,

$$(d(x) \cdot f(d(x)) \cdot d(x \cdot d(x)) = (d(x) \cdot f(d(x))).$$

(By Definition 2.1)  

$$\quad = (d(x) \cdot f(d(x))) \cdot ((d(x) \cdot f(d(x))) \land (x \cdot d(x))).$$

(By Definition 1.11)  

$$\quad = (d(x) \cdot f(d(x))).$$

(By Proposition 1.5 (5))  

$$\quad = 0.$$

Hence, $d(x) \cdot f(d(x)) \leq d(x \cdot d(x))$ for all $x \in A$.

(5) For all $x \in A$,

$$(d(d(x)) \cdot f(x)) \cdot d(d(x) \cdot x) = (d(d(x)) \cdot f(x)).$$

(By Definition 2.1)  

$$\quad = (d(d(x)) \cdot f(x)) \cdot ((d(d(x)) \cdot f(x)) \land (d(x) \cdot x)).$$

(By Definition 1.11)  

$$\quad = (d(d(x)) \cdot f(x)).$$

(By Proposition 1.5 (5))  

$$\quad = 0.$$

Hence, $d(d(x)) \cdot f(x) \leq d(d(x) \cdot x)$ for all $x \in A$. 
(6) For all $x,y \in A,$

(By Definition 1.11) \hspace{1cm} (d(y \cdot x \cdot f(x)) \cdot d(x \wedge y) = (d(y \cdot x) \cdot f(x)) \cdot d((y \cdot x) \cdot x)

= (d(y \cdot x) \cdot f(x)).

(By Definition 2.1) \hspace{1cm} ((d(y \cdot x) \cdot f(x)) \wedge ((y \cdot x) \cdot x))

= (d(y \cdot x) \cdot f(x)).

(By Definition 1.11) \hspace{1cm} ((d(y \cdot x) \cdot f(x)) \wedge (x \wedge y))

= (d(y \cdot x) \cdot f(x)).

(By Definition 1.11) \hspace{1cm} (d(y \cdot x) \cdot f(x)))

(By Proposition 1.5 (5)) \hspace{1cm} = 0.

Hence, $d(y \cdot x) \cdot f(x) \leq d(x \wedge y)$ for all $x,y \in A.$

(7) For all $x \in A,$

(By UP-2) \hspace{1cm} d(x) = 0 \cdot d(x)

(By Proposition 2.10 (1)) \hspace{1cm} = (f(x) \cdot d(x)) \cdot d(x)

(By Definition 1.11) \hspace{1cm} = d(x) \wedge f(x).

Hence, $d(x) = d(x) \wedge f(x)$ for all $x \in A.$ \hfill $\square$

**Proposition 2.11.** Let $d$ be an $r$-f-derivation of type I of $A$. Then the following properties hold:

for any $x,y \in A,$

(1) $f(x) \cdot d(y) \leq d(x \cdot y),$

(2) $f(x \cdot d(x)) \cdot d(d(x)) \leq d(d(x)),$

(3) $f(d(x)) \cdot d(x) \leq d(d(x) \cdot x),$

(4) $f(x \cdot d(d(x))) \leq d(x \cdot d(x)),$ and

(5) $f(y \cdot x \cdot d(x) \leq d(x \wedge y).$
**Proof.**  (1) For all \( x, y \in A \),

\[
(f(x) \cdot d(y)) \cdot d(x \cdot y) = (f(x) \cdot d(y)) \cdot ((f(x) \cdot d(y)) \land (x \cdot y))
\]

\[
= (f(x) \cdot d(y)) \cdot (((x \cdot y) \cdot (f(x) \cdot d(y)))).
\]

(By Definition 2.1)

\[
(f(x) \cdot d(y))
\]

(By Definition 1.11)

\[
= 0.
\]

(By Proposition 1.5 (5))

Hence, \( f(x) \cdot d(y) \leq d(x \cdot y) \) for all \( x, y \in A \).

