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NORMS OVER INTUITIONISTIC FUZZY VECTOR SPACES

RASUL RASULI*

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran

Abstract. In this paper, we have introduced and discussed of intuitionistic fuzzy vector subspaces under norms (a t-norm T and a t-conorm C). This work is an attempt to study purely algebraic properties of them.

Keywords: vector spaces; fuzzy set theory; intuitionistic mathematics; intuitionistic fuzzy set; norms.

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1. Introduction

Vector spaces are the subject of linear algebra and are well characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space. Infinite-dimensional vector spaces arise naturally in mathematical analysis, as function spaces, whose vectors are functions. These vector spaces are generally endowed with additional structure, which may be a topology, allowing the consideration of issues of proximity and continuity. Among these topologies, those that are defined by a norm or inner product are more commonly used, as having a notion of distance between two vectors. This is particularly the case of Banach spaces and Hilbert spaces, which are fundamental in mathematical analysis. Historically, the first ideas leading to vector spaces can be traced back as far as the 17th century's analytic

*Corresponding author

E-mail address: rasulirasul@yahoo.com

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geometry, matrices, systems of linear equations, and Euclidean vectors. The modern, more abstract treatment, first formulated by Giuseppe Peano in 1888, encompasses more general objects than Euclidean space, but much of the theory can be seen as an extension of classical geometric ideas like lines, planes and their higher-dimensional analogs. Today, vector spaces are applied throughout mathematics, science and engineering. They are the appropriate linear-algebraic notion to deal with systems of linear equations. They offer a framework for Fourier expansion, which is employed in image compression routines, and they provide an environment that can be used for solution techniques for partial differential equations. Furthermore, vector spaces furnish an abstract, coordinate-free way of dealing with geometrical and physical objects such as tensors. This in turn allows the examination of local properties of manifolds by linearization techniques. Vector spaces may be generalized in several ways, leading to more advanced notions in geometry and abstract algebra. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$. Fuzzy sets generalize classical sets, since the indicator functions (aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In mathematics, fuzzy sets (aka uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Lotfi A. Zadeh [24]. Intuitionistic fuzzy sets are sets whose elements have degrees of membership and non-membership. Intuitionistic fuzzy sets have been introduced by Krassimir Atanassov (1983) as an extension of Lotfi Zadeh's notion of fuzzy set, which itself extends the classical notion of a set. The triangular norm (T-norm) and the triangular conorm (C-conorm) originated from the studies of probabilistic metric spaces [5, 23] in which triangular inequalities were extended using the theory of T-norm and C-conorm. The author by using norms, investigated some properties of fuzzy algebraic structures [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. In this paper, we define intuitionistic fuzzy vector subspaces with respect to norms (a t-norm T and a t-conorm C). Next we show relainship between them and vector subspaces. Late, we characterize them under some conditions. Finally, we investigate them under increasing map and decreasing map.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal.

Definition 2.1. (See [4]) A vector space or a linear space consists of the following:
(1) a field $\mathbb{F}$ of scalars.
(2) a set V of objects called vectors.
(3) a rule (or operation) called vector addition; which associates with each pair of vectors $\alpha, \beta \in V$; $\alpha + \beta \in V$, called the sum of α and β in such a way that
(a) addition is commutative $\alpha + \beta = \beta + \alpha$,
(b) addition is associative $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
(c) there is a unique vector 0 in V, called the zero vector, such that $\alpha + 0 =$ for all $\alpha \in V$,
(d) for each vector α in V there is a unique vector$(-\alpha)$ in V such that $\alpha + (-\alpha) = 0$,
(e) a rule (or operation), called scalar multiplication, which associates with each scalar c in $\mathbb{F}$ and a vector α in V, a vector $c \bullet \alpha$ in V, called the product of c and α, in such a way that $1 \bullet \alpha = \alpha$, $(c_1 \bullet c_2) \bullet \alpha = c_1 \bullet (c_2 \bullet \alpha)$, $c \bullet (\alpha + \beta) = c \bullet \alpha + c \bullet \beta$, $(c_1 + c_2) \bullet \alpha = (c_1 \bullet \alpha) + (c_2 \bullet \alpha)$ for $\alpha, \beta \in V$ and $c, c_1, c_2 \in F$. It is important to note as the definition states that a vector space is a composite object consisting of a field, a set of vectors and two operations with certain special properties. The same set of vectors may be part of a number of distinct vectors. We simply by default of notation just say V a vector space over the field $\mathbb{F}$ and call elements of V as vectors only as matter of convenience for the vectors in V.

