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PROJECTIVE DIMENSION OF SECOND ORDER SYMMETRIC DERIVATION OF KÄHLER MODULES FOR HYPERSURFACES

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Abstract. $R = k[x_1,...,x_s]$ be a polynomial algebra and I be an ideal of R generated by $f \in R$. Then $S = R/I = \frac{k[x_1,...,x_s]}{(f)}$ be an affine domain which is called hypersurfaces. $\Omega_1(S)$ denotes the module of first order derivations of Kähler modules over S. $\vee^2(\Omega_1(S))$ denotes the module of second order derivations of symmetric algebra on $\Omega_1(S)$. In this paper, we prove that if S be an affine domain represented by $S = \frac{k[x_1,...,x_s]}{(f)}$, then projective dimension of $\vee^2(\Omega_1(S))$ is less than or equal to 1.

Keywords: Kähler module; symmetric derivation; projective dimension; hypersurfaces.

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1. Introduction

Let R be a coordinate ring related to an irreducible affine algebra.Vasconcelos gave some important results about projective dimension of $\Omega_1(R)$. He proved the following at 1968 in [9]:" Let R be an affine domain and $pd(\Omega_1(R))$ is finite, then R is normal if and only if $\Omega_1(R)$ has

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a free resolution of length less than or equal to one." Matsuoka presented that projective dimension of $\Omega_1(R)$ is infinite under some special cases in [3].The universal modules of higher differential operators of an algebra were introduced by H.Osborn in [7].Afterwards, same notion emerged in [4,8]. Erdogan studied with regard to the universal modules of higher differential operators at 1993 in [2]. In that work, he represented very important results of projective dimension of universal modules.For instance, he proved that if $R = k[x_1,...,x_s]$ and I be an ideal of R generated by $f \in R$, then $\mathrm{pd}\Omega_n(R/I) \leq 1$. Olgun defined generalized symmetric derivations on universal modules and gave some properties,examples and some interesting results by using this definition in [5] and connectedly Kähler modules and symmetric derivations introduced by H.Osborn in [7].Using these,in this work, we have found out the following:Firstly, "Let M be an R-module.Assume that N be a submodule of M and that L_N be the submodule of $\vee^2(M)$ generated by $\{m \vee n : m \in M \text{ and } n \in N\}$.Then there is a R-module isomorphism $\vee^2 M/L_N \simeq \vee^2(M/N)$.
Secondly," if S is an affine domain represented by $S = k[x_1,...,x_s]/(f)$, then projective dimension of $\vee^2(\Omega_1(S))$ is less than or equal to one."Ultimately, we have given some examples about them.

The studies about symmetric derivations of universal modules and its projective dimensions are less than the other ones.So, the purpose of this paper is to give the more informations and homological properties related to it. Throughout this paper we assume R be a commutative algebra over an algebraically closed field k with characteristic zero. Let $\Omega_q(R)$ be the module of q-th order Kähler differentials of R over k and d_q be the canonical $q-th$ order k-derivation $R \longrightarrow \Omega_q(R)$ of R.The pair $\{\Omega_q(R), d_q\}$ has the universal mapping property with regard to the $q-th$ order k-derivations of R. $\Omega_q(R)$ is generated by the set $\{d_q(r) : r \in R\}$.

2. Preliminaries

Definition 2.1. [6] Let M be an R-module,$M \otimes_R M$ be the tensor product of M with itself and K be the submodule of $M \otimes_R M$ generated by the elements of the form $x \otimes y - y \otimes x$ where

$x,y\in M$. Then the module $\vee^2(M)=M\otimes_R M/K$ is called the second order symmetric power of M. Assume that $\otimes : M\times M\longrightarrow M\otimes M$ is the canonical balanced map defined by $\otimes(x,y)=x\otimes y$ and that $\gamma : M\otimes M\longrightarrow \vee^2 M$ is the natural surjection .Then the composite of $\otimes$ and γ is a bilinear map. We denote $\gamma\otimes$ by $\vee$.

Lemma 2.1. [1] Let M,N be R-modules and $\theta : M\times M\longrightarrow N$ be a bilinear map.Then there exists an unique R-module homomorphism $f:\vee^2 M\longrightarrow N$ such that the diagram

$$\begin{array}{ccc} M\times M & \xrightarrow{\theta} & N \\ \vee\searrow & & \nearrow f \\ & \vee^2 M & \end{array}$$

commutes.