(2) For all \( x \in A \),

\[
(f(x \cdot d(x)) \cdot d(d(x))) \cdot d(d(x)) = (f(x \cdot d(x)) \cdot d(d(x))).
\]

(By Theorem 2.8 (2))

\[
d(d(x) \land x)
\]

\[
= (f(x \cdot d(x)) \cdot d(d(x))).
\]

(By Definition 1.11)

\[
d((x \cdot d(x)) \cdot d(x))
\]

\[
= (f(x \cdot d(x)) \cdot d(d(x))).
\]

\[
((f(x \cdot d(x)) \cdot d(d(x))) \land
\]

(By Definition 2.1)

\[
((x \cdot d(x)) \cdot x))
\]

\[
= (f(x \cdot d(x)) \cdot d(d(x))).
\]

\[
(((x \cdot d(x)) \cdot x) \cdot (f(x \cdot d(x)) \cdot d(d(x))))).
\]

(By Definition 1.11)

\[
(f(x \cdot d(x)) \cdot d(d(x))))
\]

(By Proposition 1.5 (5))

\[
= 0.
\]

Hence, \( f(x \cdot d(x)) \cdot d(d(x)) \leq d(d(x)) \) for all \( x \in A \).
(3) For all $x \in A$,

$$
(f(d(x)) \cdot d(x)) \cdot d(d(x) \cdot x) = (f(d(x)) \cdot d(x)) \cdot d(d(x) \cdot x).
$$

(By Definition 2.1) 

$$
((f(d(x)) \cdot d(x)) \wedge (d(x) \cdot x))
\quad = (f(d(x)) \cdot d(x)) \cdot d(d(x) \cdot x).
$$

(By Definition 1.11) 

$$
((d(x) \cdot x) \cdot (f(d(x)) \cdot d(x))) 
\quad = (f(d(x)) \cdot d(x))
$$

(By Proposition 1.5 (5)) 

$$
= 0.
$$

Hence, $f(d(x)) \cdot d(x) \leq d(d(x) \cdot x)$ for all $x \in A$.

(4) For all $x \in A$,

$$
(f(x) \cdot d(d(x))) \cdot d(x \cdot d(x)) = (f(x) \cdot d(d(x))) \cdot d(x \cdot d(x)).
$$

(By Definition 2.1) 

$$
((f(x) \cdot d(d(x))) \wedge (x \cdot d(x)))
\quad = (f(x) \cdot d(d(x))) \cdot d(x \cdot d(x)).
$$

(By Definition 1.11) 

$$
((x \cdot d(x)) \cdot (f(x) \cdot d(d(x)))
\quad = (f(x) \cdot d(d(x)))
$$

(By Proposition 1.5 (5)) 

$$
= 0.
$$

Hence, $f(x) \cdot d(d(x)) \leq d(x \cdot d(x))$ for all $x \in A$. 

(5) For all $x, y \in A$,

\[
(f(y \cdot x) \cdot d(x)) \cdot d(x \land y) = (f(y \cdot x) \cdot d(x)) \cdot d((y \cdot x) \cdot x)
\]

\[= (f(y \cdot x) \cdot d(x)).\]

(By Definition 1.11)

\[
((f(y \cdot x) \cdot d(x)) \land ((y \cdot x) \cdot x))
\]

\[= (f(y \cdot x) \cdot d(x)).
\]

(By Definition 2.1)

\[
(((y \cdot x) \cdot x) \cdot (f(y \cdot x) \cdot d(x)))
\]

\[= (f(y \cdot x) \cdot d(x)).
\]

(By Definition 1.11)

\[
(f(y \cdot x) \cdot d(x))
\]

\[= 0.
\]

(By Proposition 1.5 (5))

Hence, $f(y \cdot x) \cdot d(x) \leq d(x \land y)$ for all $x, y \in A$. □

**Definition 2.12.** Let $d$ be an $l$-derivation (resp. $r$-derivation, $f$-derivation) of type I of $A$. We define a subset $\text{Ker}_d(A)$ of $A$ by

\[
\text{Ker}_d(A) = \{ x \in A \mid d(x) = 0 \}.
\]

**Theorem 2.13.** In a UP-algebra $A$, if $d$ is an $r$-derivation of type I of $A$, then

(1) $y \land x \in \text{Ker}_d(A)$ for all $y \in \text{Ker}_d(A)$ and $x \in A$, and

(2) $x \cdot y \in \text{Ker}_d(A)$ for all $y \in \text{Ker}_d(A)$ and $x \in A$.

