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ON COUNTABLY GENERATED QTAG-MODULES

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Abstract. A right module M over an associative ring with unity is a *QTAG*-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. The main aim of this paper is to investigate the direct sums of countably generated QTAG-modules and obtain some significant results related to these modules.

Keywords: QTAG-module; summable modules; high submodules; h -pure submodules.

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1. INTRODUCTION

Different concepts for groups like purity, projectivity, injectivity, height etc. have been generalized for modules. In generalizing the results of groups for modules sometimes we find that they are not true for modules. So, for establishing such generalizations we impose some conditions either on the module or on the underlying ring. Here we impose a condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the rings are associative with unity. After these conditions many

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elegant results of groups can be proved for QTAG-modules which are not true in general. Many results of this paper are the generalization of the paper [3].

The study of QTAG-modules was initiated by Singh [8]. Mehdi et al. [2] worked a lot on these modules. They introduced many concepts and generalized different notions for these modules and contributed to the development of the theory of QTAG-modules. Yet there is much to explore.

Singh [8] proved that a QTAG-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. This indicates that M is a direct sum of uniserial modules if and only if $\text{Soc}(M) = \bigoplus_{k \in \omega} S_k$ and $H(x) = k$ for every $x \in S_k$. This motivates us to define summable modules. A reduced QTAG-module M of length σ is said to be summable if $\text{Soc}(M) = \bigoplus_{\rho < \sigma} S_\rho$ and the nonzero elements of S_ρ are contained in $(H_\rho(M) \setminus H_{\rho+1}(M))$ [1]. This implies that a QTAG-module M of length ω is a direct sum of uniserial modules if and only if it is summable.

Here we investigate the direct sums of countably generated QTAG-modules. The following example shows that a summable λ -module ($\omega \leq \lambda$) need not be a direct sum of countably generated modules. Let M_α be the countably generated QTAG-module of length α , having each of its nonzero Ulm-invariant equal to N_0 . If $\gamma < \lambda$, then for each β ($\alpha < \beta < \gamma$) M_α can be embedded in M_β such that $H_\mu(M_\beta) \cap M_\alpha = H_\mu(M_\alpha)$ for all μ . For a limit ordinal γ , $M_\gamma \simeq \bigcup_{\alpha < \gamma} M_\alpha$ and $H_\mu(M_\beta) \cap M_\alpha = H_\mu(M_\alpha)$ for $\alpha < \beta \leq \gamma$. If γ is a non limit ordinal, $\gamma - 1$ is a limit ordinal and $H_{\gamma-1}(M_\gamma)$ -high submodule of M_γ is isomorphic to $M_{\gamma-1}$. If $\gamma - 2$ exists then we consider $M_{\gamma-1}$, a direct summand of M_γ i.e. $M_\gamma = M_{\gamma-1} \oplus H_{\gamma-1}(M_\gamma)$ for $\alpha < \beta \leq \gamma$. As α ranges over the countable ordinals the modules M_α are embedded in one another. Let $M = \bigcup_{\alpha < \lambda} M_\alpha$. Now M is summable, $M/H_\alpha(M)$ is a direct sum of countably generated modules for each countable α , but M is not a direct sum of countably generated modules.

This made us investigate the other conditions as summability is not enough. The condition that $M/H_\rho(M)$ should be a direct sum of countably generated modules for all ρ less than the length of M ensures that M is a direct sum of countably generated modules. If ρ is a countable ordinal and $M/H_\rho(M)$ and $H_\rho(M)$ are h -reduced countably generated QTAG-modules then M

is a direct sum of countably generated QTAG-modules. Therefore we focused on the QTAG-modules of limit length.

In section 3 of this article we prove some significant results related to the modules which are the direct sums of countably generated modules. Section 4 deals with some applications of these results.

2. PRELIMINARIES

Throughout this paper, all rings are associative with unity and modules M are unital QTAG-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R -module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d \left(\frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M , respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k . A submodule N of M is h -pure [5] in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule N of M is said to be isotype in M , if it is σ -pure for every ordinal σ [4]. M is h -divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h -reduced if it does not contain any h -divisible submodule. In other words it is free from the elements of infinite height.

For a QTAG-module M , there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ . $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σ^{th} -Ulm submodule of M . Several results which hold for TAG-modules also hold good for QTAG-modules [8]. Notations follows the standard work of Fuchs [6, 7].

3. MAIN RESULTS

This section deals with some interesting results and consequences. We start with the following:

Definition 3.1. *A submodule N of a QTAG-module M is height finite, if the heights of the elements of N take only finitely many values.*

Remark 3.1. We may observe that if M is a summable QTAG-module of countable length then $\text{Soc}(M)$ is the union of an ascending chain of height finite submodules. Since $\text{Soc}(M) = \bigoplus_{\rho < \sigma} S_\rho$, $S_\rho \subseteq (H_\rho(M) \setminus H_{\rho+1}(M)) \cup \{0\}$ if $\rho_1, \rho_2, \dots, \rho_n, \dots$ are the ordinals less than σ , then $\text{Soc}(M) = \bigcup_{k=1}^{\infty} T_k$, $T_k = S_{\rho_1} \oplus \dots \oplus S_{\rho_k}$ and T_k are height finite.