Definition 2.2. [5] Let k is a field with characteristic zero and R be a commutative k-algebra with unit. $R\rightarrow\Omega_q(R)$ denotes q-th order Kähler derivations of R and let $\vee(\Omega_q(R))$ be the symmetric algebra $\bigoplus_{p\geq 0}\vee^p(\Omega_q(R))$ generated by $\Omega_q(R)$ over R.

D is a k-linear map of $\vee(\Omega_q(R))$ into itself is called a generalized symmetric derivations such that satisfies the following conditions:

i) $D(\vee^p(\Omega_q(R)))\subset\vee^{p+1}(\Omega_q(R))$

ii) D is a q-th order derivation over k and

iii) the restriction of D to R is the Kähler derivation

$d_q : R\rightarrow\Omega_q(R)$ where $R\simeq\vee^0(\Omega_q(R))$

Proposition 2.1. [5] Let $R=k[x_1,....,x_s]$ be a polynomial algebra with

$dimR=s$. Then $\Omega_q(R)$ is a free R-module of rank $\binom{q+s}{s}-1$ generated by the set

$\{\ d_q(x_1^{i_1}.....x_s^{i_s}) : i_1+.....+i_s\leq q\}$

$\vee^2(\Omega_q(R))$ is a free R-module of rank $\binom{t+1}{t-1}$

where $t=\binom{q+s}{s}-1$ with basis $\{\ d_q(x_1^{i_1}.....x_s^{i_s})\otimes\ d_q(x_1^{i_1}.....x_s^{i_s}) : i_1+.....+i_s\leq q\}$

Proposition 2.2. [2] Let $S=R/I$ for an ideal of R containing $f\in R$. F is a free S-module with basis $\{d_n(x^\alpha) : \mid\alpha\mid\leq n\}$ and N is a submodule of F generated by the set $\{d_n(x^\alpha f) :\mid\alpha\mid< n\}$,

from corollary 3.1.3 in [2], there is an isomorphism $\Omega_n(S) \simeq F/N$. Therefore we have
$0 \longrightarrow N \longrightarrow F \longrightarrow \Omega_n(S) \simeq F/N \longrightarrow 0$ an exact sequence of S-modules.

Lemma 2.2. [5] Let R be an affine domain with $dimR = s$. Then $\Omega_q(R)$ is a free R-module if and only if $\vee^2(\Omega_q(R))$ is a free R-module.

3. Main results

3.1. Projective dimension of the second order symmetric derivation of the first order Kähler differentials for hypersurfaces

Proposition 3.1. Let M be an R-module. Assume that N is a submodule of M and L_N is the submodule of $\vee^2 M$ generated by the set $\{x \vee y \mid x \in M \ and \ y \in N\}$. Then, we say that $\vee^2 M/L_N$ is isomorphic to $\vee^2(M/N)$.

Proof Let $\vartheta : M \times M \longrightarrow \vee^2(M/N)$ be the composite map

$$M \times M \xrightarrow{(\lambda,\lambda)} M/N \times M/N \xrightarrow{\vee} \vee^2(M/N)$$

where $\lambda : M \longrightarrow M/N$ is the natural surjection. Then, ϑ is a bilinear map.
By the universal property of $\vee^2(M)$,there exists an unique R-module homomorphism ρ : $\vee^2 M \longrightarrow \vee^2(M/N)$ such that the diagram

$$\begin{array}{ccc} M \times M & \xrightarrow{\vartheta} & \vee^2(M/N) \\ \vee \searrow & & \nearrow \rho \\ & \vee^2 M & \end{array}$$

commutes. It is obvious that $\rho(L_N) = 0$. Therefore we have an induced map
$\bar{\rho} : \vee^2 M/L_N \longrightarrow \vee^2(M/N)$ defined by
$\overline{\rho}(\sum_i \overline{r_i x_i \vee x_i'}) = \rho(\sum_i r_i x_i \vee x_i') = \sum_i r_i \overline{x_i} \vee \overline{x_i'}$ where $r_i \in R$ and $x_i, x_i' \in M$. Otherwise, the composite map