**Proof.** (1) Assume that $d$ is an $r$-derivation of type I of $A$. Let $y \in \text{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

\[
d(y \land x) = d((x \cdot y) \cdot y)
\]

(By Definition 1.11)

\[
= (f(x \cdot y) \cdot d(y)) \land ((x \cdot y) \cdot y)
\]

(By Definition 2.1)

\[
= (f(x \cdot y) \cdot 0) \land ((x \cdot y) \cdot y)
\]

(By UP-3)

\[
= 0 \land ((x \cdot y) \cdot y)
\]

(By Proposition 1.13 (1))

\[
= 0.
\]

Hence, $y \land x \in \text{Ker}_d(A)$. 
(2) Assume that $d$ is an $r$-$f$-derivation of type I of $A$. Let $y \in \text{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

\begin{align*}
\text{(By Definition 2.1)} \quad d(x \cdot y) &= (f(x) \cdot d(y)) \land (x \cdot y) \\
&= (f(x) \cdot 0) \land (x \cdot y) \\
\text{(By UP-3)} \quad &= 0 \land (x \cdot y) \\
\text{(By Proposition 1.13 (1))} \quad &= 0.
\end{align*}

Hence, $x \cdot y \in \text{Ker}_d(A)$. \hfill \Box

**Theorem 2.14.** In a meet-commutative UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type I of $A$ and for any $x, y \in A$ is such that $y \leq x$ and $y \in \text{Ker}_d(A)$, then $x \in \text{Ker}_d(A)$.

**Proof.** Assume that $y \leq x$ and $y \in \text{Ker}_d(A)$, then we get $y \cdot x = 0$ and $d(y) = 0$. Thus

\begin{align*}
\text{(By UP-2)} \quad d(x) &= d(0 \cdot x) \\
&= d((y \cdot x) \cdot x) \\
\text{(By Definition 1.12)} \quad &= d((x \cdot y) \cdot y) \\
\text{(By Definition 2.1)} \quad &= (f(x \cdot y) \cdot d(y)) \land ((x \cdot y) \cdot y) \\
\text{(By Definition 1.11)} \quad &= (f(x \cdot y) \cdot 0) \land (y \land x) \\
\text{(By UP-3)} \quad &= 0 \land (y \land x) \\
\text{(By Proposition 1.13 (1))} \quad &= 0.
\end{align*}

Hence, $x \in \text{Ker}_d(A)$. \hfill \Box

**Theorem 2.15.** In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type I of $A$, then $\text{Ker}_d(A)$ is a UP-subalgebra of $A$. 
Proof. Assume that $d$ is an $r$-$f$-derivation of type I of $A$. By Theorem 2.6 (2), we have $d(0) = 0$ and so $0 \in \ker_d(A) \neq \emptyset$. Let $x, y \in \ker_d(A)$. Then $d(x) = 0$ and $d(y) = 0$. Thus

\[(By \text{ Definition } 2.1)\]
\[d(x \cdot y) = (f(x) \cdot d(y)) \wedge (x \cdot y)\]
\[= (f(x) \cdot 0) \wedge (x \cdot y)\]
\[(By \text{ UP-3})\]
\[= 0 \wedge (x \cdot y)\]
\[(By \text{ Proposition } 1.13 (1))\]
\[= 0.\]

Hence, $x \cdot y \in \ker_d(A)$, so $\ker_d(A)$ is a UP-subalgebra of $A$. \hfill \square

On an $l$-$f$-derivation (resp. $r$-$f$-derivation and $f$-derivation) of type I of $A$, we can show that $\ker_d(A)$ is not a UP-ideal of $A$, by the following examples.