Throughout this section, $\mathbb{F}$ is any field of characteristic zero.

Example 2.2. Let $V = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then V is a vector space over $\mathbb{R}$ or $\mathbb{Q}$ but V is not a vector space over the complex field $\mathbb{C}$.

Definition 2.3. (See [4]) Let V be a vector space over the field $\mathbb{F}$. A vector subspace of V is a subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on V.

We have the following nice characterization theorem for subspaces.

Theorem 2.4. *(See [4]) Let W be a non-empty subset of a vector V over the field $\mathbb{F}$. Then W is a vector subspace of V if and only if for each pair $\alpha, \beta \in W$ and each scalar $c \in \mathbb{F}$ the vector $c\alpha + \beta \in W$.*

Example 2.5. Let $M_{n\times n} = \{(a_{ij}) \mid a_{ij} \in \mathbb{Q}\}$ be the vector space over $\mathbb{Q}$. Let $D_{n\times n} = \{(a_{ii}) \mid a_{ii} \in \mathbb{Q}\}$ be the set of all diagonal matrices with entries from $\mathbb{Q}$. Then $D_{n\times n}$ is a vector subspace of $M_{n\times n}$.

Definition 2.6. (See [4]) Let V and W be two vector spaces over the field of $\mathbb{F}$. A map $f : V \to W$ is called a linear transformation if $f(ax+y) = af(x) + f(y)$ for all $x, y \in V$ and $a \in \mathbb{F}$.

Definition 2.7. (See [3]) Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$. Denote by $[0,1]^X$, the set of all fuzzy subset of X.

Definition 2.8. (See [2]) For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 2.9. (See [2]) Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$, respectively.
We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.10. (See [2]) Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then
(1) $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Definition 2.11. (See [3]) A t-norm T is a function $T:[0,1]\times[0,1]\to[0,1]$ having the following four properties:

(T1) $T(x,1)=x$ (neutral element)

(T2) $T(x,y)\leq T(x,z)$ if $y\leq z$ (monotonicity)

(T3) $T(x,y)=T(y,x)$ (commutativity)

(T4) $T(x,T(y,z))=T(T(x,y),z)$ (associativity),

for all $x,y,z\in[0,1]$.

Recall that t-norm T is idempotent if for all $x\in[0,1]$, we have that $T(x,x)=x$.

Corollary 2.12. *Let T be a t-norm. Then for all $x\in[0,1]$*

(1) T(x,0)=0.

(2) $T(0,0)=0$.

Example 2.13. (1) Standard intersection t-norm $T_m(x,y)=\min\{x,y\}$.

(2) Bounded sum t-norm $T_b(x,y)=\max\{0,x+y-1\}$.

(3) algebraic product t-norm $T_p(x,y)=xy$.

(4) Drastic t-norm

$$T_D(x,y)=\begin{cases} y & \text{if } x=1\\ x & \text{if } y=1\\ 0 & \text{otherwise.}\end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y)=\begin{cases} \min\{x,y\} & \text{if } x+y>1\\ 0 & \text{otherwise.}\end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y)=\begin{cases} 0 & \text{if } x=y=0\\ \frac{xy}{x+y-xy} & \text{otherwise.}\end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y)\leq T(x,y)\leq T_{\min}(x,y)$ for all $x,y\in[0,1]$.

Definition 2.14. (See [3]) A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(C1) $C(x,0) = x$

(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$

(C3) $C(x,y) = C(y,x)$

(C4) $C(x,C(y,z)) = C(C(x,y),z)$,

for all $x,y,z \in [0,1]$.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, we have that $C(x,x) = x$.

Corollary 2.15. *Let C be a t-conorm. Then for all $x \in [0,1]$*

(1) $C(x,1) = 1$.

(2) $C(0,0) = 0$.

Example 2.16. (1) Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.

(2) Bounded sum t-conorm $C_b(x,y) = \min\{1,x+y\}$.

