A NOTE ON ISBELL’S ZIGZAG THEOREM FOR COMMUTATIVE SEMIGROUPS

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Abstract. We have given a new and short proof of the Isbell’s Zigzag Theorem for the category of all commutative semigroups.

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1. Introduction

In [5], Howie and Isbell have extended Isbell’s Zigzag Theorem, by using free products of commutative semigroups, for the category of all commutative semigroups. Stenstrom [8], by using tensor product of monoids, provided a new proof of the celebrated Isbell’s Zigzag Theorem in the category of all semigroups. In this paper, we provide, based on Stenstrom’s approach, a new algebraic proof of the Howie and Isbell’s result [6, Theorem 1.1] for the category of all commutative semigroups.

2. Preliminaries

Let $U$ and $S$ be any semigroups with $U$ a subsemigroup of $S$ in a category $C$ of semigroups. We say that $U$ dominates an element $d$ of $S$ in $C$ if for every semigroup $T \in C$ and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of $S$ dominated by $U$ is called the dominion of $U$ in $S$, and we denote it by $\text{Dom}_C(U, S)$. It can be easily seen that $\text{Dom}_C(U, S)$ is a subsemigroup of $S$ containing $U$. A morphism $\alpha : S \to T$ in $C$ is said to be an epimorphism (epi for short) if for all morphims $\beta, \gamma$, $\alpha \beta = \alpha \gamma$ in $C$ implies $\beta = \gamma$ (where $\beta, \gamma$ are semigroup morphisms). It

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can be easily checked that $\alpha : S \to T$ is epi in C if and only if $i : S\alpha \to T$ is epi in C and the inclusion map $i : U \to S$ is epi in C if and only if $\text{Dom}_C(U, S) = S$. Note that when C is the category of all semigroups, the above definitions which have been first given by Hall and Jones [2] are precisely those of Howie and Isbell [5] and Isbell [6].

Let $S$ be a semigroup with identity 1 and $A$ be any non-empty set. Then $A$ is said to be a right $S$-system if there exists a mapping $(x, s) \mapsto xs$ from $A \times S$ into $A$ such that $(xs)t = x(st)$ for all $x \in A, s, t \in S$ and $x1 = x$ for all $x \in A$. Dually, we may define a left $S$-system $B$.

Let $A$ be a right $S$-system and $B$ be a left $S$-system and let $\tau$ be the equivalence relation on $A \times B$ generated by the relation $T = \{((as, b), (a, sb)) : a \in A, b \in B, s \in S\}$. Then $A \times B/\tau$ is called the tensor product of $A$ and $B$ over $S$ and is denoted by $A \otimes_S B$. We also denote an element $(a, b)\tau$ of $A \otimes_S B$ by $a \otimes b$.

For any unexplained notations and conventions, one may refer to Clifford and Preston [1] and Howie [4]. We shall also use the notation $\text{Dom}(U, S)$, when it is clear from the context, for the dominion of $U$ in $S$ both in the category of all semigroups as well as in the category of all commutative semigroups.

A most useful characterization of semigroup dominions is provided by Isbell’s Zigzag Theorem.

**Result 2.1** ([4, Theorem 8.3.4]). Let $U$ be a submonoid of a monoid $S$ and let $d \in S$. Then $d \in \text{Dom}(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of $d$ as follows:

\[ d = a_1s_1 = a_1t_1b_1 = a_2s_2b_1 = a_2t_2b_2 = \ldots = a_{n-1}t_{n-1}b_{n-1} = s_nb_{n-1}, \]

where $n \geq 1$, $s_i, t_i \in U$, $a_i, b_i \in S$ and

\[
\begin{align*}
    d &= a_1s_1, & & s_1 = t_1b_1 \\
    a_it_i &= a_{i+1}s_{i+1}, & & s_{i+1}b_i = t_{i+1}b_{i+1} \quad (i = 2, \ldots, n-2) \\
    a_{n-1}t_{n-1} &= s_n, & & s_nb_{n-1} = d.
\end{align*}
\]

Such a series of factorization is called a zigzag in $S$ over $U$ with value $d$, length $n$ and spine $s_1, \ldots, s_n, t_1, \ldots, t_{n-1}$. We refer to the equations in Result 1.1 as the zigzag equations.