**Example 2.16.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \rightarrow A$ as follows:

\[f(0) = 0, f(1) = 1, f(2) = 3, \text{ and } f(3) = 2.\]

Then $f$ is a UP-endomorphism. We define a self-map $d : A \rightarrow A$ as follows:

\[d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 0.\]

Then $d$ is an $l$-$f$-derivation of type I of $A$ and $\ker_d(A) = \{0, 2, 3\}$. Since $0 \cdot (2 \cdot 1) = 0 \in \ker_d(A), 2 \in \ker_d(A)$ but $0 \cdot 1 = 1 \notin \ker_d(A)$, we have $\ker_d(A)$ is not a UP-ideal of $A$.

**Example 2.17.** Form Example 2.16, we have $d$ is an $r$-$f$-derivation of type I of $A$ and $\ker_d(A) = \{0, 2, 3\}$. Since $0 \cdot (3 \cdot 1) = 0 \in \ker_d(A), 3 \in \ker_d(A)$ but $0 \cdot 1 = 1 \notin \ker_d(A)$, we have $\ker_d(A)$ is not a UP-ideal of $A$.

**Example 2.18.** Form Example 2.16 and 2.17, we have $d$ is an $f$-derivation of type I of $A$ and $\ker_d(A)$ is not a UP-ideal of $A$. 
Definition 2.19. Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type I of $A$. We define a subset $\text{Fix}_d(A)$ of $A$ by

$$\text{Fix}_d(A) = \{x \in A \mid d(x) = x\}.$$ 

On an $r$-$f$-derivation of type I of $A$, we can show that $\text{Fix}_d(A)$ is not a UP-subalgebra of $A$ by the following example.

Example 2.20. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
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<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \to A$ as follows:

$$f(0) = 0, f(1) = 2, f(2) = 2, \text{ and } f(3) = 0.$$ 

Then $f$ is a UP-endomorphism. We define a self-map $d : A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 2, \text{ and } d(3) = 0.$$ 

Then $d$ is an $r$-$f$-derivation of type I of $A$ and $\text{Fix}_d(A) = \{0, 1, 2\}$. Since $1, 2 \in \text{Fix}_d(A)$ but $1 \cdot 2 = 3 \notin \text{Fix}_d(A)$, we have $\text{Fix}_d(A)$ is not a UP-subalgebra of $A$.

On an $l$-$f$-derivation of type I of $A$, we can show that $\text{Fix}_d(A)$ is not a UP-ideal of $A$, by the following example.

Example 2.21. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
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<tr>
<td>3</td>
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<td>3</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \to A$ as follows:
Then \( f \) is a UP-endomorphism. We define a self-map \( d : A \to A \) as follows:

\[
d(0) = 0, d(1) = 0, d(2) = 2, d(3) = 0, \text{ and } d(4) = 2.
\]

Then \( d \) is an \( l-f \)-derivation of type I of \( A \) and \( \text{Fix}_d(A) = \{0, 2\} \). Since \( 0 \cdot (2 \cdot 1) = 0 \in \text{Fix}_d(A), 2 \in \text{Fix}_d(A) \) but \( 0 \cdot 1 = 1 \notin \text{Fix}_d(A) \), we have \( \text{Fix}_d(A) \) is not a UP-ideal of \( A \).

3. Left-\( f \)-derivations and right-\( f \)-derivations of type II

In this section, we first introduce the notion of left (resp. right)-\( f \)-derivations of type II of \( A \) and study some of their basic properties. Finally, two subsets \( \text{Ker}_d(A) \) and \( \text{Fix}_d(A) \) for left (resp. right)-\( f \)-derivation \( d \) of type II of \( A \) are studied.

**Definition 3.1.** Let \( A \) be a UP-algebra and let \( f \) be a UP-endomorphism of \( A \). A self-map \( d : A \to A \) is called a \textit{left-\( f \)-derivation of type II} (in short, an \textit{l-\( f \)-derivation of type II}) of \( A \) if it satisfies the identity \( d(x \cdot y) = (x \cdot y) \land (d(x) \cdot f(y)) \) for all \( x, y \in A \). Similarly, a self-map \( d : A \to A \) is called a \textit{right-\( f \)-derivation of type II} (in short, an \textit{r-\( f \)-derivation of type II}) of \( A \) if it satisfies the identity \( d(x \cdot y) = (x \cdot y) \land (f(x) \cdot d(y)) \) for all \( x, y \in A \). Moreover, if \( d \) is both a left-\( f \)-derivation and a right-\( f \)-derivation of type II of \( A \), it is called an \textit{\( f \)-derivation of type II} of \( A \).