(3) Algebraic sum t-conorm $C_p(x,y) = x+y-xy$.

(4) Drastic t-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise.} \end{cases}$$

(5) Nilpotent maximum t-conorm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x+y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$ for any t-conorm C and all $x,y \in [0,1]$.

Lemma 2.17. *(See [1]) Let T be a t-norm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

Lemma 2.18. *(See [1]) Let* C *be a* t*-conorm. Then*

$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

3. Norms over intuitionistic fuzzy vector spases

In what follows, V is a vector space on a field $\mathbb{F}$, unless otherwise specified.

Definition 3.1. Let $A = (\mu_A, \nu_A)$ be $IFSs$ in V. Then $A = (\mu_A, \nu_A)$ is said to be intuitionistic fuzzy vector subspaces with respect to norms(a t-norm T and a t-conorm C) (in short, $IFTC(V)$) of V if

(1) $\mu_A(x+y) \geq T(\mu_A(x), \mu_A(y))$,

(2) $\mu_A(-x) \geq \mu_A(x)$,

(3) $\mu_A(ax) \geq \mu_A(x)$,

(4) $\nu_A(x+y) \leq C(\nu_A(x), \nu_A(y))$,

(5) $\nu_A(-x) \leq \nu_A(x)$,

(6) $\nu_A(ax) \leq \nu_A(x)$,

for all $x, y \in V$ and $a \in \mathbb{F}$.

Corollary 3.2. *Let* $A = (\mu_A, \nu_A) \in IFTC(V)$. *Then* $A(-x) = \mu(x)$ *for all* $x \in V$.

Proof. Let $x \in V$. Then

$$\mu_A(-x) \geq \mu_A(x) = \mu_A(-(-x)) \geq \mu_A(-x)$$

and so $\mu_A(-x) = \mu_A(x)$.

Also

$$\nu_A(-x) \leq \nu_A(x) = \nu_A(-(-x)) \leq \nu_A(-x)$$

and so $\nu_A(-x) = \nu_A(x)$.

Thus $A(-x) = (\mu_A(-x), \nu_A(-x)) = (\mu_A(x), \nu_A(x)) = A(x)$. □

Example 3.3. Let $V = \mathbb{R} \times \mathbb{R}$ be a vectorspace over a field $\mathbb{F} = \mathbb{R}$. Define $\mu_A : \mathbb{R} \times \mathbb{R} \to [0,1]$ as

$$\mu_A(x,y) = \begin{cases} 0.75 & (x,y) \in \{(0,y) \mid y \in \mathbb{R}\} \\ 0.25 & \text{otherwise} \end{cases}$$

and

$\nu_A : \mathbb{R} \times \mathbb{R} \to [0,1]$ as

$$\nu_A(x,y) = \begin{cases} 0.15 & (x,y) \in \{(x,0) \mid x \in \mathbb{R}\} \\ 0.55 & \text{otherwise.} \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ and $C(a,b) = C_p(a,b) = a+b-ab$ for all $a,b \in [0,1]$. Then $A = (\mu_A, \nu_A) \in IFTC(V)$.

Theorem 3.4. *Let V be a subspace over field $\mathbb{F}$ and W be a subspace of V and $\mu_A, \nu_A : W \to \{0,1\}$ be the characteristic functions. Then $A = (\mu_A, \nu_A) \in IFTC(V)$.*

Proof. Let $x, y \in V$ and we investigate the following conditions:

(1) If $x, y \in W$, then $x + y \in W$ and we have

$$\mu_A(x+y) = 1 \geq 1 = T(1,1) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x+y) = 1 \leq 1 = C(1,1) = C(\nu_A(x), \nu_A(y)).$$

(2) If $x \notin W$ and $y \in W$, then $x + y \notin W$ and then

$$\mu_A(x+y) = 0 \geq 0 = T(0,1) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x+y) = 0 \leq 0 = C(0, \nu_A(y)) = C(\nu_A(x), \nu_A(y)).$$

(3) Finally, if $x, y \notin W$, then

$$\mu_A(x+y) \geq 0 = T(0,0) = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x+y) = 0 \leq 0 = C(0,0) = C(\nu_A(x), \nu_A(y)).$$