We prove the following lemma:

Lemma 3.1. Let N be a submodule of a QTAG-module M such that

- (i) $H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$, for every $\alpha < \beta$;
- (ii) M/N is countably generated of length $\leq \beta$.

If K/N is finitely generated submodule of M/N , then K also satisfies these two conditions.

Proof. Assume that $H_\alpha(M/K) = \frac{H_\alpha(M) + K}{K}$, for all $\alpha < \gamma < \beta$. Consider $x + K \in H_\gamma(M/K)$. If $\gamma = \alpha + 1$, then there exists $y + K \in H_\alpha(M/K) = \frac{H_\alpha(M) + K}{K}$ such that $d\left(\frac{(y+K)R}{(x+K)R}\right) = 1$, therefore $y \in H_\alpha(M)$ and $x + K \in \frac{H_\gamma(M) + K}{K}$. Otherwise, if γ is a limit ordinal, then for each $\alpha < \gamma$, $x + K = z + K$, where $z \in H_\alpha(M)$. Thus $x - z = u + v$ where $u \in K$, $v \in N$. Since u assumes only finitely many values, there is a fixed $u_0 \in K$ and a set $T \subseteq \{\alpha \mid \alpha < \gamma\}$ such that $\sup T = \gamma$ and $x - z = u_0 + v$ where $v \in N$. Now $x - u_0 + K = z + N \in \frac{H_\alpha(M) + N}{N}$. Since $\sup T = \gamma$, $x - u_0 + N = z + N \in \frac{H_\alpha(M) + N}{N}$ for all $\alpha \in T$ and $x - u_0 + N \in H_\gamma(M/N) = \frac{H_\gamma(M) + N}{N}$. Now we may write $x - u_0 + N = z + N$ with $z \in H_\gamma(M)$ and $x + K = z + K \in \frac{H_\alpha(M) + K}{K}$.

To prove the second condition it is sufficient to prove $H_\beta(M/K) = 0$ i.e., length of M/K is β . For $x + K \in H_\beta(M/K)$ we find $u \in K$ such that $x - u + N \in H_\beta(M/N) = 0$. Therefore $x - u \in N$ or $x + K = 0$. \square

Theorem 3.1. If N is a submodule of a QTAG-module M such that

- (i) $H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$ for every $\alpha < \beta$;
- (ii) $H_\alpha(M) \cap N = H_\alpha(N)$ for every $\alpha < \beta$;
- (iii) M/N is countably generated of length $\leq \beta$;

then N is a summand of M .

Proof. Since M/N is countably generated, we have to show that if K/N is a finitely generated submodule of M/N , every height increasing homomorphism $f : K \rightarrow N$ ($f|_N = I_N$) can be extended to a height increasing homomorphism $\bar{f} : K + xR \rightarrow N$ whenever $y \in K$ such that $d\left(\frac{xR}{yR}\right) = 1$. Consider $x \notin K$. Now $H_{M/K}(x+K) = \alpha$, for some $\alpha < \beta$. Therefore $x+K = z+K$ with $z \in H_\alpha(M)$. Since $u \in K$ where $d\left(\frac{zR}{uR}\right) = 1$ and $xR + K = zR + K$, $x \in H_\alpha(M)$. Now $H_M(x+v) \leq H_{M/K}(x+K) = H_M(x) = \alpha$ for all $v \in K$. Again $\alpha + 1 = \beta$, we have $w \in H_\alpha(N)$ such that $w' = f(y) \in H_{\alpha+1}(M) \cap N = H_{\alpha+1}(N)$ where $d\left(\frac{wR}{w'R}\right) = 1$. Now we may define \bar{f} on $K + xR$ such that $\bar{f}(xr+u) = wr + f(u)$, $r \in R$, $u \in K$. Here \bar{f} is a well defined height increasing homomorphism, which is an extension of f . \square

As a corollary to the Theorem 3.1, we have the following theorem:

Theorem 3.2. *Let N be a submodule of a QTAG-module M such that*

- (i) $\frac{M/N}{H_\beta(M/N)}$ is countably generated;
- (ii) $\frac{N + H_\beta(M)}{H_\beta(M)}$ is a summand of $M/H_\beta(M)$;
- (iii) $N \cap H_\beta(M) = H_\beta(N)$ and
- (iv) $H_\beta(M) = H_\beta(N) \oplus T$;

then $M = N \oplus K$ with $K \supseteq T$.