By using Microsoft Excel, we have all examples.

**Example 3.2.** Let \( A = \{0, 1, 2, 3, 4\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 0 & 0 & 2 & 3 & 3 \\
2 & 0 & 1 & 0 & 0 & 4 \\
3 & 0 & 1 & 3 & 0 & 4 \\
4 & 0 & 1 & 3 & 0 & 0 \\
\end{array}
\]

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( f \) be an identity map on \( A \). Then \( f \) is a UP-endomorphism. We define a self-map \( d : A \to A \) as follows:
Then $d$ is an $l$-f-derivation of type II of $A$.

**Example 3.3.** Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. Let $f$ be an identity map on $A$. Then $f$ is a UP-endomorphism. We define a self-map $d : A \rightarrow A$ as follows:

$$d(0) = 0, d(1) = 0, d(2) = 0, d(3) = 0, \text{ and } d(4) = 4.$$  

Then $d$ is an $r$-f-derivation of type II of $A$.

**Example 3.4.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f : A \rightarrow A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.$$  

Then $f$ is a UP-endomorphism. We define a self-map $d : A \rightarrow A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 1.$$  

Then $d$ is an $f$-derivation of type II of $A$.

**Definition 3.5.** An $l$-f-derivation (resp. $r$-f-derivation, $f$-derivation) $d$ of type II of $A$ is called **regular** if $d(0) = 0$.

**Theorem 3.6.** In a UP-algebra $A$, the following statements hold:
(1) every \( l\)-\( f \)-derivation of type II of \( A \) is regular, and

(2) every \( r\)-\( f \)-derivation of type II of \( A \) is regular.

**Proof.** (1) Assume that \( d \) is an \( l\)-\( f \)-derivation of type II of \( A \). Then

\[
\begin{align*}
\text{(By UP-3)} & \quad d(0) = d(x \cdot 0) \\
\text{(By Definition 3.1)} & \quad = (x \cdot 0) \land (d(x) \cdot f(0)) \\
\text{(By UP-3)} & \quad = 0 \land (d(x) \cdot 0) \\
\text{(By UP-3)} & \quad = 0 \land 0 \\
\text{(By Proposition 1.13 (3))} & \quad = 0.
\end{align*}
\]

Hence, \( d \) is regular.

(2) Assume that \( d \) is an \( r\)-\( f \)-derivation of type II of \( A \). Then

\[
\begin{align*}
\text{(By UP-3)} & \quad d(0) = d(0 \cdot 0) \\
\text{(By Definition 3.1)} & \quad = (0 \cdot 0) \land (f(0) \cdot d(0)) \\
\text{(By UP-3)} & \quad = 0 \land (0 \cdot d(0)) \\
\text{(By UP-2)} & \quad = 0 \land d(0) \\
\text{(By Proposition 1.13 (1))} & \quad = 0.
\end{align*}
\]

Hence, \( d \) is regular. \( \Box \)

**Corollary 3.7.** Every \( f \)-derivation of type II of \( A \) is regular.

**Theorem 3.8.** In a UP-algebra \( A \), the following statements hold:

(1) if \( d \) is an \( l\)-\( f \)-derivation of type II of \( A \), then \( d(x) = x \land f(x) \) for all \( x \in A \), and

(2) if \( d \) is an \( r\)-\( f \)-derivation of type II of \( A \), then \( d(x) = x \land d(x) \) for all \( x \in A \).
Proof. (1) Assume that $d$ is an $l$-$f$-derivation of type II of $A$. Then, for all $x \in A$,

(By UP-2) $d(x) = d(0 \cdot x)$

(By Definition 3.1) $= (0 \cdot x) \land (d(0) \cdot f(x))$

(By Theorem 3.6 (1)) $= (0 \cdot x) \land (0 \cdot f(x))$

(By UP-2) $= x \land f(x)$.