$$M \times M \xrightarrow{\vee} \vee^2 M \xrightarrow{\Phi} \vee^2 M/L_N$$

is a bilinear map where Φ is the natural map.
Just now, $\Phi \vee (x,y) \in L_N$ for every $x \in M$ and $y \in N$. Since $x \vee y - y \vee x = 0$, $x \vee y$ belongs to L_N

and then $\Phi\vee(x,y)\in L_N$.Hence, $\Phi\vee$ produces a bilinear map $\beta : M/N\times M/N \longrightarrow \vee^2 M/L_N$ defined by $\beta(\bar{x},\overline{x'}) = \overline{x\vee x'}$. Similarly, β produces an unique R-module homomorfizm γ : $\vee^2(M/N)\longrightarrow \vee^2 M/L_N$ such that the following digram

$$\begin{array}{ccc} M/N\times M/N & \xrightarrow{\beta} & \vee^2 M/L_N \\ \vee\searrow & & \nearrow\gamma \\ & \vee^2(M/N) & \end{array}$$

commutes.It is clear that $\bar{\rho}\gamma$ and $\gamma\bar{\rho}$ are identity maps as we desired.

Proposition 3.2. Let S denote an affine algebra represented by R/I
where $R = k[x_1,...x_s]$is a polynomial algebra and $I = (f_1,...f_m)$ ideal of R. Then the map φ : $\vee^2(F/N)\longrightarrow\vee^2 F/L_N$ defined by
$\varphi(\overline{d_1(x_i)}\vee\overline{d_1(x_j)}) = \overline{d_1(x_i)\vee d_1(x_j)}$ is an isomorphism of S-modules.
Here $\vee^2 F$ is a free S-module with basis $\{d_1(x_i)\vee d_1(x_j) : 1\leq i<j\leq s\}$
L_N is a submodule of $\vee^2 F$ generated by the set $\{d(f_k)\vee d(x_j) : k = 1,...,m,$
$j = 1,...,s\}$.

Proof This is obvious by proposition 3.1

Theorem 3.1. Let $S = k[x_1,...x_s]/(f)$ be an affine domain.Then $pd(\Omega_n(S))\leq 1$.

Proof The proof is similiar to [Erdogan, A.,Theorem 3.1.4,p.35]

Theorem 3.2. Let $S = k[x_1,...x_s]/(f)$ be an affine domain.Then projective dimension of $\vee^2(\Omega_1(S))$ is less than or equal to one.

Proof Let $R = k[x_1,...x_s]$ and $d_1 : R\longrightarrow\Omega_1(R)$ be the first order k-derivation of R. Assume that m is a maximal ideal of R containing f. Then the localisation of R at m is a regular algebra of dimension s. By lemma 2.6, $\vee^2(\Omega_1(R))$ is a free R-module. If $dimR = s$, then $rank\Omega_1(R) = \binom{1+s}{s} - 1 = \binom{s}{s-1}$
Let $\binom{s}{s-1} = t$. Then $rank\vee^2(\Omega_1(R)) = \binom{t+1}{t-1}$. Since m be the maximal ideal of R, $\vee^2(\Omega_1(R))\otimes R/m$ is a R/m- vector space of dimension $\binom{t+1}{t-1}$.
$\vee^2(\Omega_1(R))\otimes R/m\simeq\vee^2(\Omega_1(R)/m(\Omega_1(R))$ then $\vee^2(\Omega_1(R)/m(\Omega_1(R))$ is an R/m- vector space

of dimension $\binom{t+1}{t-1}$ if and only if $\Omega_1(R)/m\Omega_1(R)$ is a R/m- vector space of dimension t. S is a domain with dimension $s-1$. Hence $\Omega_1(R)/f\Omega_1(R)$ is a domain of dimension $s-1$.

Let L be the field fractions of $\Omega_1(R)/f\Omega_1(R)$, then $trdeg_L\Omega_1(R)/f\Omega_1(R) = s-1$, so, $dim_L\Omega_1(L) = dim_L\frac{\Omega_1(R)}{f\Omega_1(R)} \otimes_L L = \binom{n+s-1}{s-1} - 1 = \binom{1+s-1}{s-1} - 1 = \binom{s}{s-1} - 1 = \binom{s-1}{s-2}$ Since $\binom{s}{s-1} = t$, then $\binom{s-1}{s-2} = t-1$ so, dim$\vee^2(\Omega_1(L)) = \binom{t+1-1}{t-1-1} = \binom{t}{t-2}$