Proof. We may put

$$\frac{M}{H_\beta(M)} = \frac{N + H_\beta(M)}{H_\beta(M)} \oplus \frac{Q}{H_\beta(M)}.$$

Thus, $(N + H_\alpha(M)) \cap Q = H_\alpha(M) \cap Q$ for all $\alpha \leq \beta$. Now suppose $H_\alpha(M) \cap N = H_\alpha(N)$ for $\alpha < \gamma \leq \beta$. If γ is a limit ordinal, we have $H_\gamma(M) \cap N = H_\gamma(N)$, otherwise $\gamma = \alpha + 1$. For $x \in H_\gamma(M) \cap N$, there exist $y \in N$, $z \in Q$ such that $d\left(\frac{(y+z)R}{xR}\right) = 1$ and $y+z \in H_\alpha(M)$. Then $z \in (N + H_\alpha(M)) \cap Q = H_\alpha(M) \cap Q$ and $y = y+z-z \in H_\alpha(M) \cap N = H_\alpha(N)$. If $d\left(\frac{yR}{y'R}\right) = 1$, then $x - y' \in N \cap Q = N \cap H_\beta(M) = H_\beta(N)$ and $x = (x - y') + y' \in H_\gamma(N)$. In order to show that

$$H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}, \text{ for all } \alpha \leq \beta,$$

consider

$$\begin{aligned}
\frac{M}{N + H_\alpha(M)} &= \frac{N + H_\alpha(M) + Q}{N + H_\alpha(M)} \\
&\cong \frac{Q}{Q \cap (N + H_\alpha(M))} = \frac{Q}{Q \cap H_\alpha(M)} \\
&\cong \frac{H_\alpha(M) + Q}{H_\alpha(M)} \subseteq \frac{M}{H_\alpha(M)}, \text{ for all } \alpha \leq \beta.
\end{aligned}$$

Now we apply Theorem 3.1 to $\frac{N \oplus T}{T} = \frac{N + H_\beta(M)}{T}$ and get

$$\begin{aligned}
H_\alpha \left(\frac{(M/T)}{(N \oplus T)/T} \right) &\cong H_\alpha \left(\frac{M}{N \oplus T} \right) = H_\alpha \left(\frac{M}{(N + H_\beta(M))} \right) \\
&\cong H_\alpha \left(\frac{(M/N)}{H_\beta(M/N)} \right) = \frac{H_\alpha(M/N)}{H_\beta(M/N)} \\
&\cong \frac{H_\alpha(M) + N}{H_\beta(M) + N} \cong \frac{(H_\alpha(M) + N)/T}{(H_\beta(M) + N)/T} \\
&= \frac{(H_\alpha(M/T) + N \oplus T)/T}{(N \oplus T)/T}.
\end{aligned}$$

These isomorphisms give identity on $\frac{M/T}{(N \oplus T)/T}$ when combined together and we have

$$\frac{((H_\alpha(M) + N)/T)}{((H_\beta(M) + N)/T)} = \frac{(H_\alpha(M/T) + N \oplus T)/T}{(N \oplus T)/T}, \text{ for all } \alpha < \beta.$$

Now,

$$\begin{aligned}
H_\alpha(M/T) \cap ((N \oplus T)/T) &= (H_\alpha(M)/T) \cap ((N \oplus T)/T) \\
&= ((H_\alpha(M) \cap N) \oplus T)/T = (H_\alpha(N) \oplus T)/T \\
&\subseteq H_\alpha((N \oplus T)/T), \text{ for all } \alpha \leq \beta.
\end{aligned}$$

Also,

$$\begin{aligned} \frac{(M/T)}{(N \oplus T)/T} &\cong \frac{M}{N + H_\beta(M)} \cong \frac{(M/H_\beta(M))}{(N + H_\beta(M))/(H_\beta(M))} \\ &= \frac{M/N}{H_\beta(M/N)} \end{aligned}$$

is countably generated module of length at most β . Again by Theorem 3.1 we have

$$\frac{M}{T} = \left(\frac{N \oplus T}{T} \right) \oplus \left(\frac{K}{T} \right)$$

and therefore $M = N \oplus K$. \square

In order to prove some important results, we need the following lemmas:

Lemma 3.2. *For a QTAG-module M , a submodule $N \subseteq M$ and a limit ordinal λ if*

$$\text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}, \text{ for every } \alpha < \lambda, \text{ then}$$

$$H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}, \text{ for all } \alpha < \lambda.$$

Proof. Assume that $\beta < \lambda$ and $H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$, for all $\alpha < \beta$. If $\beta = \alpha + 1$ and $x + N \in H_\beta(M/N)$, then $d\left(\frac{(y+N)R}{(x+N)R}\right) = 1$ for some $y \in H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N}$. Without loss of generality, we may assume that $y \in H_\alpha(M)$ and $x + N = y' + N \in \frac{H_\beta(M) + N}{N}$, where $d\left(\frac{yR}{y'R}\right) = 1$. Otherwise, if β is a limit ordinal and we assume $H^k(H_\beta(M/N)) \subseteq \frac{H_\beta(M) + N}{N}$ and $x + N \in H^{k+1}(H_\beta(M/N))$. If $d\left(\frac{(x+N)R}{(y+N)R}\right) = k$, then $y + N \in \text{Soc}(H_{\beta+k}(M/N)) = \frac{\text{Soc}(H_{\beta+k}(M)) + N}{N}$ as $\beta + k < \lambda$. Therefore $y + N = z + N$ such that $z' \in H^{k+1}(H_\beta(M))$, $d\left(\frac{z'R}{zR}\right) = k$ and $(x - z') + N \in H^k(H_\beta(M/N)) \subseteq \frac{H_\beta(M) + N}{N}$. Therefore $x - z' + N = u + N$ with $u \in H_\beta(M)$ and $x + N = (z' + u) + N \in \frac{H_\beta(M) + N}{N}$. \square

