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# A NEW DERIVATION OF UP-ALGEBRAS BY MEANS OF UP-ENDOMORPHISMS

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**Abstract.** The notions of left (resp. right)-$f$-derivations of type I and of left (resp. right)-$f$-derivations of type II of UP-algebras are introduced, some useful examples are discussed, and related properties are investigated. Moreover, we show that the kernel of right-$f$-derivations of type I and of right-$f$-derivations of type II of UP-algebras is a UP-subalgebra, and also give examples to show that the the kernel of left (resp. right)-$f$-derivations of type I and of left (resp. right)-$f$-derivations of type II of UP-algebras is not a UP-ideal, the fixed set of right-$f$-derivations of type I and of left (resp. right)-$f$-derivations of type II of UP-algebras is not a UP-subalgebra, and the fixed set of left-$f$-derivations of type I of UP-algebras is not a UP-ideal in general.

**Keywords:** UP-algebra; UP-subalgebra; UP-ideal; left-$f$-derivation; right-$f$-derivation.

**2010 AMS Subject Classification:** 03G25.

## 1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [14], BCI-algebras [15], BCH-algebras [11], KU-algebras

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[34], SU-algebras [19] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [15] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [14, 15] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

In the theory of rings and near rings, the properties of derivations is an important topic to study [32, 22]. In 2004, Jun and Xin [18] applied the notions of rings and near rings theory to BCI-algebras and obtained some properties. Zhan and Liu [38] introduced the notion of left-right (resp. right-left) $f$-derivations of BCI-algebras, investigated some related properties by using the idea of regular $f$-derivations and they gave characterizations of $p$-semisimple BCI-algebras. Several researches were conducted on the generalizations of the notion of derivations and application to many logical algebras such as: In 2005, Zhan and Liu [38] introduced the notion of left-right (resp. right-left) $f$-derivations of BCI-algebras. In 2006, Abujabal and Al-shehri [1] studied derivations of $p$-semisimple BCI-algebras and proved that for any derivations $d_1, d_2$ of a $p$-semisimple BCI-algebra $X$, $d_1 \circ d_2$ is also a derivation of $X$, $d_1 \circ d_2 = d_2 \circ d_1$ and $d_1 * d_2 = d_2 * d_1$. In 2007, Abujabal and Al-shehri [2] introduced the notion of left derivations of BCI-algebras, investigated regular left derivations and studied left derivations on $p$-semisimple BCI-algebras. In 2009, Javed and Aslam [17] studied derivations of $p$-semisimple BCI-algebras and proved that for any $f$-derivations $d_f$, $d'_f$ of a $p$-semisimple BCI-algebra $X$, $d_f \circ d'_f$ is also an $f$-derivation of $X$ and $d_f \circ d'_f = d'_f \circ d_f$. Nisar [31] introduced the notions of right $F$-derivations and left $F$-derivations of BCI-algebras. Nisar [30] characterized $f$-derivations of BCI-algebras. Prabpayak and Leerawat [33] studied left-right derivations and right-left derivations of BCC-algebras and also considered regular derivations of BCC-algebras. In 2010, Al-shehri [4] applied the notion of derivations in ring and near-ring theory to MV-algebras and investigated some of its properties. They introduced additive derivations of MV-algebras, investigated several properties and proved that an additive derivation of a linearly ordered MV-algebra is an isotone by used the notion of an isotone derivation and characterized derivations of MV-algebras.

Kim [20] introduced the notion of $f$-derivations which is a generalization of derivations in subtraction algebras, and some related properties are investigated. In 2011, Thomys [36] described $f$-derivations of weak BCC-algebras in which the condition $(xy)z=(xz)y$ for any $x,y,z$ in a BCI-algebra $X$ when $x,y$ belong to the same branch. In 2012, Al-shehri and Bawazeer [5] studied the notion of left-right (resp. right-left) $t$-derivations of BCC-algebras, investigated some properties on $t$-derivations of BCC-algebras and considered $t$-regular, $t$-derivations and the $d_t$-invariant on ideals of BCC-algebras. Lee and Kim [23] considered the properties of $f$-derivations of BCC-algebras and also characterized Ker$d$ by $f$-derivations. Muhiuddin and Al-roqi [27] introduced the notion of $t$-derivations of BCI-algebras and proved that for any $t$-derivation $d_t$ of a BCI-algebra $X$, $d_t \circ d'_t$ is also a $t$-derivation of $X$ and $d_t \circ d'_t = d'_t \circ d_t$, and for any $t$-derivation $d_t$ of a $p$-semisimple BCI-algebra $X$, $d_t * d'_t = d'_t * d_t$. Muhiuddin and Al-roqi [26] introduced the notion of (regular) $(\alpha,\beta)$-derivations of BCI-algebras. In 2013, Bawazeer, Al-shehri and Babusal [9] introduced the notion of generalized derivations of BCC-algebras. Lee [21] introduced a new kind of derivations of BCI-algebras. Ardekani and Davvaz [6] extend the notion of derivations of MV-algebras, introduced the notion of $f$-derivations and $(f,g)$-derivations of MV-algebras and investigated some properties of them. Muhiuddin, Al-roqi, Jun and Ceven [29] introduced the notion of symmetric left bi-derivations of BCI-algebras. Ghorbani, Torkzadeh and Motamed [10] introduced the notion of $(\odot,\oplus)$-derivations and $(\ominus,\odot)$-derivations for MV-algebras and studied the connection between these derivations on MV-algebras. They characterized the isotone $(\odot,\oplus)$-derivations and proved that $(\ominus,\odot)$-derivations are isotone. And they determined the relationship between $(\odot,\oplus)$-derivations and $(\ominus,\odot)$-derivations for MV-algebras. Leerawat and Bunphan [24] introduced the notion of $f$-derivations of Boolean algebras, namely Boolean $f$-derivations, investigated some related properties and proved that the fixed set and the kernel of Boolean $f$-derivations are ideals in Boolean algebras. Torkzadeh and Abbasian [37] defined the concept of studied $(\odot,\vee)$-derivations for BL-algebras and discussed some related results, studied $(\odot,\vee)$-derivations on boolean center $B(A)$ of a BL-algebras $A$, investigated some properties of isotone $(\odot,\vee)$-derivations on a BL-algebras $A$ and characterized the $(\odot,\vee)$-derivations on the Gödel structure [0,1]. In 2014, Al-roqi [3] introduced the notion of generalized (regular) $(\alpha,\beta)$-derivations of BCI-algebras. Muhiuddin and Al-roqi [28] introduced the notion of generalized left derivations of BCI-algebras. Ardekani and Davvaz [7] introduced the notion of $(f,g)$-derivations of BCI-algebras. Min, Xioao-long and Yi-jun [25] introduced the notion of $f$-derivations and $g$-derivations of MV-algebras. In 2015, Asawasamrit [8] introduced the notion of $f$-derivations of KK-algebras and investigated some related properties. Jana, Senapati and Pal [16] introduced the notion of left-right (resp. right-left) derivations, $f$-derivations, generalized derivations of KUS-algebras and proved that $(\mathrm{Der}(X),\wedge)$ refers to a semigroup for any $p$-semisimple KUS-algebra $X$ and defined the relationship between left-right derivations, right-left derivations and generalized derivations of KUS-algebras. In 2016, Sawika, Intasan, Kaewwasri and Iampan [35] introduced the notions of $(l,r)$-derivations, $(r,l)$-derivations and derivations of UP-algebras and investigated some related properties. Iampan [12] introduced the notion of $f$-derivations of UP-algebras which is the generalization of the notion of derivations [35].