(2) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Then, for all $x \in A$,

(By UP-2) $d(x) = d(0 \cdot x)$

(By Definition 3.1) $= (0 \cdot x) \land (f(0) \cdot d(x))$

(By UP-2) $= x \land (0 \cdot d(x))$

(By UP-2) $= x \land d(x)$.

\[\square\]

Corollary 3.9. If $d$ is an $f$-derivation of type II of $A$, then $d(x) = x \land f(x) = x \land d(x)$ for all $x \in A$.

Proposition 3.10. Let $d$ be an $l$-$f$-derivation of type II of $A$. Then the following properties hold: for any $x, y \in A$,

(1) $x \leq d(x)$,
(2) $x \cdot y \leq d(x \cdot y)$,
(3) $x \land y \leq d(x \land y)$, and
(4) $d(x) = d(x) \land x$.

Proof. (1) For all $x \in A$,

(By Theorem 3.8 (1)) $x \cdot d(x) = x \cdot (x \land f(x))$

(By Definition 1.11) $= x \cdot ((f(x) \cdot x) \cdot x)$

(By Proposition 1.5 (5)) $= 0$.

Hence, $x \leq d(x)$ for all $x \in A$. 
(2) For all \( x, y \in A \),

\[
(x \cdot y) \cdot d(x \cdot y) = (x \cdot y) \cdot ((x \cdot y) \land (d(x) \cdot f(y)))
\]

(By Definition 3.1)

\[
= (x \cdot y) \cdot (((d(x) \cdot f(y)) \cdot (x \cdot y)) \cdot (x \cdot y))
\]

(By Definition 1.11)

\[
= 0.
\]

(By Proposition 1.5 (5))

Hence, \( x \cdot y \leq d(x \cdot y) \) for all \( x, y \in A \).

(3) For all \( x, y \in A \),

\[
(x \land y) \cdot d(x \land y) = (x \land y) \cdot d((y \cdot x) \cdot x)
\]

(By Definition 1.11)

\[
= (x \land y) \cdot (((y \cdot x) \cdot x) \land (d(y \cdot x) \cdot f(x)))
\]

(By Definition 3.1)

\[
= (x \land y) \cdot ((x \land y) \land (d(y \cdot x) \cdot f(x)))
\]

(By Definition 1.11)

\[
= (x \land y) \cdot (((d(y \cdot x) \cdot f(x)) \cdot x) \land x)
\]

(By Definition 1.11)

\[
= 0.
\]

(By Proposition 1.5 (5))

Hence, \( x \land y \leq d(x \land y) \) for all \( x, y \in A \).

(4) For all \( x \in A \),

\[
d(x) = 0 \cdot d(x)
\]

(By UP-2)

\[
= (x \cdot d(x)) \cdot d(x)
\]

(By Proposition 3.10 (1))

\[
= d(x) \land x.
\]

(By Definition 1.11)

Hence, \( d(x) = d(x) \land x \) for all \( x, y \in A \).

\[\square\]

**Proposition 3.11.** Let \( d \) be an \( r \cdot f \)-derivation of type II of \( A \). Then the following properties hold: for any \( x, y \in A \),

1. \( x \leq d(x) \),
2. \( x \cdot y \leq d(x \cdot y) \),
3. \( x \land y \leq d(x \land y) \), and
4. \( d(x) = d(x) \land x \).
Proof. (1) For all $x \in A$, 

(By Theorem 3.8 (2)) \[ x \cdot d(x) = x \cdot (x \land d(x)) \]

(By Definition 1.11) \[ = x \cdot ((d(x) \cdot x) \cdot x) \]

(By Proposition 1.5 (5)) \[ = 0. \]

Hence, $x \leq d(x)$ for all $x \in A$.

(2) For all $x, y \in A$, 

(By Definition 3.1) \[ (x \cdot y) \cdot d(x \cdot y) = (x \cdot y) \cdot ((x \cdot y) \land (f(x) \cdot d(y))) \]

(By Definition 1.11) \[ = (x \cdot y) \cdot (((f(x) \cdot d(y)) \cdot (x \cdot y)) \cdot (x \cdot y)) \]

(By Proposition 1.5 (5)) \[ = 0. \]

Hence, $x \cdot y \leq d(x \cdot y)$ for all $x, y \in A$.