Lemma 3.3. *Let $N \subseteq K$ be submodules of a QTAG-module M such that K/N is h -divisible. If*

$$\text{Soc}\left(H_\alpha\left(\frac{M}{N}\right)\right) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}, \text{ then}$$

$$\text{Soc}\left(H_\alpha\left(\frac{M}{K}\right)\right) = \frac{\text{Soc}(H_\alpha(M)) + K}{K}.$$

Proof. Since K/N is h -divisible, it is a direct summand of M/N and $\frac{M}{N} = \frac{K}{N} \oplus \frac{Q}{N}$, for some $Q \supseteq N$. For $x + K \in \text{Soc}(H_\alpha(M/K))$, $x = y + z$, $y \in Q$, $z \in N$ and we may define a homomorphism $f : M/K \rightarrow Q/N$ such that $f(x + K) = y + N$. Therefore $y + N \in \text{Soc}(H_\alpha(Q/N)) \subseteq \text{Soc}(H_\alpha(M/N)) = (\text{Soc}(H_\alpha(M)) + N)/N$ and there exists a $u \in \text{Soc}(H_\alpha(M))$ such that $y + N = u + N$. Therefore

$$x + K = y + K = u + K \in \frac{\text{Soc}(H_\alpha(M)) + K}{K}.$$

which completes the proof. \square

Lemma 3.4. *For $k < \omega$, let K be a $H_{\alpha+k}(M)$ -high submodule of M . Then $H^i(M) \subseteq H^i(K) + H_\alpha(M)$ for every $i \leq k + 1$.*

Proof. Since $\text{Soc}(M) = \text{Soc}(K) \oplus \text{Soc}(H_{\alpha+k}(M)) \subseteq \text{Soc}(K) + H_\alpha(M)$, we suppose that $H^i(M) \subseteq H^i(K) + H_\alpha(M)$ for every $i \leq k$. Consider $x' \in H^{i+1}(M)$. Then there exists $x \in H^i(M)$ such that $d\left(\frac{x'R}{xR}\right) = i$ and $x = y + z'$ where $y \in \text{Soc}(K)$ and $d\left(\frac{z'R}{z'R}\right) = k$ with $z \in H_\alpha(M)$. Now $y = x - z' \in H_i(M) \cap K = H_i(K)$, as K is h -pure in M . Therefore $d\left(\frac{y'R}{yR}\right) = i$ for some $y' \in K$. Now if $d\left(\frac{z''R}{z'R}\right) = i$, then $x - y' - z'' \in H^i(M) \subseteq H^i(K) + H_\alpha(M)$ which implies that $x \in H^{i+1}(K) + H_\alpha(M)$. \square

The above discussion leads to the following:

Remark 3.2. *For a submodule N of a QTAG-module M , if $\text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}$ for all $\alpha \leq \beta$, then $H_\alpha(M) \cap N = H_\alpha(N)$ for all $\alpha \leq \beta$. For $\beta \leq \omega$, the converse holds.*

Lemma 3.5. *If N is a direct summand of a $H_\alpha(M)$ -high submodule of M , then $\text{Soc}(H_\gamma(M/N)) = \frac{\text{Soc}(H_\gamma(M)) + N}{N}$, for all $\gamma \leq \alpha$.*

Proof. Being a direct summand of a h -pure submodule of M , N is also h -pure in M . Remark 3.2 suggests that for all $\gamma < \omega$, $\text{Soc}(H_\gamma(M/N)) = \frac{\text{Soc}(H_\gamma(M)) + N}{N}$. Without loss of generality, we may assume that $\alpha \geq \omega$. Let $\alpha = \beta + k$, $k < \omega$, β a limit ordinal and $Q \oplus N$, a $H_\alpha(M)$ -high submodule of M . Suppose the result holds for all ordinals $\delta < \alpha$. If $\omega \leq \delta < \alpha$ and if T is a $H_\delta(N)$ -high submodule of N , then T is a direct summand of $H_\delta(M)$ -high submodule of M . Therefore if K is $H_\delta(Q)$ -high in Q , then $T \oplus K$ is $H_\delta(M)$ -high in M . Inductively,

$\text{Soc}(H_\delta(M/T)) = \frac{\text{Soc}(H_\delta(M)) + T}{T}$. As $\frac{N}{T}$ is h -divisible by Theorem 3.2, $\frac{\text{Soc}(H_\delta(M)) + N}{N} = \text{Soc}(H_\delta(M/N))$ and the result holds for all $\gamma < \alpha$.

In order to prove that $\text{Soc}(H_\alpha(M/N)) = \frac{\text{Soc}(H_\alpha(M)) + N}{N}$, we have to show that $H^{k+1}(H_\beta(M/N)) = \frac{H^{k+1}(H_\beta(M)) + N}{N}$. Consider $x + N \in H^{k+1}\left(H_\beta\left(\frac{M}{N}\right)\right)$. Since N is h -pure, $x \in H^{k+1}(M)$ and by Lemma 3.4, $x = x_1 + x_2 + x_3$ where $x_1 \in H^{k+1}(N)$, $x_2 \in H^{k+1}(Q)$ and $x_3 \in H_\beta(M)$. As β is a limit ordinal, for every ordinal $\gamma < \beta$, by Lemma 3.2, we may write $x + N = z + N$, where $z \in H_\gamma(M)$. Thus for each $\gamma < \beta$, we have $x = x_1 + x_2 + x_3 = z + z_1$ where $z_1 \in N$.