The notion of derivations play an important role in studying the many logical algebras. In this paper, we introduce the notions of left (resp. right)-$f$-derivations of type I and of left (resp. right)-$f$-derivations of type II of UP-algebras, some useful examples are discussed, and related properties are investigated.

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 1.1.** [13] An algebra $A=(A;\cdot,0)$ of type $(2,0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x,y,z\in A$,

**(UP-1):** $(y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z))=0$,

**(UP-2):** $0\cdot x=x$,

**(UP-3):** $x\cdot 0=0$, and

**(UP-4):** $x\cdot y=y\cdot x=0$ implies $x=y$.

From [13], we know that the notion of UP-algebras is a generalization of KU-algebras.

**Example 1.2.** [13] Let $X$ be a universal set. Define a binary operation $\cdot$ on the power set of $X$ by putting $A\cdot B=B\cap A'=A'\cap B=B-A$ for all $A,B\in\mathscr{P}(X)$. Then $(\mathscr{P}(X);\cdot,\emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*.

**Example 1.3.** [13] Let $X$ be a universal set. Define a binary operation $*$ on the power set of $X$ by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathscr{P}(X)$. Then $(\mathscr{P}(X); *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

**Example 1.4.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 2 | 3 | 4 |
| 2 | 0 | 1 | 0 | 0 | 3 |
| 3 | 0 | 1 | 3 | 0 | 3 |
| 4 | 0 | 1 | 3 | 0 | 0 |

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let $A$ and $B$ denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

**Proposition 1.5.** [13] *In a UP-algebra $A$, the following properties hold: for any $x, y, z \in A$,*

*(1)* $x \cdot x = 0$,

*(2)* $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

*(3)* $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

*(4)* $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

*(5)* $x \cdot (y \cdot x) = 0$,

*(6)* $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

*(7)* $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on $A$ [13] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

**Proposition 1.6.** [13] *In a UP-algebra $A$, the following properties hold: for any $x, y, z \in A$,*

*(1)* $x \leq x$,

*(2)* $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

*(3)* $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$*,*

*(4)* $x \leq y$ *implies* $z \cdot x \leq z \cdot y$*,*

*(5)* $x \leq y$ *implies* $y \cdot z \leq x \cdot z$*,*

*(6)* $x \leq y \cdot x$*, and*

*(7)* $x \leq y \cdot y$*.*

**Definition 1.7.** [13] A nonempty subset $B$ of $A$ is called a *UP-ideal* of $A$ if it satisfies the following properties:

(1) the constant 0 of $A$ is in $B$, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, $A$ and $\{0\}$ are UP-ideals of $A$.

**Definition 1.8.** [13] A subset $S$ of $A$ is called a *UP-subalgebra* of $A$ if the constant 0 of $A$ is in $S$, and $(S; \cdot, 0)$ it self forms a UP-algebra. Clearly, $A$ and $\{0\}$ are UP-subalgebras of $A$.

**Proposition 1.9.** [13] *A nonempty subset* $S$ *of a UP-algebra* $A = (A; \cdot, 0)$ *is a UP-subalgebra of* $A$ *if and only if* $S$ *is closed under the* $\cdot$ *multiplication on* $A$*.*

**Definition 1.10.** [13] Let $(A; \cdot, 0)$ and $(A'; \cdot', 0')$ be UP-algebras. A mapping $f$ from $A$ to $A'$ is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f \colon A \to A'$ is called a *UP-endomorphism* of $A$ if $A' = A$.

**Definition 1.11.** [35] For any $x, y \in A$, we define a binary operation $\wedge$ on $A$ by $x \wedge y = (y \cdot x) \cdot x$.

**Definition 1.12.** [35] A *UP-algebra* $A$ is called *meet-commutative* if $x \wedge y = y \wedge x$ for all $x, y \in A$, that is, $(y \cdot x) \cdot x = (x \cdot y) \cdot y$ for all $x, y \in A$.

**Proposition 1.13.** [35] *In a UP-algebra* $A$*, the following properties hold : for any* $x \in A$*,*

*(1)* $0 \wedge x = 0$*,*

*(2)* $x \wedge 0 = 0$*, and*

*(3)* $x \wedge x = x$*.*

## 2. Left-$f$-derivations and right-$f$-derivations of type I

In this section, we first introduce the notion of left (resp. right)-$f$-derivations of type I of $A$ and study some of their basic properties. Finally, two subsets $\mathrm{Ker}_d(A)$ and $\mathrm{Fix}_d(A)$ for left (resp. right)-$f$-derivation $d$ of type I of $A$ are studied.

**Definition 2.1.** Let $A$ be a UP-algebra and let $f$ be a UP-endomorphism of $A$. A self-map $d\colon A \to A$ is called a *left-$f$-derivation of type I* (in short, an *$l$-$f$-derivation of type I*) of $A$ if it satisfies the identity $d(x \cdot y) = (d(x) \cdot f(y)) \wedge (x \cdot y)$ for all $x, y \in A$. Similarly, a self-map $d\colon A \to A$ is called a *right-$f$-derivation of type I* (in short, an *$r$-$f$-derivation of type I*) of $A$ if is satisfies the identity $d(x \cdot y) = (f(x) \cdot d(y)) \wedge (x \cdot y)$ for all $x, y \in A$. Moreover, if $d$ is both a left-$f$-derivation and a right-$f$-derivation of type I of $A$, it is called an *$f$-derivation of type I of $A$*.

By using Microsoft Excel, we have all examples.

**Example 2.2.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 3 |
| 3 | 0 | 1 | 2 | 0 |

Then $(A; \cdot, 0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 3, \text{ and } f(3) = 2.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 0.$$

Then $d$ is an $l$-$f$-derivation of type I of $A$.

