ON \( \chi \)-INJECTIVE MODULES

SAUGATA PURKAYASTHA*, HELEN K. SAIKIA

Department of Mathematics, Gauhati University, Guwahati-781014, India

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Abstract. In this paper, we introduce the notion of \( \chi \)-injective modules where \( \chi \) denotes a collection of right ideals of a ring \( R \). We establish various important properties of this module.

Keywords: \( \chi \)-injective module, essential ideal, direct summand, divisible module.

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1. Introduction

The notion of injective modules was first introduced by Baer in 1940 in [2] in the form of divisible abelian groups. A right \( R \)-module \( M \) is said to be injective if it satisfies Baer’s criteria of injectivity: every homomorphism from any right ideal \( I \) of \( R \) to \( M \) can be extended to whole of \( R \). Since then many researchers have embarked on to determine a class of ideals of a ring \( R \) such that an \( R \)-module \( M \) is injective if and only if it satisfies Baer’s criteria of injectivity for such a class. For instance, Smith [11] showed that if \( R \) is a commutative Noetherian ring, then the collection of all prime ideals of \( R \) is such a class. Later on, Vamos [12] termed such a class

*Corresponding author

E-mail address: sau.pur@rediffmail.email

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as a test set for injectivity of a module. Beachy et. al. [3] finally showed that for a piecewise Noetherian ring a set of prime ideals is a test set if and only if it contains all essential prime ideals. In the same spirit, in our present work, we introduce the notion of $\chi$-injective module. Let $R$ be a ring and $\chi$ be a collection of right ideals of $R$. A right $R$ module $M$ is said to be $\chi$-injective if for every ideal $I \in \chi$, every homomorphism $f : I \to M$ can be extended to whole of $R$. Unlike the authors listed above, we study the properties of such a module rather than emphasizing on the collection $\chi$.

We have also related various other notions like pure-exact sequence, multiplication module with the notion of $\chi$-injective module in [8] and [9].

2. Preliminaries

Definition 1.1. An essential (large) submodule of a module $B$ is any submodule $A$ which has non-zero intersection with every non-zero submodule of $B$. We write $A \leq_e B$ to denote the situation. Moreover we say that $B$ is an essential extension of $A$.

Definition 1.2. A ring $R$ is said to be Baer if the left annihilator of any subset of $R$ is generated as a left ideal by an idempotent of $R$.

Definition 1.3. For a ring $R$ a right $R$ module $M$ is called semisimple (or completely reducible) if it is a direct sum of simple modules. Thus, a ring $R$ is said to be left (right) semisimple if it is semisimple as a left (right) $R$ module.

Definition 1.4. A short exact sequence is an exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if $\exists$ a homomorphism $j : C \to B$ with $gj = 1_C$.

Definition 1.5. A ring $R$ is said to be von-Neumann regular if for each $r \in R$, $\exists$ $r' \in R$ with $rr' r = r$.

Definition 1.6. A right $R$ module $P$ is said to be projective if whenever $p$ is a surjective homomorphism from $A$ to $B$ and $h$ is any homomorphism from $P$ to $B$, there exists another homomorphism $g$ from $P$ to $A$ such that $pg = h$. 
Definition 1.7. An $R$ module $M$ is said to be divisible if for any $u \in M$ and $a \in R$ such that $\text{ann}_r(a) \subseteq \text{ann}(u)$, $u$ is divisible by $a$, i.e. $\exists v \in M$ such that $u = va$, where $\text{ann}_r(a)$ denotes the right annihilator of the element $a$.

Any other terminology or result relevant to the present work can be found in [4], [5],[6], [7] and [10].

3. Main results

Definition 3.1. Let $M$ be a right $R$ module and $\chi$ be a collection of right ideals of $R$. Then $M$ is said to be $\chi$-injective if every homomorphism $f : I \to M, I \in \chi$ can be extended to whole of $R$.

Example 3.1. Let $M$ be a right $R$ module where $R$ is a commutative Noetherian ring. If we let $\chi$ to be collection of all prime ideals of $R$, then by [11] it follows that $M$ is $\chi$-injective.

Theorem 3.1. Let $M$ be a right $R$ module and $\chi$ be a collection of right ideals of $R$. Then the following are equivalent:

1) $M$ is $\chi$-injective.

2) for any $I \in \chi$ and for every homomorphism $f : I \to M$, there exists $m \in M$ such that $f(a) = ma$.