Now, $x_1 - z_1 + x_2 \in H_\gamma(M) \cap (N \oplus Q) = H_\gamma(N) \oplus H_\gamma(Q)$ and $x_2 \in \bigcap_{\gamma < \beta} H_\gamma(Q) = H_\beta(M)$ implying that $x + N = (x_2 + x_3) + N \in \frac{H^{k+1}(H_\beta(M)) + N}{N}$. \square

We need some more lemmas to prove our main results:

Lemma 3.6. *Let $K = \bigoplus_{i \in I} K_i$ be a submodule of a QTAG-module M such that each K_i is countably generated. If N is a submodule of M , $N \cap K = \bigoplus_{i \in I'} K_i$, $I' \subset I$ and if T/N is a countably generated submodule of $\frac{M}{N}$, then there is a submodule $Q \subseteq M$ such that $T \subseteq Q$, $\frac{Q}{N}$ is countably generated and $Q \cap K = \bigoplus_{i \in J} K_i$ for some subset J of I . If $\frac{M}{K}$ and $\frac{N}{N \cap K}$ are h -divisible, then M can be chosen so that $\frac{Q}{Q \cap K}$ is also h -divisible.*

Proof. Since $\frac{T \cap K}{N \cap K}$ is countably generated, there is a countable subset I'' of I such that $\bigoplus_{i \in I''} K_i$ contains a complete set of representatives of $\frac{T \cap K}{N \cap K}$. Set $Q = T + \left(\bigoplus_{i \in I''} K_i\right) = T + \left(\bigoplus_{i \in J} K_i\right)$, where $J = I' \cup I''$.

Now, suppose $\frac{M}{K}$ and $\frac{N}{N \cap K}$ are h -divisible. Consider a submodule $L \supset N$ such that $\frac{L}{N}$ is countably generated *i.e.*, $\frac{L}{N} = \Sigma(b_i + N)R$. Consider the elements $x_1, x_2, \dots, x_n, \dots$ such that $b_i - y_i \in K$ where $d\left(\frac{x_i R}{y_i R}\right) = 1$.

If $P = L + \Sigma x_i R$, then $\frac{P}{N}$ is countably generated and $L \subseteq H_1(P) + K$. In this way we can choose two ascending sequences of submodules $T \subseteq Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_k \subseteq \dots$ and $P_1 \subseteq P_2 \subseteq \dots \subseteq P_k \subseteq \dots$, where $\frac{M_k}{N}$ and $\frac{P_k}{N}$ are countably generated for every k , $Q_k \cap K = \bigoplus_{i \in I_k} K_i$, $Q_k \subseteq P_k \subseteq Q_{k+1}$ and

$Q_k \subseteq H_1(P_k) + K$ for each k . If we put $Q = \bigcup_{k=1}^{\infty} Q_k = \bigcup_{k=1}^{\infty} P_k$ and $J = \bigcup_{k=1}^{\infty} I_k$, then $P_k \subseteq H_1(P_{k+1}) + K$ ensures that $\frac{Q}{K \cap Q}$ is h -divisible. \square

Lemma 3.7. *Let $\{K_k\}_{k=1}^{\infty}$ be a sequence of submodules of a QTAG-module M and $K_k = \bigoplus_{i \in I_k} K_k^{(i)}$, where each $K_k^{(i)}$ is countably generated. Let A be a set of positive integers such that M/K_k is h -divisible, for every $k \in A$. If N is a submodule of M such that $N \cap K_k = \bigoplus_{i \in I_k} K_k^{(i)}$ for each k and $\frac{N}{N \cap K_k}$ is h -divisible, whenever $k \in A$ and if $\frac{T}{N}$ is a countably generated submodule of $\frac{M}{N}$, then there is a submodule Q of M such that $T \subseteq Q$, $\frac{Q}{N}$ is countably generated, $Q \cap K_k = \bigoplus_{i \in J_k} K_k^{(i)}$ and $\frac{Q}{Q \cap K_k}$ is h -divisible, whenever $k \in A$.*

Proof. Let n_1, n_2, \dots be a sequence of positive integers such that for every pair of positive integers k and n there is an integer $j > k$ such that $n_j = n$. By Lemma 3.6, we may construct a chain of submodules of M containing T , $Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_k \subseteq \dots$ such that for each k , $\frac{Q_k}{N}$ is countably generated. $Q_k \cap K_{n_k} = \bigoplus_{i \in I_{n_k}^k} K_{n_k}^i$ and $\frac{Q_k}{Q_k \cap K_{n_k}}$ is h -divisible, whenever $n_k \in A$. Set $Q = \bigcup_{k=1}^{\infty} Q_k$ and for each n , let J_n be the union of all $I_{n_k}^k$ for which $n_k = n$. Since for each n , Q is the union of those Q_k such that $n_k = n$, Q is the required submodule. \square