**Example 2.3.** Let $A = \{0,1,2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 0 | 0 | 2 |
| 2 | 0 | 0 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 0, \text{ and } f(2) = 1.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, \text{ and } d(2) = 1.$$

Then $d$ is an $r$-$f$-derivation of type I of $A$.

**Example 2.4.** Let $A = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 3 | 0 |
| 3 | 0 | 1 | 0 | 0 | 0 |
| 4 | 0 | 1 | 2 | 3 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3, \text{ and } f(4) = 0.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 2, d(3) = 3, \text{ and } d(4) = 0.$$

Then $d$ is an $f$-derivation of type I of $A$.

**Definition 2.5.** An $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) $d$ of type I of $A$ is called *regular* if $d(0) = 0$.

**Theorem 2.6.** *In a UP-algebra A, the following statements hold:*

*(1) every $l$-$f$-derivation of type I of A is regular, and*

*(2) every $r$-$f$-derivation of type I of A is regular.*

**Proof.** (1) Assume that $d$ is an $l$-$f$-derivation of type I of $A$. Then

| | |
|---|---|
| (By UP-3) | $d(0) = d(x \cdot 0)$ |
| (By Definition 2.1) | $= (d(x) \cdot f(0)) \wedge (x \cdot 0)$ |
| (By UP-3) | $= (d(x) \cdot 0) \wedge 0$ |
| (By UP-3) | $= 0 \wedge 0$ |
| (By Proposition 1.13 (3)) | $= 0.$ |

Hence, $d$ is regular.

(2) Assume that $d$ is an $r$-$f$-derivation of type I of $A$. Then

| | |
|---|---|
| (By UP-3) | $d(0) = d(0 \cdot 0)$ |
| (By Definition 2.1) | $= (f(0) \cdot d(0)) \wedge (0 \cdot 0)$ |
| (By UP-3) | $= (0 \cdot d(0)) \wedge 0$ |
| (By UP-2) | $= d(0) \wedge 0$ |
| (By Proposition 1.13 (2)) | $= 0.$ |

Hence, $d$ is regular. □

**Corollary 2.7.** *Every $f$-derivation of type I of $A$ is regular.*

**Theorem 2.8.** *In a UP-algebra $A$, the following statements hold:*

*(1) if $d$ is an $l$-$f$-derivation of type I of $A$, then $d(x) = f(x) \wedge x$ for all $x \in A$, and*

*(2) if $d$ is an $r$-$f$-derivation of type I of $A$, then $d(x) = d(x) \wedge x$ for all $x \in A$.*

**Proof.** (1) Assume that $d$ is an $l$-$f$-derivation of I of $A$. Then, for all $x \in A$,

| | |
|---|---|
| (By UP-2) | $d(x) = d(0 \cdot x)$ |
| (By Definition 2.1) | $= (d(0) \cdot f(x)) \wedge (0 \cdot x)$ |
| (By Theorem 2.6 (1)) | $= (0 \cdot f(x)) \wedge (0 \cdot x)$ |
| (By UP-2) | $= f(x) \wedge x.$ |

(2) Assume that $d$ is an $r$-$f$-derivation of type I of $A$. Then, for all $x \in A$,

$$
\begin{aligned}
&\text{(By UP-2)} && d(x) = d(0 \cdot x) \\
&\text{(By Definition 2.1)} && = (f(0) \cdot d(x)) \wedge (0 \cdot x) \\
&\text{(By UP-2)} && = (0 \cdot d(x)) \wedge x \\
&\text{(By UP-2)} && = d(x) \wedge x.
\end{aligned}
$$

□

**Corollary 2.9.** *If $d$ is an $f$-derivation of type I of $A$, then $d(x) = f(x) \wedge x = d(x) \wedge x$ for all $x \in A$.*

**Proposition 2.10.** *Let $d$ be an $l$-$f$-derivation of type I of $A$. Then the following properties hold: for any $x, y \in A$,*

*(1)* $f(x) \leq d(x)$,
*(2)* $d(x) \cdot f(y) \leq d(x \cdot y)$,
*(3)* $d(x \cdot f(x)) \cdot f(f(x)) \leq d(d(x))$,
*(4)* $d(x) \cdot f(d(x)) \leq d(x \cdot d(x))$,
*(5)* $d(d(x)) \cdot f(x) \leq d(d(x) \cdot x)$,
*(6)* $d(y \cdot x) \cdot f(x) \leq d(x \wedge y)$, *and*
*(7)* $d(x) = d(x) \wedge f(x)$.

**Proof.** (1) For all $x \in A$,

$$
\begin{aligned}
&\text{(By Theorem 2.8 (1))} && f(x) \cdot d(x) = f(x) \cdot (f(x) \wedge x) \\
&\text{(By Definition 1.11)} && = f(x) \cdot ((x \cdot f(x)) \cdot f(x)) \\
&\text{(By Proposition 1.5 (5))} && = 0.
\end{aligned}
$$

Hence, $f(x) \leq d(x)$ for all $x \in A$.

(2) For all $x,y\in A$,

$$
\begin{aligned}
&\text{(By Definition 2.1)} & (d(x)\cdot f(y))\cdot d(x\cdot y) &= (d(x)\cdot f(y))\cdot((d(x)\cdot f(y))\wedge(x\cdot y) \\
& & &= (d(x)\cdot f(y))\cdot(((x\cdot y)\cdot(d(x)\cdot f(y)))\cdot \\
&\text{(By Definition 1.11)} & &\quad (d(x)\cdot f(y))) \\
&\text{(By Proposition 1.5 (5))} & &= 0.
\end{aligned}
$$

Hence, $d(x)\cdot f(y)\leq d(x\cdot y)$ for all $x,y\in A$.

(3) For all $x\in A$,

$$
\begin{aligned}
& & (d(x\cdot f(x))\cdot f(f(x)))\cdot d(d(x)) &= (d(x\cdot f(x))\cdot f(f(x)))\cdot \\
&\text{(By Definition 2.1)} & &\quad (d(f(x)\wedge x)) \\
& & &= (d(x\cdot f(x))\cdot f(f(x)))\cdot \\
&\text{(By Definition 1.11)} & &\quad (d((x\cdot f(x))\cdot f(x))) \\
& & &= (d(x\cdot f(x))\cdot f(f(x)))\cdot \\
& & &\quad ((d((x\cdot f(x))\cdot f(f(x)))\wedge \\
&\text{(By Definition 2.1)} & &\quad ((x\cdot f(x))\cdot f(x)))) \\
& & &= (d(x\cdot f(x))\cdot f(f(x)))\cdot \\
& & &\quad ((d((x\cdot f(x))\cdot f(f(x)))\wedge \\
&\text{(By Definition 1.11)} & &\quad (f(x)\wedge x))) \\
& & &= (d(x\cdot f(x))\cdot f(f(x)))\cdot \\
& & &\quad (((f(x)\wedge x)\cdot(d(x\cdot f(x))\cdot f(f(x))))\cdot \\
&\text{(By Definition 1.11)} & &\quad (d(x\cdot f(x))\cdot f(f(x)))) \\
&\text{(By Proposition 1.5 (5))} & &= 0.
\end{aligned}
$$

Hence, $d(x\cdot f(x))\cdot f(f(x))\leq d(d(x))$ for all $x\in A$.