Proof. (1) $\implies$ (2) Let $i$ be a natural embedding from $I$ to $R$ and $f : I \to M$ be any homomorphism such that there exists another homomorphism $\varphi : R \to M$ such that $f = \varphi i$. As $f, \varphi$ are module homomorphisms, for $a \in I$, we have,

$$f(a) = \varphi(a)$$

$$= \varphi(1 \cdot a)$$

$$= \varphi(1)a$$

$$= ma$$

where $\varphi(1) = m$ for $m \in M$. Thus, (2) follows.

(2) $\implies$ (1). Let for a right ideal $I \in \chi$ and for a homomorphism $f : I \to M$, there exists $m \in M$
such that \( f(a) = ma \). If we define \( \varphi : R \rightarrow M \) by \( \varphi(a) = ma \) for \( a \in R \), then clearly \( \varphi \) is a module homomorphism and \( \varphi_I = f \). This shows that \( M \) is \( \chi \)-injective.

From the definition of \( \chi \)-injective modules, it is clear that an injective module is \( \chi \)-injective. However a \( \chi \)-injective module need not be injective. We consider the following example.

**Example 3.1.** We recall from [1] that a right ideal of a ring \( R \) is said to be pure if and only if for every \( x \in I \), \( \exists y \in I \) such that \( x = xy \). We consider the ring of integers \( \mathbb{Z} \) as a module over itself. If \( \chi \) denotes the collection of all non-zero proper pure ideals, then \( \mathbb{Z} \) as a module over itself is \( \chi \)-injective. Infact \( \mathbb{Z} \) does not possess any non-zero proper pure ideal in this case as \( \mathbb{Z} \) is free from non-zero one sided zero divisors. But \( \mathbb{Z}_{\mathbb{Z}} \) is not injective.

We now establish a condition under which a \( \chi \)-injective module is injective.

**Theorem 3.2.** Let \( Q \) be a \( \chi \)-injective module, where \( \chi \) is the collection of all essential right ideals of \( R \). Let \( M, N \) be right \( R \) modules. Then \( Q \) is injective if \( M \leq_e N \) and any homomorphism \( \varphi : M \rightarrow Q \) can be extended to \( N \).

**Proof.** Let a module \( Q \) be \( \chi \)-injective, and let us consider the following diagram, where \( M \leq_e N \)

\[
\begin{array}{ccc}
O & \rightarrow & M \\
\downarrow & & \downarrow & \phi \\
M & \rightarrow & N \\
\end{array}
\]

We now consider a set \( \kappa \) of extensions, i.e. the set of all pairs \((C, h)\) where \( M \leq C \leq N \) and \( h : C \rightarrow Q \) such that \( h |_{C} = \varphi \). Then clearly \( \kappa \neq \emptyset \) as \((M, \varphi) \in \kappa \). We now introduce an ordering relation by setting \((C_1, h_1) \leq (C_2, h_2)\) if and only if \( C_1 \subseteq C_2 \) and \( h_2 \) extends \( h_1 \). This can be easily verified to be a partial ordering on \( \kappa \). Every non-empty increasing chain \( \{ (C_i, h_i) | i \in I \} \) in \( \kappa \) has a upper bound \((C', h')\), where \( C' = \bigcup_{i \in I} C_i \) and \( h'|_{C'} = h_i \). Thus, in view of Zorn’ lemma, \( \exists \) a maximal element \((C^*, h^*)\) in \( \kappa \). By construction, \( M \leq C^* \leq N \). We need to show \( C^* = N \) i.e. \( N \subseteq C^* \)

Suppose \( \exists \) a non-zero \( b \in N \) such that \( b \notin C^* \). We set \( I = \{ a \in R : ba \in C^* \} \). Then \( I \) is an essential right ideal of \( R \) and hence \( I \in \chi \). Thus \( \exists \) a homomorphism \( f : I \rightarrow R \) defined by \( f(a) = h^*(ba) \). By assumption, \( \exists q \in Q \) such that \( f(a) = qa = h^*(ba) \) (as \( Q \) is \( \chi \)-injective) \( \forall a \in I \). Then we can define a homomorphism \( g : C^* + bR \rightarrow Q \) by setting \( g(c + ba) = h^*(c) + qa \) \( \forall c \in C^* \) and \( a \in R \). It extends to a homomorphism \( h^* \) and is well defined. For suppose,
\( c_1 + ba_1 = c_2 + ba_2 \) for \( c_1, c_2 \in C^* \) and \( a_1, a_2 \in R \). Then \( a_1 - a_2 \in I \) and hence \( f(a_1 - a_2) = f(a_1) - f(a_2) = qa_1 - qa_2 \). On the other hand \( f(a_1) - f(a_2) = h^*(ba_1) - h^*(ba_2) = h^*(ba_1 - ba_2) = h^*(c_2 - c_1) = h^*(c_2) - h^*(c_1) \). Hence we have \( h^*(c_2) - h^*(c_1) = qa_1 - qa_2 \). Thus \( g(c_1 + ba_1) = h^*(c_1) + qa_1 = h^*(c_2) + qa_2 = g(c_2 + ba_2) \), as required i.e. the function is well-defined.