**Result 2.2** ([4, Theorem 8.1.8]). Two elements $a \otimes b$ and $c \otimes d$ in $A \otimes_S B$ are equal if and only if $(a, b) = (c, d)$ or there exist $a_1, a_2, \ldots, a_{n-1}$ in $A, b_1, b_2, \ldots, b_{n-1}$ in $B, s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1}$ in $S$ such that

\[
\begin{align*}
    a &= a_1s_1, & & s_1b = t_1b_1 \\
    a_1t_1 &= a_2s_2, & & s_2b_1 = t_2b_2 \\
    a_it_i &= a_{i+1}s_{i+1}, & & s_{i+1}b_i = t_{i+1}b_{i+1} \quad (i = 2, \ldots, n-2) \\
    a_{n-1}t_{n-1} &= cs_n, & & s_nb_{n-1} = d.
\end{align*}
\]
3. Main Result

**Theorem 3.1.** Let \( U \) be a submonoid of a commutative monoid \( S \), Then \( d \) is in \( \text{Dom}(U,S) \) if and only if either \( d \in U \) or there exists a zigzag in \( S \) over \( U \) with value \( d \).

**Proof.** To prove the theorem, we, by Result 1.2, essentially show that if \( d \in S \), then \( d \in \text{Dom}(U,S) \) if and only if \( d \otimes 1 = 1 \otimes d \) in \( A = S \otimes_U S \), where 1 is the identity of \( S \). So let us suppose first that \( d \in S \) and \( d \otimes 1 = 1 \otimes d \) in \( A = S \otimes_U S \). Then, by Result 1.2, we have

\[
\begin{align*}
    d &= a_1 s_1, & s_1 &= t_1 b_1 \\
    a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\
    a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} & (i = 2, \ldots, n - 2) \\
    a_{n-1} t_{n-1} &= s_n, & s_n b_{n-1} &= d;
\end{align*}
\]

where \( a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S \) and \( s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U \).

Let \( T \) be a semigroup and let \( \alpha, \beta : S \to T \) be homomorphisms agreeing on \( U \); i.e.

\[
\alpha \mid U = \beta \mid U
\]

Now, by using zigzag equations, we have

\[
\begin{align*}
    \alpha(d) &= \alpha(a_1 s_1) = \alpha(a_1) \alpha(s_1) = \alpha(a_1) \beta(t_1 b_1) = \alpha(a_1) \beta(t_1) \beta(b_1) = \alpha(a_1 t_1) \beta(b_1) \\
    &= \cdots = \alpha(a_{n-1} t_{n-1}) \beta(b_{n-1}) = \alpha(s_n) \beta(b_{n-1}) = \beta(s_n b_{n-1}) = \beta(d) \\
    \Rightarrow d &\in \text{Dom}(U,S).
\end{align*}
\]

To prove the converse, we first show that for a commutative monoid, the equivalence relation \( \tau \) is a congruence; i.e.

\[
(a, b) \tau (c, d) \tau = (ac, bd) \tau.
\]

For this, we have to show that \( \tau \) is compatible; i.e.

if \( (a, b) \tau = (c, d) \tau \) and \( (a', b') \tau = (c', d') \tau \), then \( ((a, b)(a', b')) \tau = ((c, d)(c', d')) \tau \).