(4) For all $x \in A$,

$$
\begin{aligned}
(d(x) \cdot f(d(x)) \cdot d(x \cdot d(x)) &= (d(x) \cdot f(d(x)))\cdot \\
\text{(By Definition 2.1)} \quad & \quad ((d(x) \cdot f(d(x))) \wedge (x \cdot d(x))) \\
&= (d(x \cdot f(d(x)))\cdot \\
& \quad (((x \cdot d(x) \cdot (d(x) \cdot f(d(x))))\cdot \\
\text{(By Definition 1.11)} \quad & \quad (d(x) \cdot f(d(x)))) \\
\text{(By Proposition 1.5 (5))} \quad &= 0.
\end{aligned}
$$

Hence, $d(x) \cdot f(d(x)) \leq d(x \cdot d(x))$ for all $x \in A$.

(5) For all $x \in A$,

$$
\begin{aligned}
(d(d(x)) \cdot f(x)) \cdot d(d(x) \cdot x) &= (d(d(x)) \cdot f(x))\cdot \\
\text{(By Definition 2.1)} \quad & \quad ((d(d(x)) \cdot f(x)) \wedge (d(x) \cdot x)) \\
&= (d(d(x)) \cdot f(x))\cdot \\
& \quad (((d(x) \cdot x) \cdot (d(d(x)) \cdot f(x)))\cdot \\
\text{(By Definition 1.11)} \quad & \quad (d(d(x)) \cdot f(x))) \\
\text{(By Proposition 1.5 (5))} \quad &= 0.
\end{aligned}
$$

Hence, $d(d(x)) \cdot f(x) \leq d(d(x) \cdot x)$ for all $x \in A$.

(6) For all $x, y \in A$,

$$\begin{aligned}
&\text{(By Definition 1.11)} & (d(y\cdot x)\cdot f(x))\cdot d(x\wedge y) &= (d(y\cdot x)\cdot f(x))\cdot d((y\cdot x)\cdot x)\\
& & &= (d(y\cdot x)\cdot f(x))\cdot\\
&\text{(By Definition 2.1)} & &\quad ((d(y\cdot x)\cdot f(x))\wedge((y\cdot x)\cdot x))\\
& & &= (d(y\cdot x)\cdot f(x))\cdot\\
&\text{(By Definition 1.11)} & &\quad ((d(y\cdot x)\cdot f(x))\wedge(x\wedge y))\\
& & &= (d(y\cdot x)\cdot f(x))\cdot\\
& & &\quad (((x\wedge y)\cdot(d(y\cdot x)\cdot f(x)))\cdot\\
&\text{(By Definition 1.11)} & &\quad (d(y\cdot x)\cdot f(x)))\\
&\text{(By Proposition 1.5 (5))} & &= 0.
\end{aligned}$$

Hence, $d(y\cdot x)\cdot f(x) \leq d(x\wedge y)$ for all $x, y \in A$.

(7) For all $x \in A$,

$$\begin{aligned}
&\text{(By UP-2)} & d(x) &= 0\cdot d(x)\\
&\text{(By Proposition 2.10 (1))} & &= (f(x)\cdot d(x))\cdot d(x)\\
&\text{(By Definition 1.11)} & &= d(x)\wedge f(x).
\end{aligned}$$

Hence, $d(x) = d(x)\wedge f(x)$ for all $x \in A$. □

**Proposition 2.11.** *Let $d$ be an r-$f$-derivation of type I of $A$. Then the following properties hold: for any $x, y \in A$,*

*(1)* $f(x)\cdot d(y) \leq d(x\cdot y)$,

*(2)* $f(x\cdot d(x))\cdot d(d(x)) \leq d(d(x))$,

*(3)* $f(d(x))\cdot d(x) \leq d(d(x)\cdot x)$,

*(4)* $f(x)\cdot d(d(x)) \leq d(x\cdot d(x))$, *and*

*(5)* $f(y\cdot x)\cdot d(x) \leq d(x\wedge y)$.

**Proof.** (1) For all $x, y \in A$,

$$
\begin{aligned}
&\text{(By Definition 2.1)} & (f(x)\cdot d(y))\cdot d(x\cdot y) &= (f(x)\cdot d(y))\cdot((f(x)\cdot d(y))\wedge(x\cdot y)) \\
& & &= (f(x)\cdot d(y))\cdot(((x\cdot y)\cdot(f(x)\cdot d(y)))\cdot \\
&\text{(By Definition 1.11)} & &\quad (f(x)\cdot d(y))) \\
&\text{(By Proposition 1.5 (5))} & &= 0.
\end{aligned}
$$

Hence, $f(x)\cdot d(y) \leq d(x\cdot y)$ for all $x, y \in A$.

(2) For all $x \in A$,

$$
\begin{aligned}
& & (f(x\cdot d(x))\cdot d(d(x)))\cdot d(d(x)) &= (f(x\cdot d(x))\cdot d(d(x)))\cdot \\
&\text{(By Theorem 2.8 (2))} & &\quad d(d(x)\wedge x) \\
& & &= (f(x\cdot d(x))\cdot d(d(x)))\cdot \\
&\text{(By Definition 1.11)} & &\quad d((x\cdot d(x))\cdot d(x)) \\
& & &= (f(x\cdot d(x))\cdot d(d(x)))\cdot \\
& & &\quad ((f(x\cdot d(x))\cdot d(d(x)))\wedge \\
&\text{(By Definition 2.1)} & &\quad ((x\cdot d(x))\cdot x)) \\
& & &= (f(x\cdot d(x))\cdot d(d(x)))\cdot \\
& & &\quad ((((x\cdot d(x))\cdot x)\cdot(f(x\cdot d(x))\cdot d(d(x))))\cdot \\
&\text{(By Definition 1.11)} & &\quad (f(x\cdot d(x))\cdot d(d(x)))) \\
&\text{(By Proposition 1.5 (5))} & &= 0.
\end{aligned}
$$

Hence, $f(x\cdot d(x))\cdot d(d(x)) \leq d(d(x))$ for all $x \in A$.