Thus we have \( (C^*, h^*) \leq (C^* + bR, g) \) i.e. we have obtained a contradiction regarding the maximality of \( (C^*, h^*) \). This completes the proof.

**Remark 3.1.** At this point, we note that in Theorem 3.2, the condition of essentiality is a sufficient condition for a \( \chi \)-injective module to be injective. However, the condition is not necessary. For instance, let us consider the \( \mathbb{Z}/ < 2 > \) as a module over \( \mathbb{Z}/ < 6 > \). \( \mathbb{Z}/ < 6 > \) has two non-trivial ideals, \( < 3 > = \{ 0, 3 \} \simeq \mathbb{Z}/ < 2 > \) and \( < 2 > = \{ 0, 2, 4 \} \simeq \mathbb{Z}/ < 3 > \). Since there is no non-zero homomorphism from \( \mathbb{Z}/ < 3 > \) to \( \mathbb{Z}/ < 2 > \) so the only ideal at stake is \( \mathbb{Z}/ < 2 > \).

The homomorphism \( f \) from \( \mathbb{Z}/ < 2 > \) to itself is determined by \( f(1) \). Also, the inclusion map \( i : \mathbb{Z}/ < 2 > \to \mathbb{Z}/ < 6 > \) can be defined as \( f(1) = 3 \). Thus if \( \tilde{f} : \mathbb{Z}/ < 6 > \to \mathbb{Z}/ < 2 > \) then we have \( \tilde{f} \circ i(1) = \tilde{f}(3) = 3 \tilde{f}(1) = \tilde{f}(1) \). Thus if we define \( \tilde{f}(1) = f(1) \) then \( \tilde{f} \) is an extension of \( f \). Consequently \( \mathbb{Z}/ < 2 > \) is injective over \( \mathbb{Z}/ < 6 > \) but none of \( < 3 > \) or \( < 2 > \) is essential in \( \mathbb{Z}/ < 6 > \).

**Theorem 3.3.** Let \( R \) be a semisimple ring. Let \( M \) be a \( \chi \)-injective right \( R \) module. Then the following hold:

1. any submodule \( K \) of \( M \) is \( \chi \)-injective.
2. the homomorphic image of \( M \) is \( \chi \)-injective module.
3. the quotient of \( M \) is \( \chi \)-injective module.

**Proof.** (1) Since \( R \) is semisimple, we have \( K, M \) and \( M/K \) all are projective. So, the short exact sequence \( 0 \to K \to M \to M/K \to 0 \) splits. Thus, if \( i : K \to M \) be the inclusion, then there exists a homomorphism \( k : M \to K \) such that \( ki = id_K \). Let \( I \in \chi \) and \( f : I \to K \) a homomorphism. Then the composite \( if : I \to M \) extends to a homomorphism \( g : R \to M \) as \( M \) is \( \chi \)-injective. If we take \( h : R \to K \) to be the composite \( kg \), then the restriction of \( h \) on \( I \) is equal to the composite \( k \) (restriction of \( g \) on \( I \)) = \( kif = f \). So. \( h \) is an extension of \( f \). This proves that \( K \) is \( \chi \)-injective.

(2) Let \( M' \) be the homomorphic image of \( M \) and we consider the following diagram
Using the $\chi$-injectivity of $M$, we get

$$gi = \bar{f} \quad (1)$$

Again $R$ being semisimple and $I$ as a module over $R$ is projective. Thus, we have

$$h\bar{f} = f \quad (2)$$

By similar arguments, we have

$$hg = \bar{g} \quad (3)$$

Then, (1) gives;

$$(gi) = h\bar{f}$$

$$(hg)i = h\bar{f}$$

$$\bar{g}i = f \text{ (using (2) and (3))}$$

as required.

(3) Let $M$ be $\chi$-injective and $K$ be a submodule of $M$. Then to show that $M' = \frac{M}{K}$ is also $\chi$-injective. We consider the following diagram
where $\pi$ denotes the natural homomorphism. Using the fact that,

$$gi = \bar{f}$$

and that the homomorphic image of a $\chi$-injective module is $\chi$-injective, we get the desired result.