Now consider

$0 \longrightarrow ker\theta \longrightarrow \vee^2(\frac{\Omega_1(R)}{f\Omega_1(R)}) \stackrel{\theta}{\longrightarrow} \vee^2(\Omega_1(S)) \longrightarrow 0$

By tensoring with S_M,we have

$0 \longrightarrow (ker\theta)_M \longrightarrow \vee^2(\frac{\Omega_1(R)}{f\Omega_1(R)})_M \stackrel{\theta_M}{\longrightarrow} \vee^2(\Omega_1(S))_M \longrightarrow 0$ where M is the image of m in S. Observe that $\vee^2(\frac{\Omega_1(R)}{f\Omega_1(R)})_M$ is free of rank$\binom{t+1}{t-1}$, where $t = \binom{s}{s-1}$. So, we get

$dim_L(ker\theta_M \otimes_L L) = \binom{t+1}{t-1} - \binom{t}{t-2} = \binom{t}{t-1}$

on the other side, $ker\theta$ is generated by the set $\xi = \{d_1(f) \vee d_1(x_j) : j = 1,...s\}$ which has $\binom{t}{t-1}$ elements. Therefore the set ξ be a free basis of $(ker\theta)_M$, since m is arbitrary, we see that $ker\theta$ is locally free so, $ker\theta$ is a free S-module. By prop.2.3.5 in [6], $ker\theta$ is a projective S-module.Consequently, $pd \vee^2 (\Omega_1(S)) \leq 1$.

Example 3.1. $R = k[x,y,z]$ be the polynomial algebra with $y^2 = xz$. Then it can be calculated the projective dimension of $\Omega_1(S)$ and $\vee^2(\Omega_1(S))$.

F is a free S-module with basis $\{d_1(x), d_1(y), d_1(z)\}$ and N is a submodule of F generated by

$d_1(f) = d_1(y^2 - xz) = 2yd_1(y) - xd_1(z) - zd_1(x)$

By corollary 3.1.3 in [2], $\Omega_1(S) \simeq F/N$. Since rank $\Omega_1(S) = \binom{s-1}{s-2} = 2$

then, $rankN = rankF - rank\Omega_1(S) = 3-2 = 1$. By lemma 2.6.2 in [2],

N is a free S-module. Therefore,we get $0 \longrightarrow N \longrightarrow F \longrightarrow \Omega_1(S) \simeq F/N \longrightarrow 0$

a free resolution of $\Omega_1(S)$ and projective dimension of $\Omega_1(S)$ is less than or equal to one.Using these modules,by prop. 3.1 and prop. 3.2, $\vee^2(\Omega_1(S)) \simeq \vee^2F/L_N$ and $\vee^2F$ is a free module with basis

$\{d_1(x) \vee d_1(x), d_1(x) \vee d_1(y), d_1(x) \vee d_1(z), d_1(y) \vee d_1(y), d_1(y) \vee d_1(z), d_1(z) \vee d_1(z)\}$ and L_N is a submodule of $\vee^2F$ generated by

$d_(f) \vee d_1(x) = 2yd_1(y) \vee d_1(x) - xd_1(z) \vee d_1(x) - zd_1(x) \vee d_1(x)$

$d_1(f) \vee d_1(y) = 2yd_1(y) \vee d_1(y) - xd_1(z) \vee d_1(y) - zd_1(x) \vee d_1(y)$

$d_1(f)\ d_1(z) = 2yd_1(y) \vee d_1(z) - xd_1(z) \vee d_1(z) - zd_1(x) \vee d_1(z)$

since $rank \vee^2 (\Omega_1(S)) = \binom{t}{t-2} = \binom{3}{3-2} = 3$ where $t = \binom{s}{s-1} = \binom{3}{2} = 3$

we get $rankL_N = rank \vee^2 F - rank \vee^2 (\Omega_1(S) = 6 - 3 = 3$

so, L_N is a free S-module of rank 3.Therefore we have

$0 \longrightarrow L_N \longrightarrow \vee^2 F \longrightarrow \vee^2(\Omega_1(S)) \simeq \vee^2 F/L_N \longrightarrow 0$

a free resolution of $\vee^2(\Omega_1(S))$ and projective dimension of $\vee^2(\Omega_1(S))$ is less than or equal to one.

Conflict of Interests

The authors declare that there is no conflict of interests.

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