(3) For all $x \in A$,

$$
\begin{aligned}
(f(d(x)) \cdot d(x)) \cdot d(d(x) \cdot x) &= (f(d(x)) \cdot d(x)) \cdot \\
&\quad ((f(d(x)) \cdot d(x)) \wedge (d(x) \cdot x)) && \text{(By Definition 2.1)} \\
&= (f(d(x)) \cdot d(x)) \cdot \\
&\quad (((d(x) \cdot x) \cdot (f(d(x)) \cdot d(x))) \cdot \\
&\quad (f(d(x)) \cdot d(x))) && \text{(By Definition 1.11)} \\
&= 0. && \text{(By Proposition 1.5 (5))}
\end{aligned}
$$

Hence, $f(d(x)) \cdot d(x) \leq d(d(x) \cdot x)$ for all $x \in A$.

(4) For all $x \in A$,

$$
\begin{aligned}
(f(x) \cdot d(d(x))) \cdot d(x \cdot d(x)) &= (f(x) \cdot d(d(x))) \cdot \\
&\quad ((f(x) \cdot d(d(x))) \wedge (x \cdot d(x))) && \text{(By Definition 2.1)} \\
&= (f(x) \cdot d(d(x))) \cdot \\
&\quad (((x \cdot d(x)) \cdot (f(x) \cdot d(d(x))) \cdot \\
&\quad (f(x) \cdot d(d(x)))) && \text{(By Definition 1.11)} \\
&= 0. && \text{(By Proposition 1.5 (5))}
\end{aligned}
$$

Hence, $f(x) \cdot d(d(x)) \leq d(x \cdot d(x))$ for all $x \in A$.

(5) For all $x, y \in A$,

$$
\begin{aligned}
&\text{(By Definition 1.11)} && (f(y\cdot x)\cdot d(x))\cdot d(x\wedge y) = (f(y\cdot x)\cdot d(x))\cdot d((y\cdot x)\cdot x)\\
& && = (f(y\cdot x)\cdot d(x))\cdot\\
&\text{(By Definition 2.1)} && \quad ((f(y\cdot x)\cdot d(x))\wedge((y\cdot x)\cdot x))\\
& && = (f(y\cdot x)\cdot d(x))\cdot\\
& && \quad ((((y\cdot x)\cdot x)\cdot(f(y\cdot x)\cdot d(x)))\cdot\\
&\text{(By Definition 1.11)} && \quad (f(y\cdot x)\cdot d(x))\\
&\text{(By Proposition 1.5 (5))} && = 0.
\end{aligned}
$$

Hence, $f(y\cdot x)\cdot d(x) \leq d(x\wedge y)$ for all $x, y \in A$. □

**Definition 2.12.** Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type I of $A$. We define a subset $\mathrm{Ker}_d(A)$ of $A$ by

$$\mathrm{Ker}_d(A) = \{x \in A \mid d(x) = 0\}.$$

**Theorem 2.13.** *In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type I of $A$, then*

*(1) $y\wedge x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$, and*

*(2) $x\cdot y \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$.*

**Proof.** (1) Assume that $d$ is an $r$-$f$-derivation of type I of $A$. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

$$
\begin{aligned}
&\text{(By Definition 1.11)} && d(y\wedge x) = d((x\cdot y)\cdot y)\\
&\text{(By Definition 2.1)} && = (f(x\cdot y)\cdot d(y))\wedge((x\cdot y)\cdot y)\\
& && = (f(x\cdot y)\cdot 0)\wedge((x\cdot y)\cdot y)\\
&\text{(By UP-3)} && = 0\wedge((x\cdot y)\cdot y)\\
&\text{(By Proposition 1.13 (1))} && = 0.
\end{aligned}
$$

Hence, $y\wedge x \in \mathrm{Ker}_d(A)$.

(2) Assume that $d$ is an $r$-$f$-derivation of type I of $A$. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

$$\begin{aligned}
&\text{(By Definition 2.1)} && d(x \cdot y) = (f(x) \cdot d(y)) \wedge (x \cdot y) \\
& && = (f(x) \cdot 0) \wedge (x \cdot y) \\
&\text{(By UP-3)} && = 0 \wedge (x \cdot y) \\
&\text{(By Proposition 1.13 (1))} && = 0.
\end{aligned}$$

Hence, $x \cdot y \in \mathrm{Ker}_d(A)$. □

**Theorem 2.14.** *In a meet-commutative UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type I of $A$ and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$.*

**Proof.** Assume that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then we get $y \cdot x = 0$ and $d(y) = 0$. Thus

$$\begin{aligned}
&\text{(By UP-2)} && d(x) = d(0 \cdot x) \\
& && = d((y \cdot x) \cdot x) \\
&\text{(By Definition 1.12)} && = d((x \cdot y) \cdot y) \\
&\text{(By Definition 2.1)} && = (f(x \cdot y) \cdot d(y)) \wedge ((x \cdot y) \cdot y) \\
&\text{(By Definition 1.11)} && = (f(x \cdot y) \cdot 0) \wedge (y \wedge x) \\
&\text{(By UP-3)} && = 0 \wedge (y \wedge x) \\
&\text{(By Proposition 1.13 (1))} && = 0.
\end{aligned}$$

Hence, $x \in \mathrm{Ker}_d(A)$. □

**Theorem 2.15.** *In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type I of $A$, then $\mathrm{Ker}_d(A)$ is a UP-subalgebra of $A$.*

**Proof.** Assume that $d$ is an $r$-$f$-derivation of type I of $A$. By Theorem 2.6 (2) , we have $d(0)=0$ and so $0\in \mathrm{Ker}_d(A)\neq\emptyset$. Let $x,y\in \mathrm{Ker}_d(A)$. Then $d(x)=0$ and $d(y)=0$. Thus

$$
\begin{aligned}
&\text{(By Definition 2.1)} & d(x\cdot y) &= (f(x)\cdot d(y))\wedge(x\cdot y)\\
&& &= (f(x)\cdot 0)\wedge(x\cdot y)\\
&\text{(By UP-3)} & &= 0\wedge(x\cdot y)\\
&\text{(By Proposition 1.13 (1))} & &= 0.
\end{aligned}
$$

Hence, $x\cdot y\in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a UP-subalgebra of $A$. □

On an $l$-$f$-derivation (resp. $r$-$f$-derivation and $f$-derivation) of type I of $A$, we can show that $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$, by the following examples.