Thus from (1)-(3) we have that

$$\mu_A(x+y) \geq T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x+y) \leq C(\nu_A(x), \nu_A(y)).$$

Also we have that the following conditions:

(1) If $x \in W$, then $-x \in W$ and then

$$\mu_A(-x) = 1 \geq 1 = \mu_A(x)$$

and

$$\nu_A(-x) = 0 \leq 0 = \nu_A(x).$$

(2) If $x \notin W$, then $-x \notin W$ and then

$$\mu_A(-x) = 0 \geq 0 = \mu_A(x)$$

and

$$\nu_A(-x) = 0 \leq 0 = \nu_A(x).$$

Thus from (1)-(2) we have that

$$\mu_A(-x) \geq \mu_A(x)$$

and

$$\nu_A(-x) \leq \nu_A(x).$$

Now let $a \in \mathbb{F}$:

(1) If $x \in W$, then $ax \in W$ and so

$$\mu_A(ax) = 1 \geq 1 = \mu_A(x)$$

and

$$\nu_A(ax) = 1 \leq 1 = \nu_A(x).$$

(2) If $x \notin W$, then $ax \notin W$ so

$$\mu_A(ax) = 0 \geq 0 = \mu_A(x)$$

and

$$\nu_A(ax) = 0 \leq 0 = \nu_A(x).$$

Then from (1) and (2) we obtain

$$\mu_A(ax) \geq \mu_A(x)$$

and

$$\nu_A(ax) \leq \nu_A(x).$$

Therefore from the above conditions we get that $A = (\mu_A, \nu_A) \in IFTC(V)$. □

Theorem 3.5. *Let* $A = (\mu_A, \nu_A) \in IFTC(V)$. *Then*

$$W = \{x \mid x \in V : A(x) = (\mu_A(x), \nu_A(x)) = (1,0) = 1_{\sim}(x)\}$$

is either empty or is a vector subspace of V.

Proof. Let $x, y \in W$ and $a \in \mathbb{F}$. Then

(1)

$$\mu_A(ax+y) \geq T(\mu_A(ax), \mu_A(y)) \geq T(\mu_A(x), \mu_A(y)) = T(1,1) = 1$$

and then $\mu_A(ax+y) = 1$. Also

(2)

$$\nu_A(ax+y) \leq C(\nu_A(ax), \nu_A(y)) \leq C(\nu_A(x), \nu_A(y)) = C(0,0) = 0$$

and then $\nu_A(ax+y) = 0$. Then from (1) and (2) we get that

$$A(ax+y) = (\mu_A(ax+y), \nu_A(ax+y)) = (1,0) = 1_{\sim}(x)$$

which means that $ax+y \in W$.

Now from Theorem 2.4 we get W is a vector subspace of V. □

Theorem 3.6. *Let* $A = (\mu_A, \nu_A) \in IFTC(V)$ *such that* T *and* C *be idempotent* t*-norm and* t*-conorm, resprctively. Then*

$$W = \{x \mid x \in V : A(x) = A(0)\}$$

is either empty or is a vector subspace of V.

Proof. Let $x, y \in W$ and $a \in \mathbb{F}$. Now

$$\mu_A(ax+y) \geq T(\mu_A(ax), \mu_A(y)) \geq T(\mu_A(x), \mu_A(y))$$

$$= T(\mu_A(0), \mu_A(0)) = \mu_A(0) \geq \mu_A(ax+y) \text{ (by Theorem 3.6)}$$

and thus $\mu_A(ax+y) = \mu(0)$.

Also

$$\nu_A(ax+y) \leq C(\nu_A(ax), \nu_A(y)) \leq C(\nu_A(x), \nu_A(y))$$

$$= C(\nu_A(0), \nu_A(0)) = \nu_A(0) \leq \nu_A(ax+y) \text{ (by Theorem 3.6)}$$

and so $\nu_A(ax+y) = \nu_A(0)$. Thus $A(ax+y) = (\mu_A(ax+y), \nu_A(ax+y)) = (0,0) = A(0)$. Now Theorem 2.4 gives us that W is a vector subspace of V. □

Theorem 3.7. *If $A = (\mu_A, \nu_A) \in IFTC(V)$ such that T and C be idempotent t-norm and t-conorm, resprctively. Then $A(0) \supseteq A(x)$ for all $x \in V$.*

Proof. As $A = (\mu_A, \nu_A) \in IFTC(V)$ so

$$\mu_A(0) = \mu_A(x-x) = \mu_A(x+(-x))$$

$$\geq T(\mu_A(x), \mu_A(-x)) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x).$$

Thus $\mu_A(0) \geq \mu_A(x)$.