Lemma 3.8. *Let N be a submodule of M such that $\text{Soc}\left(H_{\alpha}\left(\frac{M}{N}\right)\right) = \frac{\text{Soc}(H_{\alpha}(M)) + N}{N}$ for all $\alpha < \lambda$. If $\text{Soc}(M) = \text{Soc}(N) \oplus T$, where T is a summable subsocle such that $T \cap H_{\lambda}(M) = 0$, then $\frac{M}{N}$ is a summable module of length at most λ .*

Proof. Let $T = \bigoplus_{\alpha < \lambda} T_{\alpha}$, where $S_{\alpha} \subseteq H_{\alpha}(M)$, $S_{\alpha} \cap H_{\alpha+1}(M) = 0$, for every α . Then

$$\text{Soc}(M/N) = \frac{\text{Soc}(M) + N}{N} = \frac{T \oplus N}{N} = \bigoplus_{\alpha < \lambda} \left(\frac{T_{\alpha} \oplus N}{N} \right).$$

We have to show that for $x+N(\neq 0) \in \frac{T_\alpha+N}{N}$, $H_{M/N}(x+N) = \alpha$. Since $\text{Soc}(M) = \text{Soc}(N) \oplus T$, $\text{Soc}(H_\alpha(M)) = (\text{Soc}(N) \cap H_\alpha(M)) \oplus (T \cap H_\alpha(M)) = (\text{Soc}(N) \cap H_\alpha(M)) \oplus (\bigoplus_{\beta \geq \alpha} S_\beta)$. Therefore

$$\begin{aligned} \text{Soc}(H_\alpha(M/N)) &= (\text{Soc}(H_\alpha(M)) + N)/N \\ &= ((T \cap H_\alpha(M)) + N)/N = \bigoplus_{\beta \geq \alpha} ((T_\beta \oplus N)/N) \\ &= ((T_\alpha \oplus N)/N) \oplus (\bigoplus_{\beta \geq \alpha+1} ((T_\beta \oplus N)/N)) \\ &= ((T_\alpha \oplus N)/N) \oplus \text{Soc}(H_{\alpha+1}(M/N)). \end{aligned}$$

and we are done. \square

Now we are able to prove one of our main results.

Theorem 3.3. *Let M be a QTAG-module of length σ , where σ is a countable limit ordinal. Then M is a direct sum of countably generated modules if M is summable and for each $\rho < \sigma$, M contains a $H_\rho(M)$ -high submodule which is a direct sum of countably generated modules.*

Proof. For an ordinal ρ such that $\omega \leq \rho < \sigma$, let K_ρ be a $H_\rho(M)$ -high submodule of M which is a direct sum of countably generated submodules. Put $K_\rho = \bigoplus_{i \in I_\rho} K_\rho^i$ such that each K_ρ^i is countably generated and $\text{Soc}(M) = \bigoplus_{i \in I} T_i$, where each T_i is countably generated. Now the elements of the minimal generating set of M , $\{x_\alpha\}$ can be well ordered $\alpha < \delta$. Since σ is countable, K_ρ 's can be enumerated and by Lemma 3.7, we may define submodules $\{N_\alpha\}$ satisfying the following conditions:

- (i) $N_0 = 0, N_\alpha \subseteq N_\beta$ for every $\alpha < \beta$ and $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$, if α is a limit ordinal;
- (ii) $x_\alpha \in N_{\alpha+1}$;
- (iii) $N_\alpha \cap K_\rho = \bigoplus_{i \in I_\rho^\alpha} K_\rho^i$, for each α ;
- (iv) $N_{\alpha+1}/N_\alpha$ is countably generated;
- (v) $N_\alpha/(N_\alpha \cap K_\rho)$ is h -divisible, for each α ;
- (vi) $\text{Soc}(N_\alpha) = \bigoplus_{i \in I^\alpha} T_i$.

By Lemma 3.5,

$$\text{Soc}(H_\rho(M/N_\alpha)) = \frac{\text{Soc}(H_\rho(M)) + N_\alpha}{N_\alpha}, \text{ for all } \rho < \sigma.$$

Therefore by Remark 3.2, each N_α is isotype and again by Lemma 3.2,

$$H_\rho(M/N_\alpha) = \frac{H_\rho(M) + N_\alpha}{N_\alpha}, \text{ for all } \rho < \sigma.$$

Now we infer that

$$H_\rho(N_{\alpha+1}/N_\alpha) = \frac{H_\rho(N_{\alpha+1}) + N_\alpha}{N_\alpha}, \text{ for all } \rho < \sigma.$$

By Lemma 3.8, $\frac{N_{\alpha+1}}{N_\alpha}$ has length at most σ and

$$H_\rho(N_{\alpha+1}) \cap N_\alpha = (H_\rho(M) \cap N_{\alpha+1}) \cap N_\alpha = H_\rho(M) \cap N_\alpha = H_\rho(N_\alpha), \text{ for all } \rho \leq \sigma.$$

Now using Theorem 3.1, we get $N_{\alpha+1} = N_\alpha + Q_\alpha$, for each $\alpha < \delta$. Consequently, $M = \bigoplus_{\alpha} Q_\alpha$, $\alpha < \delta$ is a direct sum of countably generated modules. \square

To prove some other interesting results, we need some more lemmas.