**Example 2.16.** Let $A=\{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 3 |
| 2 | 0 | 0 | 0 | 3 |
| 3 | 0 | 0 | 2 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A\to A$ as follows:

$$f(0)=0, f(1)=1, f(2)=3, \text{ and } f(3)=2.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A\to A$ as follows:

$$d(0)=0, d(1)=1, d(2)=0, \text{ and } d(3)=0.$$

Then $d$ is an $l$-$f$-derivation of type I of $A$ and $\mathrm{Ker}_d(A)=\{0,2,3\}$. Since $0\cdot(2\cdot 1)=0\in \mathrm{Ker}_d(A), 2\in \mathrm{Ker}_d(A)$ but $0\cdot 1=1\notin \mathrm{Ker}_d(A)$, we have $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Example 2.17.** Form Example 2.16, we have $d$ is an $r$-$f$-derivation of type I of $A$ and $\mathrm{Ker}_d(A)=\{0,2,3\}$. Since $0\cdot(3\cdot 1)=0\in \mathrm{Ker}_d(A), 3\in \mathrm{Ker}_d(A)$ but $0\cdot 1=1\notin \mathrm{Ker}_d(A)$, we have $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Example 2.18.** Form Example 2.16 and 2.17, we have $d$ is an $f$-derivation of type I of $A$ and $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Definition 2.19.** Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type I of $A$. We define a subset $\mathrm{Fix}_d(A)$ of $A$ by

$$\mathrm{Fix}_d(A) = \{x \in A \mid d(x) = x\}.$$

On an $r$-$f$-derivation of type I of $A$, we can show that $\mathrm{Fix}_d(A)$ is not a UP-subalgebra of $A$ by the following example.

**Example 2.20.** Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 3 | 3 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 1 | 2 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 2, f(2) = 2, \text{ and } f(3) = 0.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 2, \text{ and } d(3) = 0.$$

Then $d$ is an $r$-$f$-derivation of type I of $A$ and $\mathrm{Fix}_d(A) = \{0,1,2\}$. Since $1,2 \in \mathrm{Fix}_d(A)$ but $1 \cdot 2 = 3 \notin \mathrm{Fix}_d(A)$, we have $\mathrm{Fix}_d(A)$ is not a UP-subalgebra of $A$.

On an $l$-$f$-derivation of type I of $A$, we can show that $\mathrm{Fix}_d(A)$ is not a UP-ideal of $A$, by the following example.

**Example 2.21.** Let $A = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 2 | 0 | 4 |
| 2 | 0 | 0 | 0 | 0 | 3 |
| 3 | 0 | 3 | 2 | 0 | 4 |
| 4 | 0 | 0 | 1 | 0 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 0, f(2) = 2, f(3) = 0, \text{and } f(4) = 2.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 0, d(2) = 2, d(3) = 0, \text{ and } d(4) = 2.$$

Then $d$ is an $l$-$f$-derivation of type I of $A$ and $\text{Fix}_d(A) = \{0,2\}$. Since $0 \cdot (2 \cdot 1) = 0 \in \text{Fix}_d(A), 2 \in \text{Fix}_d(A)$ but $0 \cdot 1 = 1 \notin \text{Fix}_d(A)$, we have $\text{Fix}_d(A)$ is not a UP-ideal of $A$.

## 3. Left-$f$-derivations and right-$f$-derivations of type II

In this section, we first introduce the notion of left (resp. right)-$f$-derivations of type II of $A$ and study some of their basic properties. Finally, two subsets $\text{Ker}_d(A)$ and $\text{Fix}_d(A)$ for left (resp. right)-$f$-derivation $d$ of type II of $A$ are studied.

**Definition 3.1.** Let $A$ be a UP-algebra and let $f$ be a UP-endomorphism of $A$. A self-map $d\colon A \to A$ is called a *left-$f$-derivation of type II* (in short, an *$l$-$f$-derivation of type II*) of $A$ if it satisfies the identity $d(x \cdot y) = (x \cdot y) \wedge (d(x) \cdot f(y))$ for all $x, y \in A$. Similarly, a self-map $d\colon A \to A$ is called a *right-$f$-derivation of type II* (in short, an *$r$-$f$-derivation of type II*) of $A$ if is satisfies the identity $d(x \cdot y) = (x \cdot y) \wedge (f(x) \cdot d(y))$ for all $x, y \in A$. Moreover, if $d$ is both a left-$f$-derivation and a right-$f$-derivation of type II of $A$, it is called an *$f$-derivation of type II of $A$*.

By using Microsoft Excel, we have all examples.

**Example 3.2.** Let $A = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 2 | 3 | 3 |
| 2 | 0 | 1 | 0 | 0 | 4 |
| 3 | 0 | 1 | 3 | 0 | 4 |
| 4 | 0 | 1 | 3 | 0 | 0 |

Then $(A; \cdot, 0)$ is a UP-algebra. Let $f$ be an identity map on $A$. Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 0, d(2) = 2, d(3) = 3, \text{ and } d(4) = 4.$$

Then $d$ is an $l$-$f$-derivation of type II of $A$.

**Example 3.3.** Let $A = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 3 | 4 |
| 3 | 0 | 1 | 0 | 0 | 4 |
| 4 | 0 | 1 | 2 | 3 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. Let $f$ be an identity map on $A$. Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 0, d(2) = 0, d(3) = 0, \text{ and } d(4) = 4.$$

Then $d$ is an $r$-$f$-derivation of type II of $A$.

**Example 3.4.** Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 2 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 1.$$

Then $d$ is an $f$-derivation of type II of $A$.

**Definition 3.5.** An $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) $d$ of type II of $A$ is called *regular* if $d(0) = 0$.

**Theorem 3.6.** *In a UP-algebra A, the following statements hold:*

*(1) every $l$-$f$-derivation of type II of A is regular, and*

*(2) every $r$-$f$-derivation of type II of A is regular.*

**Proof.** (1) Assume that $d$ is an $l$-$f$-derivation of type II of $A$. Then

$$
\begin{aligned}
&\text{(By UP-3)} && d(0) = d(x \cdot 0) \\
&\text{(By Definition 3.1)} && = (x \cdot 0) \wedge (d(x) \cdot f(0)) \\
&\text{(By UP-3)} && = 0 \wedge (d(x) \cdot 0) \\
&\text{(By UP-3)} && = 0 \wedge 0 \\
&\text{(By Proposition 1.13 (3))} && = 0.
\end{aligned}
$$

Hence, $d$ is regular.