Also

$$\nu_A(0) = \nu_A(x-x) = \nu_A(x+(-x))$$

$$\leq C(\nu_A(x), \nu_A(-x)) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x).$$

Thus $\nu_A(0) \leq \nu_A(x)$.

Then $A(0) = (\mu_A(0), \nu_A(0)) \supseteq (\mu_A(x), \nu_A(x)) = A(x)$ for all $x \in V$. □

Theorem 3.8. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ such that T and C be idempotent t-norm and t-conorm, resprctively. If $A(x-y) = A(0)$, then $A(x) = A(y)$ for all $x, y \in V$.*

Proof. Let $A=(\mu_A,\nu_A)\in IFTC(V)$ and $x,y\in V$. As $A(x-y)=A(0)$ so

$$A(x-y)=(\mu_A(x-y),\nu_A(x-y))=(\mu_A(0),\nu_A(0))$$

and then $\mu_A(x-y)=\mu_A(0)$ and $\nu_A(x-y)=\nu_A(0)$. Now

(1)

$$\mu_A(x)=\mu_A(x-y+y)\geq T(\mu_A(x-y),\mu_A(y))\geq T(\mu_A(0),\mu_A(y))$$

$$\geq T(\mu_A(y),\mu_A(y))=\mu_A(y)=\mu_A(y+x-x)=\mu_A(-(x-y)+x)$$

$$\geq T(\mu_A(-(x-y)),\mu_A(x))\geq T(\mu_A(x-y),\mu_A(x))=T(\mu_A(0),\mu_A(x))$$

$$\geq T(\mu_A(x),\mu_A(x))=\mu_A(x).$$

Therefore $\mu_A(x)=\mu_A(y)$.

(2)

$$\nu_A(x)=\nu_A(x-y+y)\leq C(\nu_A(x-y),\nu_A(y))\leq C(\nu_A(0),\nu_A(y))$$

$$\leq C(\nu_A(y),\nu_A(y))=\nu_A(y)=\nu_A(y+x-x)=\nu_A(-(x-y)+x)$$

$$\leq C(\nu_A(-(x-y)),\nu_A(x))\leq C(\nu_A(x-y),\nu_A(x))=C(\nu_A(0),\nu_A(x))$$

$$\leq C(\nu_A(x),\nu_A(x))=\nu_A(x).$$

Therefore $\nu_A(x)=\nu_A(y)$.

Then from (1) and (2) we get that $A(x)=(\mu_A(x),\nu_A(x))=(\mu_A(y),\nu_A(y))=A(y)$. □

Theorem 3.9. *Let T and C be idempotent t-norm and t-conorm, resprctively.*

Then $A=(\mu_A,\nu_A)\in IFTC(V)$ If and only if

(1) $\mu_A(x-y)\geq T(\mu_A(x),\mu_A(y))$,

(2) $\mu_A(ax)\geq \mu_A(x)$,

(3) $\nu_A(x-y)\leq C(\nu_A(x),\nu_A(y))$,

(4) $\nu_A(ax)\leq \nu_A(x)$,

for all $x,y\in V$ and $a\in\mathbb{F}$.

Proof. Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $x, y \in V$. Then

$$\mu_A(x-y) = \mu_A(x+(-y)) \geq T(\mu_A(x), \mu_A(-y)) \geq T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x-y) = \nu_A(x+(-y)) \leq C(\nu_A(x), \nu_A(-y)) \leq C(\nu_A(x), \nu_A(y)).$$

Conversely, if

(1) $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(x))$,

(2) $\mu_A(ax) \geq \mu_A(x)$, then

$$\mu_A(0) = \mu_A(x-x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x).$$

Now

$$\mu_A(-x) = \mu_A(0-x) \geq T(\mu_A(0), \mu_A(x)) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

and

$$\mu_A(x+y) = \mu_A(x-(-y)) \geq T(\mu_A(x), \mu_A(-y)) \geq T(\mu_A(x), \mu_A(y)).$$