**Theorem 3.4.** The direct sum of two $\chi$-injective modules is again $\chi$-injective.

**Proof.** Let $M_1$ and $M_2$ be $\chi$-injective modules. Then to show that $M_1 \oplus M_2$ is $\chi$-injective. Since $M_1$ and $M_2$ are $\chi$-injective, so given $I \in \chi$ and homomorphisms $f_1 : I \to M_1$ and $f_2 : I \to M_2$, we have extensions $g_1 : R \to M_1$ and $g_2 : R \to M_2$ respectively. Again any homomorphism $f : I \to M_1 \oplus M_2$ can be written as $f = (f_1, f_2)$, where $f_1 = p_1 f$ and $f_2 = p_2 f$, $p_1$ and $p_2$ being the projections of $M_1$ and $M_2$ to $M_1 \oplus M_2$ respectively. Now if we take $(g_1, g_2) : R \to M_1 \oplus M_2$, then $(g_1, g_2)$ extends $f = (f_1, f_2)$. Consequently, $M_1 \oplus M_2$ is $\chi$-injective.

**Theorem 3.5.** Let $R$ be a right Noetherian Von-Neumann regular ring. Then a right $R$ module $I$ is $\chi$-injective if and only if it is divisible, $\chi$ being the collection of all right ideals of $R$ of the type $\{aR : a \in R\}$.

**Proof.** Let $I$ be divisible. Let $f : aR \to I$ be a homomorphism where $aR \in \chi$. Let

$$u = f(a) \in I$$

. Then by definition

$$x \in \text{ann}_r(a)$$

$$ax = 0$$

$$f(ax) = 0$$

$$f(a)x = 0$$

$$ux = 0$$

$$x \in \text{ann}(u)$$
Then, by definition \( u = va \) for some \( v \in I \) Then, if we take \( g : R_R \to I \) defined by \( g(1) = v \)

Then;

\[
g(1)a = va
\]

\[
g(a) = va
\]

\[
= u
\]

\[
= f(a), \forall a \in aR
\]

i.e.

\[
g|_{aR} = f
\]

or that \( f \) extends \( R_R \). Consequently \( I \) is \( \chi \)-injective.

Conversely, let \( I \) be \( \chi \)-injective. Then any homomorphism \( f : aR \to I, aR \in \chi \) extends to \( R_R \).

We now show that \( I \) is divisible, i.e. \( \text{ann}^l(\text{ann}_r(a)) = Ia \), where \( \text{ann}^l(\text{ann}_r(a)) \) denotes the annihilator of \( \text{ann}_r(a) \) taken in \( I \). We first show that

\[
Ia \subseteq \text{ann}^l(\text{ann}_r(a))
\]

Let \( x \in Ia \). Then \( x = ra \) for some \( r \in I \). Then, we have,

\[
\text{ann}_r(a) \cdot a = 0
\]

\[
r \cdot \text{ann}_r(a) \cdot a = 0
\]

\[
\text{ann}_r(a) \cdot ra = 0
\]

\[
\text{ann}_r(a) \cdot x = 0
\]

\[
x \in \text{ann}(\text{ann}_r(a))
\]

Since \( x \in I \), we have

\[
x \in \text{ann}^l(\text{ann}_r(a))
\]

i.e.

\[
Ia \subseteq \text{ann}^l(\text{ann}_r(a)).
\]

Now to show that
\[ \text{ann}^l(\text{ann}_r(a)) \subseteq Ia \]

Let

\[ x \in \text{ann}^l(\text{ann}_r(a)). \]

As \( I \) is \( \chi \)-injective, the homomorphism \( f : aR \to I \) extends \( g : R_R \to I \). Then \( f(aR) = xR \) is a well-defined homomorphism as for

\[ ar = as \]
\[ a(r - s) = 0 \]
\[ r - s \in \text{ann}_r(a) \]
\[ x(r - s) = 0 \]
\[ xr = xs \]

Again,

\[ x = f(a) = g(a) = g(1)a = va \]

for some \( g(1) = v \) Thus

\[ \text{ann}^l(\text{ann}_r(a)) \subseteq Ia \]

consequently,

\[ \text{ann}^l(\text{ann}_r(a)) = Ia \]

or that, \( I \) is divisible.

**Corollary 3.1** Over a Baer ring \( R \), a right \( R \) module is \( \chi \)-injective if and only if it is divisible.

**Proof.** We need to establish that a right Noetherian von Neumann regular is Baer, since in that case the result will follow from proposition 5. If \( R \) is right Noetherian, it follows that the ideals of \( R \) are finitely generated [6]. Also if \( R \) is von-Neumann regular, every finitely generated ideal is principal and is generated by an idempotent [10]. Thus, we may conclude that \( R \) is Baer.

**Conflict of Interests**

The authors declare that there is no conflict of interests.
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