(3) For all $x, y \in A$, 

(By Definition 1.11) \[ (x \land y) \cdot d(x \land y) = (x \land y) \cdot d((y \cdot x) \cdot x) \]

(By Definition 3.1) \[ = (x \land y) \cdot (((y \cdot x) \cdot x) \land (f(y \cdot x) \cdot d(x))) \]

(By Definition 1.11) \[ = (x \land y) \cdot ((x \land y) \land (f(y \cdot x) \cdot d(x))) \]

\[ = (x \land y) \cdot (((f(y \cdot x) \cdot d(x))) \cdot (x \land y)) \]

(By Definition 1.11) \[ = 0. \]

Hence, $x \land y \leq d(x \land y)$ for all $x, y \in A$.

(4) For all $x \in A$, 

(By UP-2) \[ d(x) = 0 \cdot d(x) \]

(By Proposition 3.11 (1)) \[ = (x \cdot d(x)) \cdot d(x) \]

(By Definition 1.11) \[ = d(x) \land x. \]

Hence, $d(x) = d(x) \land x$ for all $x, y \in A$. $\blacksquare$
**Corollary 3.12.** Let $d$ be an $f$-derivation of type II of $A$. Then the following properties hold:

for any $x, y \in A$,

1. $x \leq d(x)$,
2. $x \cdot y \leq d(x \cdot y)$,
3. $x \land y \leq d(x \land y)$, and
4. $d(x) = d(x) \land x$.

**Definition 3.13.** Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type II of $A$. We define a subset $\text{Ker}_d(A)$ of $A$ by

$$\text{Ker}_d(A) = \{ x \in A \mid d(x) = 0 \}.$$  

**Theorem 3.14.** In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type II of $A$, then

1. $y \land x \in \text{Ker}_d(A)$ for all $y \in \text{Ker}_d(A)$ and $x \in A$, and
2. $x \cdot y \in \text{Ker}_d(A)$ for all $y \in \text{Ker}_d(A)$ and $x \in A$.

**Proof.** (1) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Let $y \in \text{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

\[
(y \land x) = d((x \cdot y) \cdot y)
\]

(By Proposition 1.11)

\[
= ((x \cdot y) \cdot y) \land (f(x \cdot y) \cdot d(y))
\]

(By Definition 3.1)

\[
= (y \land x) \land (f(x \cdot y) \cdot 0)
\]

(By Proposition 1.11)

\[
= (y \land x) \land 0
\]

(UP-3)

\[
= 0.
\]

(By Proposition 1.13 (2))

Hence, $y \land x \in \text{Ker}_d(A)$.  

(2) Assume that \( d \) is an \( r-f \)-derivation of type II of \( A \). Let \( y \in \text{Ker}_d(A) \) and \( x \in A \). Then \( d(y) = 0 \). Thus

\[
(x \cdot y) = (x \cdot y) \wedge (f(x) \cdot d(y)) \\
= (x \cdot y) \wedge (f(x) \cdot 0)
\]

(By Definition 3.1)

\[
= (x \cdot y) \wedge 0
\]

(By UP-3)

\[
= 0.
\]

(By Proposition 1.13 (2))

Hence, \( x \cdot y \in \text{Ker}_d(A) \).

\[\square\]

**Theorem 3.15.** In a meet-commutative UP-algebra \( A \), if \( d \) is an \( r-f \)-derivation of type II of \( A \) and for any \( x, y \in A \) is such that \( y \leq x \) and \( y \in \text{Ker}_d(A) \), then \( x \in \text{Ker}_d(A) \).