Also if

(3) $\mu(x-y) \leq C(\mu(x), \mu(x))$

(4) $\mu(ax) \leq \mu(x)$, then

$$\nu_A(0) = \nu_A(x-x) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x).$$

Thus

$$\nu_A(-x) = \nu_A(0-x) \leq C(\nu_A(0), \nu_A(x)) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x)$$

and

$$\nu_A(x+y) = \nu_A(x-(-y)) \leq C(\nu_A(x), \nu_A(-y)) \leq C(\nu_A(x), \nu_A(y)).$$

Therefore $A = (\mu_A, \nu_A) \in IFTC(V)$. □

Theorem 3.10. *Let $A = (\mu_A, \nu_A)$ be IFSs in V. Let $A(0) = 1_{\sim}(x) = (1, 0)$ and*

(1) $\mu_A(x-y) \geq T(\mu_A(x), \mu_A(y))$,

(2) $\mu_A(ax) \geq \mu_A(x)$,

(3) $\nu_A(x-y) \leq C(\nu_A(x), \nu_A(y))$,

(4) $\nu_A(ax) \leq \nu_A(x)$,

for all $x,y \in V$ and $a \in \mathbb{F}$.

Then $A = (\mu_A, \nu_A) \in IFTC(V)$.

Proof. Since $A(0) = 1_{\sim}(x) = (1,0)$ so $A(0) = (\mu_A(0), \nu_A(0)) = (1,0)$ and then $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Now we get that

$$\mu_A(-x) = \mu_A(0-x) \geq T(\mu_A(0), \mu_A(x)) = T(1, \mu_A(x)) = \mu_A(x)$$

and

$$\mu_A(x+y) = \mu_A(x-(-y)) \geq T(\mu_A(x), \mu_A(-y)) \geq T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(-x) = \nu_A(0-x) \leq C(\nu_A(0), \nu_A(x)) = C(0, \nu_A(x)) = \nu_A(x)$$

and

$$\nu_A(x+y) = \nu_A(x-(-y)) \leq C(\nu_A(x), \nu_A(-y)) \leq C(\nu_A(x), \nu_A(y)).$$

Therefore $A = (\mu_A, \nu_A) \in IFTC(V)$. □

Theorem 3.11. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $A(x-y) = 1_{\sim}(x) = (1,0)$. Then $A(x) = A(y)$ for all $x,y \in V$.*

Proof. As $A(x-y) = 1_{\sim}(x) = (1,0)$ so $A(x-y) = (\mu_A(x-y), \nu_A(x-y)) = (1,0)$ and then $\mu_A(x-y) = 1$ and $\nu_A(x-y) = 0$ for all $x,y \in V$. Then

$$\mu_A(x) = \mu_A(x-y+y) \geq T(\mu_A(x-y), \mu_A(y))$$

$$= T(1, \mu_A(y)) = \mu_A(y) = \mu_A(-y) \textit{ (by Corollary 3.2)}$$

$$= \mu_A(x-x-y) = \mu_A(x-y-x) \geq T(\mu_A(x-y), \mu_A(-x))$$

$$= T(1, \mu_A(-x)) = \mu_A(-x) = \mu_A(x) \textit{ (by Corollary 3.2)}$$

and then $\mu_A(x) = \mu_A(y)$.

Also

$$\nu_A(x) = \nu_A(x-y+y) \leq C(\nu_A(x-y), \nu_A(y))$$

$$= T(0, \nu_A(y)) = \nu_A(y) = \nu_A(-y) \textit{ (by Corollary 3.2)}$$

$$= \nu_A(x-x-y) = \nu_A(x-y-x) \leq C(\nu_A(x-y), \nu_A(-x))$$

$$= C(0, \nu_A(-x)) = \nu_A(-x) = \nu_A(x) \text{ (by Corollary 3.2)}$$

therefore $\nu_A(x) = \nu_A(y)$.