Since \( a \otimes b = c \otimes d \), by Result 1.2, we have

\[
\begin{align*}
    a &= a_1 s_1, & s_1 b &= t_1 b_1 \\
    a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\
    a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} & (i = 2, \ldots, n - 2) \\
    a_{n-1} t_{n-1} &= c s_n, & s_n b_{n-1} &= d; & (A)
\end{align*}
\]

for some \( a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S \) and \( s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U \).
Similarly, as \( a' \otimes b' = c' \otimes d' \), we have

\[
\begin{align*}
    a' &= a_1' s_1', \\
    a_1't'_1 &= a_2's_2', \\
    a_1't'_2 &= a_1's_{i+1}', \\
    a_{n-1}'t_{n-1}' &= c's_n', \\
    s_1'b' &= t_1'b_1', \\
    s_2'b_1' &= t_2'b_2', \\
    s_{i+1}'b_i' &= t_{i+1}'b_{i+1}' & (i = 2, \ldots, n - 2) \\
    s_n'b_{n-1}' &= d', \\
\end{align*}
\]

for some \( a_1', a_2', \ldots, a_{n-1}'b_1', b_1', b_2', \ldots, b_{n-1}' \in S \) and \( s_1', s_2', \ldots, s_n', t_1', t_2', \ldots, t_{n-1}' \in U \).

Now, from equations (A) and (B), we have

\[
\begin{align*}
    aa' &= (a_1s_1)(a_1's_1'), \\
    (a_1t_1)(a_1't_1') &= (a_2s_2)(a_2's_2'), \\
    (a_1t_{i+1})(a_1's_{i+1}') &= (a_{i+1}s_{i+1})(a_{i+1}'s_{i+1}'), \\
    (a_{n-1}t_{n-1})(a_{n-1}'t_{n-1}') &= (cs_n)(c's_n'),
\end{align*}
\]

Since, in the above system of equalities all members belong to \( S \), so, by using commutativity of \( S \), we have

\[
\begin{align*}
    aa' &= (a_1a_1')(s_1s_1'), \\
    (a_1a_1')(t_1t_1') &= (a_2a_2')(s_2s_2'), \\
    (a_1a_{i+1})(t_{i+1}t_{i+1}') &= (a_{i+1}a_{i+1}')(s_{i+1}s_{i+1}'), \\
    (a_{n-1}a_{n-1}')(t_{n-1}t_{n-1}') &= (cs_n)(cs_n'),
\end{align*}
\]

where \( a_1a_1', a_2a_2', \ldots, a_{n-1}a_{n-1}', b_1b_1', b_2b_2', \ldots, b_{n-1}b_{n-1}' \in S \) and \( s_1s_1', s_2s_2', \ldots, s_n's_n', t_1t_1', t_2t_2', \ldots, t_{n-1}t_{n-1}' \in U \).

Thus, by Result 1.2, we have

\[
aa' \otimes bb' = cc' \otimes dd' \Rightarrow (aa', bb')\tau = (cc', dd')\tau \Rightarrow ((a, b)(a', b'))\tau = ((c, d)(c', d'))\tau \Rightarrow \tau \text{ is a congruence}.
\]

Now define \( \alpha : S \to S \times A \) and \( \beta : S \to S \times A \) by

\[
\alpha(s) = (s, s \otimes 1), \quad \beta(s) = (s, 1 \otimes s).
\]

Then \( \alpha, \beta \) are, clearly, semigroup morphisms.
Since, \( u \otimes 1 = 1 \otimes u \), we have

\[
\alpha(u) = \beta(u), \text{ for all } u \in U.
\]

Therefore \( \alpha(d) = \beta(d) \)

\[
\Rightarrow (d, d \otimes 1) = (d, 1 \otimes d)
\]

\[
\Rightarrow d \otimes 1 = 1 \otimes d.
\]

This completes the proof of the theorem.

Thus we have the following:

**Theorem 3.2.** If \( U \) is a submonoid of a commutative monoid \( S \), then \( d \) is in \( \text{Dom}(U,S) \) if and only if either \( d \in U \) or there exists a zigzag in \( S \) over \( U \) with value \( d \).

It may easily be verified that the arguments employed by Howie[4] in proving Theorems 8.3.4 to 8.3.5 work through to complete the proof for the following Isbell’s Zigzag Theorem for the category of all commutative semigroups.

**Theorem 3.3.** Let \( U \) be a subsemigroup of a commutative semigroup \( S \). Then \( d \in \text{Dom}(U,S) \) if and only if either \( d \in U \) or there exists a zigzag in \( S \) over \( U \) with value \( d \).

References