Lemma 3.9. *If ρ is a countable ordinal and $x \in H_\rho(M)$, then there is a countably generated submodule N of M such that $x \in H_\rho(N)$.*

Proof. The statement holds if $\rho = 0$. We shall prove the result inductively by assuming that it holds for all ordinals $\sigma < \rho$. If $\rho = \sigma + 1$, then there exists an element $y \in H_\rho(M)$ such that $d\left(\frac{yR}{xR}\right) = 1$. By assumption, there is a countably generated submodule N such that $y \in H_\sigma(N)$. Now $x \in H_{\sigma+1}(N) = H_\rho(N)$ as $d\left(\frac{yR}{xR}\right) = 1$. Otherwise, if ρ is a limit ordinal, then $x \in H_\sigma(M)$ for all $\sigma < \rho$. Inductively, there exists a countable submodule N_σ for each $\sigma < \rho$, such that $x \in H_\sigma(N_\sigma)$. If $N = \sum N_\sigma$, then N is countably generated and $x \in \bigcap_{\sigma < \rho} H_\sigma(N_\rho) \subseteq \bigcap_{\sigma < \rho} H_\sigma(N) = H_\rho(N)$. \square

Lemma 3.10. *Let σ be a countable ordinal and N a submodule of a QTAG-module M such that $H_\sigma(M) \cap N = H_\sigma(N)$. If T/N is a countably generated submodule of M/N , then there exists a submodule $K \subseteq M$ such that $T \subseteq K$, K/N is countably generated and $H_\sigma(M) \cap K = H_\sigma(K)$.*

Proof. As

$$\frac{T \cap H_\sigma(M)}{H_\sigma(N)} \cong \frac{(T \cap H_\sigma(N)) + N}{N} \subseteq \frac{T}{N},$$

we may select elements $x_1, x_2, \dots, x_k, \dots$ such that $\frac{T \cap H_\sigma(M)}{H_\sigma(N)} = \Sigma x_i R$. By Lemma 3.9, for each k , there is a countably generated submodule N_k such that $x_k \in H_\sigma(N_k)$. Put $K_1 = T + \Sigma N_k$. Now $\frac{K_1}{N}$ is countably generated and $T \cap H_\sigma(M) \subseteq H_\sigma(K_1)$. On repeating this process, we may construct a chain of submodules $K_1 \subseteq K_2 \subseteq \dots$ such that $\frac{K_k}{N}$ is countably generated and $K_k \cap H_\sigma(M) \subseteq H_\sigma(K_{k+1})$. Now $K = \bigcup_{k=1}^{\infty} K_k$ is the required module. \square

Lemma 3.11. *Let σ be a countable ordinal and T a submodule of M such that $T = \bigoplus_{i \in I} T_i$, $\frac{M}{T} = \bigoplus_{j \in J} Q_j$ and each T_i and Q_j is countably generated. Suppose N is a submodule of M such that $N \cap T = \bigoplus_{i \in I'} T_i$, $\frac{N+T}{T} = \bigoplus_{j \in J'} Q_j$ and $H_\sigma(M) \cap N = H_\sigma(N)$. If $\frac{Q}{N}$ is a countably generated submodule of $\frac{M}{N}$, then there exists a submodule P of M such that $Q \subseteq P$, $\frac{P}{N}$ is countably generated, $P \cap T = \bigoplus_{i \in I''} T_i$, $\frac{P+T}{T} = \bigoplus_{j \in J''} Q_j$ and $H_\sigma(M) \cap P = H_\sigma(P)$.*

Proof. Since Q/N is a countably generated module, so $\frac{Q+T}{N+T}$ is also countably generated. Therefore there exists a subset $\bar{J} \subset J$ such that $\bigoplus_{j \in \bar{J}} Q_j$ contains a set of representatives of $\left(\frac{Q+T}{T}\right) / \left(\frac{N+T}{T}\right)$. If L is generated by a complete set of representatives of $\bigoplus_{j \in \bar{J}} Q_j$, then L is a countably generated submodule of M such that $\frac{L+T}{T} = \bigoplus_{j \in \bar{J}} Q_j$. If $P_1 = Q + L$ and $J_1 = J' \cup \bar{J}$, then $\frac{P_1}{N}$ is countably generated and $\frac{P_1+T}{T} = \bigoplus_{j \in J_1} Q_j$. Now by Lemma 3.10 and Lemma 3.6, there exist ascending chains of submodules $P_1 \subseteq P_2 \subseteq \dots \subseteq P_k \subseteq \dots$; $L_1 \subseteq L_2 \subseteq \dots \subseteq L_k \subseteq \dots$ and $K_1 \subseteq K_2 \subseteq \dots \subseteq K_k \subseteq \dots$ such that $N \subseteq P_k \subseteq L_k \subseteq K_k \subseteq P_{k+1}$, P_k/N is countably generated for each k , $\frac{P_k+T}{T} = \bigoplus_{j \in J_k} Q_j$, $L_k \cap T = \bigoplus_{i \in I_k} T_i$ and $H_\sigma(M) \cap K_k = H_\sigma(K_k)$. Now $P = \bigcup_{k=1}^{\infty} P_k$, $I'' = \bigcup_{k=1}^{\infty} I_k$ and $J'' = \bigcup_{k=1}^{\infty} J_k$ and we are done. \square

Now we are able to prove the following result.