(2) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Then

$$
\begin{aligned}
&\text{(By UP-3)} && d(0) = d(0 \cdot 0) \\
&\text{(By Definition 3.1)} && = (0 \cdot 0) \wedge (f(0) \cdot d(0)) \\
&\text{(By UP-3)} && = 0 \wedge (0 \cdot d(0)) \\
&\text{(By UP-2)} && = 0 \wedge d(0) \\
&\text{(By Proposition 1.13 (1))} && = 0.
\end{aligned}
$$

Hence, $d$ is regular. □

**Corollary 3.7.** *Every $f$-derivation of type II of A is regular.*

**Theorem 3.8.** *In a UP-algebra A, the following statements hold:*

*(1) if $d$ is an $l$-$f$-derivation of type II of A, then $d(x) = x \wedge f(x)$ for all $x \in A$, and*

*(2) if $d$ is an $r$-$f$-derivation of type II of A, then $d(x) = x \wedge d(x)$ for all $x \in A$.*

**Proof.** (1) Assume that $d$ is an $l$-$f$-derivation of type II of $A$. Then, for all $x \in A$,

(By UP-2) $d(x) = d(0 \cdot x)$

(By Definition 3.1) $= (0 \cdot x) \wedge (d(0) \cdot f(x))$

(By Theorem 3.6 (1)) $= (0 \cdot x) \wedge (0 \cdot f(x))$

(By UP-2) $= x \wedge f(x)$.

(2) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Then, for all $x \in A$,

(By UP-2) $d(x) = d(0 \cdot x)$

(By Definition 3.1) $= (0 \cdot x) \wedge (f(0) \cdot d(x))$

(By UP-2) $= x \wedge (0 \cdot d(x))$

(By UP-2) $= x \wedge d(x)$.

□

**Corollary 3.9.** *If $d$ is an $f$-derivation of type II of $A$, then $d(x) = x \wedge f(x) = x \wedge d(x)$ for all $x \in A$.*

**Proposition 3.10.** *Let $d$ be an $l$-$f$-derivation of type II of $A$. Then the following properties hold: for any $x, y \in A$,*

*(1)* $x \leq d(x)$,
*(2)* $x \cdot y \leq d(x \cdot y)$,
*(3)* $x \wedge y \leq d(x \wedge y)$, *and*
*(4)* $d(x) = d(x) \wedge x$.

**Proof.** (1) For all $x \in A$,

(By Theorem 3.8 (1)) $x \cdot d(x) = x \cdot (x \wedge f(x))$

(By Definition 1.11) $= x \cdot ((f(x) \cdot x) \cdot x)$

(By Proposition 1.5 (5)) $= 0$.

Hence, $x \leq d(x)$ for all $x \in A$.

(2) For all $x,y \in A$,

(By Definition 3.1) $(x \cdot y) \cdot d(x \cdot y) = (x \cdot y) \cdot ((x \cdot y) \wedge (d(x) \cdot f(y)))$

(By Definition 1.11) $= (x \cdot y) \cdot (((d(x) \cdot f(y)) \cdot (x \cdot y)) \cdot (x \cdot y))$

(By Proposition 1.5 (5)) $= 0.$

Hence, $x \cdot y \leq d(x \cdot y)$ for all $x,y \in A$.

(3) For all $x,y \in A$,

(By Definition 1.11) $(x \wedge y) \cdot d(x \wedge y) = (x \wedge y) \cdot d((y \cdot x) \cdot x)$

(By Definition 3.1) $= (x \wedge y) \cdot (((y \cdot x) \cdot x) \wedge (d(y \cdot x) \cdot f(x)))$

(By Definition 1.11) $= (x \wedge y) \cdot ((x \wedge y) \wedge (d(y \cdot x) \cdot f(x)))$

$= (x \wedge y) \cdot (((d(y \cdot x) \cdot f(x)) \cdot$

(By Definition 1.11) $(x \wedge y)) \cdot (x \wedge y))$

(By Proposition 1.5 (5)) $= 0.$

Hence, $x \wedge y \leq d(x \wedge y)$ for all $x,y \in A$.

(4) For all $x \in A$,

(By UP-2) $d(x) = 0 \cdot d(x)$

(By Proposition 3.10 (1)) $= (x \cdot d(x)) \cdot d(x)$

(By Definition 1.11) $= d(x) \wedge x.$

Hence, $d(x) = d(x) \wedge x$ for all $x,y \in A$. □

**Proposition 3.11.** *Let $d$ be an $r$-$f$-derivation of type II of $A$. Then the following properties hold: for any $x,y \in A$,*

*(1)* $x \leq d(x)$,

*(2)* $x \cdot y \leq d(x \cdot y)$,

*(3)* $x \wedge y \leq d(x \wedge y)$, *and*

*(4)* $d(x) = d(x) \wedge x$.

**Proof.** (1) For all $x \in A$,

$$
\begin{aligned}
&\text{(By Theorem 3.8 (2))} && x \cdot d(x) = x \cdot (x \wedge d(x)) \\
&\text{(By Definition 1.11)} && = x \cdot ((d(x) \cdot x) \cdot x) \\
&\text{(By Proposition 1.5 (5))} && = 0.
\end{aligned}
$$

Hence, $x \leq d(x)$ for all $x \in A$.

(2) For all $x, y \in A$,

$$
\begin{aligned}
&\text{(By Definition 3.1)} && (x \cdot y) \cdot d(x \cdot y) = (x \cdot y) \cdot ((x \cdot y) \wedge (f(x) \cdot d(y))) \\
&\text{(By Definition 1.11)} && = (x \cdot y) \cdot (((f(x) \cdot d(y)) \cdot (x \cdot y)) \cdot (x \cdot y)) \\
&\text{(By Proposition 1.5 (5))} && = 0.
\end{aligned}
$$

Hence, $x \cdot y \leq d(x \cdot y)$ for all $x, y \in A$.

(3) For all $x, y \in A$,

$$
\begin{aligned}
&\text{(By Definition 1.11)} && (x \wedge y) \cdot d(x \wedge y) = (x \wedge y) \cdot d((y \cdot x) \cdot x) \\
&\text{(By Definition 3.1)} && = (x \wedge y) \cdot (((y \cdot x) \cdot x) \wedge (f(y \cdot x) \cdot d(x))) \\
&\text{(By Definition 1.11)} && = (x \wedge y) \cdot ((x \wedge y) \wedge (f(y \cdot x) \cdot d(x))) \\
& && = (x \wedge y) \cdot (((f(y \cdot x) \cdot d(x)) \cdot \\
&\text{(By Definition 1.11)} && \quad (x \wedge y)) \cdot (x \wedge y)) \\
&\text{(By Proposition 1.5 (5))} && = 0.
\end{aligned}
$$

Hence, $x \wedge y \leq d(x \wedge y)$ for all $x, y \in A$.

(4) For all $x \in A$,

$$
\begin{aligned}
&\text{(By UP-2)} && d(x) = 0 \cdot d(x) \\
&\text{(By Proposition 3.11 (1))} && = (x \cdot d(x)) \cdot d(x) \\
&\text{(By Definition 1.11)} && = d(x) \wedge x.
\end{aligned}
$$

Hence, $d(x) = d(x) \wedge x$ for all $x, y \in A$. □

**Corollary 3.12.** *Let $d$ be an $f$-derivation of type II of $A$. Then the following properties hold: for any $x, y \in A$,*

*(1)* $x \leq d(x)$,

*(2)* $x \cdot y \leq d(x \cdot y)$,

*(3)* $x \wedge y \leq d(x \wedge y)$, *and*

*(4)* $d(x) = d(x) \wedge x$.