Then $A(x) = (\mu_A(x), \nu_A(x)) = (\mu_A(y), \nu_A(y)) = A(y)$. □

Theorem 3.12. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $A(x) \supseteq A(y)$ for some $x, y \in V$. If T and C be idempotent t-norm and t-conorm, resprctively, then $A(x+y) = A(y)$ for all $x, y \in V$.*

Proof. Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $A(x) \supseteq A(y)$ which means that $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$ for all $x, y \in V$. Now by setting $y = x+y$, then we have that $\mu_A(x) \geq \mu_A(x+y)$ and $\nu_A(x) \leq \nu_A(x+y)$. Then

$$\mu_A(x+y) \geq T(\mu_A(x), \mu_A(y)) \geq T(\mu_A(y), \mu_A(y)) = \mu_A(y)$$

$$= \mu_A(x+y-x) \geq T(\mu_A(x+y), \mu_A(-x)) \geq T(\mu_A(x+y), \mu_A(x))$$

$$\geq T(\mu_A(x+y), \mu_A(x+y)) = \mu_A(x+y)$$

and so $\mu_A(x+y) = \mu_A(y)$.

Also

$$\nu_A(x+y) \leq C(\nu_A(x), \nu_A(y)) \leq C(\nu_A(y), \nu_A(y)) = \nu_A(y)$$

$$= \nu_A(x+y-x) \leq C(\nu_A(x+y), \nu_A(-x)) \leq C(\nu_A(x+y), \nu_A(x))$$

$$\leq C(\nu_A(x+y), \nu_A(x+y)) = \nu_A(x+y)$$

thus $\nu_A(x+y) = \nu_A(y)$. Therefore

$$A(x+y) = (\mu_A(x+y), \nu_A(x+y)) = (\mu_A(y), \nu_A(y)) = A(y).$$

□

Theorem 3.13. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $A(x) \subseteq A(y)$ for some $x, y \in V$. If T and C be idempotent t-norm and t-conorm, resprctively, then $A(x+y) = A(x)$ for all $x, y \in V$.*

Proof. It is trivial. □

Theorem 3.14. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and T and C be idempotent t-norm and t-conorm, resprctively. If $A(x) \neq A(y)$, then $A(x+y) = (T(\mu_A(x), \mu_A(y)), C(\nu_A(x), \nu_A(y)))$ for all $x, y \in V$.*

Proof. As $A(x) \neq A(y)$ so $A(x) \supset A(y)$ or $A(x) \subset A(y)$ for all $x, y \in V$.

(1) If $A(x) \supset A(y)$, then $\mu_A(x) > \mu_A(y)$ and $\nu_A(x) < \nu_A(y)$ and then $\mu_A(y) = T(\mu_A(x), \mu_A(y))$ and $\nu_A(y) = C(\nu_A(x), \nu_A(y))$ for all $x, y \in V$. Now from Theorem 3.12 we have that $\mu_A(x+y) = \mu_A(y)$ and then $\mu_A(x+y) = \mu_A(y) = T(\mu(x), \mu(y))$ and $\nu_A(x+y) = \nu_A(y) = C(\mu(x), \mu(y))$. Thus

$$A(x+y) = (\mu_A(x+y), \nu_A(x+y)) = (T(\mu_A(x), \mu_A(y)), C(\nu_A(x), \nu_A(y))).$$

(2) If $A(x) \subset A(y)$, then $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$ and then $\mu_A(x) = T(\mu_A(x), \mu_A(y))$ and $\nu_A(x) = C(\nu_A(x), \nu_A(y))$ for all $x, y \in V$. Now from Theorem 3.13 we have that $\mu_A(x+y) = \mu_A(x)$ and then $\mu_A(x+y) = \mu_A(x) = T(\mu(x), \mu(y))$ and $\nu_A(x+y) = \nu_A(x) = C(\mu(x), \mu(y))$. Then

$$A(x+y) = (\mu_A(x+y), \nu_A(x+y)) = (T(\mu_A(x), \mu_A(y)), C(\nu_A(x), \nu_A(y))).$$

□

Theorem 3.15. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and T and C be idempotent t-norm and t-conorm, resprctively. Then $A(x-y) = A(y)$ if and only if $A(x) = A(0)$ for all $x, y \in V$.*

Proof. Let $A(x-y) = A(y)$ then from letting $y = 0$ we have that $A(x) = A(0)$.