**Proof.** Assume that \( y \leq x \) and \( y \in \text{Ker}_d(A) \), then we get \( y \cdot x = 0 \) and \( d(y) = 0 \). Thus

\[
d(x) = d(0 \cdot x) \\
= d((y \cdot x) \cdot x) \\
= d((x \cdot y) \cdot y) \\
= ((x \cdot y) \cdot y) \wedge (f(x \cdot y) \cdot d(y)) \\
= (y \wedge x) \wedge (f(x \cdot y) \cdot 0) \\
= (y \wedge x) \wedge 0 \\
= 0.
\]

(By Definition 1.12)

(By Definition 3.1)

(By Definition 1.11)

(By UP-3)

(By Proposition 1.13 (1))

Hence, \( x \in \text{Ker}_d(A) \).

\[\square\]

**Theorem 3.16.** In a UP-algebra \( A \), if \( d \) is an \( r-f \)-derivation of type II of \( A \), then \( \text{Ker}_d(A) \) is a UP-subalgebra of \( A \).
Proof. Assume that $d$ is an $r$-$f$-derivation of type II of $A$. By Theorem 3.6 (2), we have $d(0) = 0$ and so $0 \in \text{Ker}_d(A) \neq \emptyset$. Let $x, y \in \text{Ker}_d(A)$. Then $d(x) = 0$ and $d(y) = 0$. Thus

\[
\begin{align*}
\text{(By Definition 3.1)} & \quad d(x \cdot y) = (x \cdot y) \land (f(x) \cdot d(y)) \\
& \quad = (x \cdot y) \land (f(x) \cdot 0) \\
\text{(By UP-3)} & \quad = (x \cdot y) \land 0 \\
\text{(By Proposition 1.13 (2))} & \quad = 0.
\end{align*}
\]

Hence, $x \cdot y \in \text{Ker}_d(A)$, so $\text{Ker}_d(A)$ is a UP-subalgebra of $A$. □

On an $l$-$f$-derivation (resp. $r$-$f$-derivation and $f$-derivation) of type II of $A$, we can show that $\text{Ker}_d(A)$ is not a UP-ideal of $A$, by the following examples.

Example 3.17. Form Example 2.16, we have $d$ is an $l$-$f$-derivation of type II of $A$ and $\text{Ker}_d(A)$ is not a UP-ideal of $A$.

Example 3.18. Form Example 2.17, we have $d$ is an $r$-$f$-derivation of type II of $A$ and $\text{Ker}_d(A)$ is not a UP-ideal of $A$.

Example 3.19. Form Example 3.17 and 3.18, we have $d$ is an $f$-derivation of type II of $A$ and $\text{Ker}_d(A)$ is not a UP-ideal of $A$.

Definition 3.20. Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type II of $A$. We define a subset $\text{Fix}_d(A)$ of $A$ by

\[
\text{Fix}_d(A) = \{ x \in A \mid d(x) = x \}.
\]

On an $l$-$f$-derivation (resp. $r$-$f$-derivation) of type II of $A$, we can show that $\text{Fix}_d(A)$ is not a UP-subalgebra of $A$ by the following examples.
Example 3.21. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 2 & 2 \\
2 & 0 & 1 & 0 & 3 \\
3 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then $(A;\cdot, 0)$ is a UP-algebra. We define a self-map $f: A \rightarrow A$ as follows:

\[
f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.
\]

Then $f$ is a UP-endomorphism. We define a self-map $d: A \rightarrow A$ as follows:

\[
d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 1.
\]

Then $d$ is an $l$-f-derivation of type II of $A$ and $\text{Fix}_d(A) = \{0, 1, 3\}$. Since $1, 3 \in \text{Fix}_d(A)$ but $1 \cdot 3 = 2 \notin \text{Fix}_d(A)$, we have $\text{Fix}_d(A)$ is not a UP-subalgebra of $A$.

Example 3.22. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 3 \\
3 & 0 & 2 & 2 & 0 \\
\end{array}
\]

Then $(A;\cdot, 0)$ is a UP-algebra. We define a self-map $f: A \rightarrow A$ as follows:

\[
f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.
\]

Then $f$ is a UP-endomorphism. We define a self-map $d: A \rightarrow A$ as follows:

\[
d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 3.
\]

Then $d$ is an $r$-f-derivation of type II of $A$ and $\text{Fix}_d(A) = \{0, 1, 3\}$. Since $1, 3 \in \text{Fix}_d(A)$ but $3 \cdot 1 = 2 \notin \text{Fix}_d(A)$, we have $\text{Fix}_d(A)$ is not a UP-subalgebra of $A$.

Conflict of Interests

The authors declare that there is no conflict of interests.
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   tion, October 2016.