**Definition 3.13.** Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type II of $A$. We define a subset $\mathrm{Ker}_d(A)$ of $A$ by

$$\mathrm{Ker}_d(A) = \{x \in A \mid d(x) = 0\}.$$

**Theorem 3.14.** *In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type II of $A$, then*

*(1)* $y \wedge x \in \mathrm{Ker}_d(A)$ *for all* $y \in \mathrm{Ker}_d(A)$ *and* $x \in A$, *and*

*(2)* $x \cdot y \in \mathrm{Ker}_d(A)$ *for all* $y \in \mathrm{Ker}_d(A)$ *and* $x \in A$.

**Proof.** (1) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

| | |
|---|---|
| (By Proposition 1.11) | $(y \wedge x) = d((x \cdot y) \cdot y)$ |
| (By Definition 3.1) | $= ((x \cdot y) \cdot y) \wedge (f(x \cdot y) \cdot d(y))$ |
| (By Proposition 1.11) | $= (y \wedge x) \wedge (f(x \cdot y) \cdot 0)$ |
| (UP-3) | $= (y \wedge x) \wedge 0$ |
| (By Proposition 1.13 (2)) | $= 0.$ |

Hence, $y \wedge x \in \mathrm{Ker}_d(A)$.

(2) Assume that $d$ is an $r$-$f$-derivation of type II of $A$. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

$$
\begin{aligned}
&\text{(By Definition 3.1)} && d(x \cdot y) = (x \cdot y) \wedge (f(x) \cdot d(y)) \\
& && = (x \cdot y) \wedge (f(x) \cdot 0) \\
&\text{(By UP-3)} && = (x \cdot y) \wedge 0 \\
&\text{(By Proposition 1.13 (2))} && = 0.
\end{aligned}
$$

Hence, $x \cdot y \in \mathrm{Ker}_d(A)$. □

**Theorem 3.15.** *In a meet-commutative UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type II of $A$ and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$.*

**Proof.** Assume that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then we get $y \cdot x = 0$ and $d(y) = 0$. Thus

$$
\begin{aligned}
&\text{(By UP-2)} && d(x) = d(0 \cdot x) \\
& && = d((y \cdot x) \cdot x) \\
&\text{(By Definition 1.12)} && = d((x \cdot y) \cdot y) \\
&\text{(By Definition 3.1)} && = ((x \cdot y) \cdot y) \wedge (f(x \cdot y) \cdot d(y)) \\
&\text{(By Definition 1.11)} && = (y \wedge x) \wedge (f(x \cdot y) \cdot 0) \\
&\text{(By UP-3)} && = (y \wedge x) \wedge 0 \\
&\text{(By Proposition 1.13 (1))} && = 0.
\end{aligned}
$$

Hence, $x \in \mathrm{Ker}_d(A)$. □

**Theorem 3.16.** *In a UP-algebra $A$, if $d$ is an $r$-$f$-derivation of type II of $A$, then $\mathrm{Ker}_d(A)$ is a UP-subalgebra of $A$.*

**Proof.** Assume that $d$ is an $r$-$f$-derivation of type II of $A$. By Theorem 3.6 (2), we have $d(0) = 0$ and so $0 \in \mathrm{Ker}_d(A) \neq \emptyset$. Let $x, y \in \mathrm{Ker}_d(A)$. Then $d(x) = 0$ and $d(y) = 0$. Thus

$$
\begin{aligned}
&\text{(By Definition 3.1)} & d(x \cdot y) &= (x \cdot y) \wedge (f(x) \cdot d(y)) \\
& & &= (x \cdot y) \wedge (f(x) \cdot 0) \\
&\text{(By UP-3)} & &= (x \cdot y) \wedge 0 \\
&\text{(By Proposition 1.13 (2))} & &= 0.
\end{aligned}
$$

Hence,$x \cdot y \in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a UP-subalgebra of $A$. □

On an $l$-$f$-derivation (resp. $r$-$f$-derivation and $f$-derivation) of type II of $A$, we can show that $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$, by the following examples.

**Example 3.17.** Form Example 2.16, we have $d$ is an $l$-$f$-derivation of type II of $A$ and $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Example 3.18.** Form Example 2.17, we have $d$ is an $r$-$f$-derivation of type II of $A$ and $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Example 3.19.** Form Example 3.17 and 3.18, we have $d$ is an $f$-derivation of type II of $A$ and $\mathrm{Ker}_d(A)$ is not a UP-ideal of $A$.

**Definition 3.20.** Let $d$ be an $l$-$f$-derivation (resp. $r$-$f$-derivation, $f$-derivation) of type II of $A$. We define a subset $\mathrm{Fix}_d(A)$ of $A$ by

$$\mathrm{Fix}_d(A) = \{x \in A \mid d(x) = x\}.$$

On an $l$-$f$-derivation (resp. $r$-$f$-derivation) of type II of $A$, we can show that $\mathrm{Fix}_d(A)$ is not a UP-subalgebra of $A$ by the following examples.

**Example 3.21.** Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 2 |
| 2 | 0 | 1 | 0 | 3 |
| 3 | 0 | 0 | 0 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 1.$$

Then $d$ is an $l$-$f$-derivation of type II of $A$ and $\mathrm{Fix}_d(A) = \{0,1,3\}$. Since $1,3 \in \mathrm{Fix}_d(A)$ but $1 \cdot 3 = 2 \notin \mathrm{Fix}_d(A)$, we have $\mathrm{Fix}_d(A)$ is not a UP-subalgebra of $A$.

**Example 3.22.** Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 3 |
| 3 | 0 | 2 | 2 | 0 |

Then $(A;\cdot,0)$ is a UP-algebra. We define a self-map $f\colon A \to A$ as follows:

$$f(0) = 0, f(1) = 1, f(2) = 0, \text{ and } f(3) = 1.$$

Then $f$ is a UP-endomorphism. We define a self-map $d\colon A \to A$ as follows:

$$d(0) = 0, d(1) = 1, d(2) = 0, \text{ and } d(3) = 3.$$

Then $d$ is an $r$-$f$-derivation of type II of $A$ and $\mathrm{Fix}_d(A) = \{0,1,3\}$. Since $1,3 \in \mathrm{Fix}_d(A)$ but $3 \cdot 1 = 2 \notin \mathrm{Fix}_d(A)$, we have $\mathrm{Fix}_d(A)$ is not a UP-subalgebra of $A$.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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