Conversely, suppose that $A(x) = A(0)$ and from Theorem 3.6 we get that $A(x) = A(0) \supseteq A(x-y)$ and $A(x) = A(0) \supseteq A(-y)$ and then we obtain that $\mu_A(x) = \mu_A(0) \geq \mu_A(x-y)$ and $\mu_A(x) = \mu_A(0) \geq \mu_A(-y)$ and $\nu_A(x) = \nu_A(0) \leq \nu_A(x-y)$ and $\nu_A(x) = \nu_A(0) \leq \nu_A(-y)$ for all $x, y \in V$. Now

$$\mu_A(x-y) = \mu_A(x+(-y)) \geq T(\mu_A(x), \mu_A(-y)) = T(\mu_A(0), \mu_A(-y))$$

$$\geq T(\mu_A(-y), \mu_A(-y)) = \mu_A(-y) = \mu_A(x-y-x) = \mu_A(x-y+(-x))$$

$$\geq T(\mu_A(x-y), \mu_A(-x)) \geq T(\mu_A(x-y), \mu_A(x)) = T(\mu_A(x-y), \mu_A(0))$$

$$\geq T(\mu_A(x-y), \mu_A(x-y)) = \mu_A(x-y)$$

and thus $\mu_A(x-y) = \mu_A(-y) = \mu_A(y)$.

Also

$$\nu_A(x-y) = \nu_A(x+(-y)) \leq C(\nu_A(x), \nu_A(-y)) = C(\nu_A(0), \nu_A(-y))$$

$$\leq C(\nu_A(-y), \nu_A(-y)) = \nu_A(-y) = \nu_A(x-y-x) = \nu_A(x-y+(-x))$$

$$\leq C(\nu_A(x-y), \nu_A(-x)) \leq C(\nu_A(x-y), \nu_A(x)) = C(\nu_A(x-y), \nu_A(0))$$

$$\leq C(\nu_A(x-y), \nu_A(x-y)) = \nu_A(x-y)$$

and thus $\nu_A(x-y) = \nu_A(-y)$ and by Corollary 3.2 we have that $\nu_A(y) = \nu_A(-y) = \nu_A(x-y)$. Therefore

$$A(x-y) = (\mu_A(x-y), \nu_A(x-y)) = (\mu_A(y), \nu_A(y)) = A(y).$$

$\square$

Theorem 3.16. *Let $A = (\mu_A, \nu_A) \in IFTC(V)$ and $f : [0, \mu_A(0)] \to [0,1]$ be an increasing map. Define a fuzzy set $\mu_A^f : V \to [0,1]$ by $\mu_A^f(x) = f(\mu_A(x))$. Let $g : [0, \nu_A(0)] \to [0,1]$ be a decreasing map and define a fuzzy set $\nu_A^g : V \to [0,1]$ by $\nu_A^g(x) = g(\nu_A(x))$.*
Then $A^{f,g} = (\mu_A^f, \nu_A^g) \in IFTC(V)$.

Proof. Let $x, y \in V$ and $a \in \mathbb{F}$. Then

(1)

$$\mu_A^f(x+y) = f(\mu_A(x+y)) \geq f(T(\mu_A(x), \mu_A(y)))$$

$$= T(f(\mu_A(x)), f(\mu_A(y))) = T(\mu_A^f(x), \mu_A^f(y)).$$

(2)

$$\mu_A^f(-x) = f(\mu_A(-x)) \geq f(\mu_A(x)) = \mu_A^f(x).$$

(3)

$$\mu_A^f(ax) = f(\mu_A(ax)) \geq f(\mu_A(x)) = \mu^f(x).$$

(4)

$$\nu_A^g(x+y) = g(\nu_A(x+y)) \leq g(C(\nu_A(x), \nu_A(y)))$$

$$= C(g(\nu_A(x)), g(\nu_A(y))) = C(\nu_A^g(x), \nu_A^g(y)).$$

(5)

$$\nu_A^g(-x) = g(\nu_A(-x)) \leq g(\nu_A(x)) = \nu_A^g(x).$$

(6)

$$\nu_A^g(ax) = g(\nu_A(ax)) \leq g(\nu_A(x)) = \nu_A^g(x).$$

Then $A^{f,g} = (\mu_A^f, \nu_A^g) \in IFTC(V)$. $\square$

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Conflict of Interests

The authors declare that there is no conflict of interests.

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