Theorem 3.4. *If σ is a countable ordinal and if M is a h -reduced QTAG-module such that $M/H_\sigma(M)$ and $H_\sigma(M)$ are direct sums of countably generated modules, then M is a direct sum of countably generated modules.*

Proof. Let $\frac{M}{H_\sigma(M)} = \bigoplus_{j \in J} Q_j$ and $H_\sigma(M) = \bigoplus_{i \in I} P_i$, where each Q_j and P_i is countably generated. We may define a well ordering relation on the set of generators of M , $\{x_\alpha\}$, $\alpha < \beta$. By Lemma 3.11, we may construct a well ordered family of the submodules $\{N_\alpha\}$ of M satisfying the following conditions:

- (i) $N_0 = 0, N_\alpha \subseteq N_\beta$ if $\alpha < \beta$ and $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$, if α is a limit ordinal;
- (ii) $x_\alpha \in N_{\alpha+1}$;
- (iii) $N_\alpha \cap H_\sigma(M) = \bigoplus_{i \in I_\alpha} T_i$;
- (iv) $\frac{N_\alpha + H_\sigma(M)}{H_\sigma(M)} = \bigoplus_{j \in J_\alpha} Q_j$;
- (v) $H_\sigma(M) \cap N_\alpha = H_\sigma(N_\alpha)$;
- (vi) $N_{\alpha+1}/N_\alpha$ is countably generated.

Now $H_\sigma(N_{\alpha+1}) \cap N_\alpha = H_\sigma(N_\alpha)$ and $H_\sigma(N_\alpha) = \bigoplus_{i \in I_\alpha} T_i$, is a direct summand of $H_\sigma(N_{\alpha+1}) = \bigoplus_{i \in I_{\alpha+1}} T_i$. If we define the canonical map $f : \frac{N_{\alpha+1} + H_\sigma(M)}{H_\sigma(M)} \rightarrow \frac{N_{\alpha+1}}{H_\sigma(N_{\alpha+1})}$, then f is an isomorphism and $f \left(\frac{N_\alpha + H_\sigma(M)}{H_\sigma(M)} \right) = \frac{N_\alpha + H_\sigma(N_{\alpha+1})}{H_\sigma(N_{\alpha+1})}$. As $\frac{N_\alpha + H_\sigma(M)}{H_\sigma(M)}$ is a direct summand of $\frac{N_{\alpha+1} + H_\sigma(M)}{H_\sigma(M)}$, $\frac{N_\alpha + H_\sigma(N_{\alpha+1})}{H_\sigma(N_{\alpha+1})}$ is a direct summand of $\frac{N_{\alpha+1}}{H_\sigma(N_{\alpha+1})}$ and all the conditions of Theorem 3.2 are satisfied, therefore for each $\alpha < \beta$ we have $N_{\alpha+1} = N_\alpha \oplus N'_\alpha$. Thus $M = \bigoplus_{\alpha < \beta} N'_\alpha$ is a direct sum of countably generated modules. \square

Since h -pure, isotype and high submodules are very significant in the study of QTAG-modules. We prove some results related to these submodules.

Proposition 3.1. *For a countable ordinal σ and a QTAG-module M , if $M/H_\sigma(M)$ is a direct sum of countably generated modules, then every $H_\sigma(M)$ -high submodule of M is a direct sum of countably generated submodules.*

Proof. Suppose the statement holds for all the ordinal less than σ and $\sigma \geq \omega$. Now we put $\sigma = \rho + k$, where ρ is a limit ordinal and $k < \omega$. Let N be a $H_\sigma(M)$ -high submodule of M . Then

by Lemma 3.5 and Lemma 3.2,

$$M/N = H_\alpha(M/N) = \frac{H_\alpha(M) + N}{N} \text{ for all } \alpha < \rho.$$

Therefore,

$$\frac{N}{H_\alpha(N)} \cong \frac{H_\alpha(M) + N}{H_\alpha(M)} = \frac{M}{H_\alpha(M)} \cong \frac{(M/H_\alpha(M))}{H_\alpha(M/H_\sigma(M))}$$

is a direct sum of countably generated submodules for all $\alpha < \rho$. Now, by induction, every $H_\alpha(M)$ -high submodule of $N/H_\rho(N)$ is a direct sum of countably generated submodules. Since $\frac{N + H_\rho(M)}{H_\rho(M)} \cong \frac{N}{H_\rho(N)}$ and $\frac{N + H_\rho(M)}{H_\rho(M)}$ is isotype in the summable module $M/H_\rho(M)$, $N/H_\rho(N)$ is also summable.

Now by Theorem 3.3, $N/H_\rho(N)$ is a direct sum of countably generated modules, therefore by Theorem 3.4, N is a direct sum of countably generated modules. \square

Theorem 3.5. *Let N be an isotype submodule of a QTAG-module M of countable length and M , a direct sum of countably generated h -reduced QTAG-modules. Then N is also a direct sum of countably generated modules.*

Proof. Let σ be the length of N . As $\frac{N + H_\sigma(M)}{H_\sigma(M)}$ is isotype in $\frac{M}{H_\sigma(M)}$, without loss of generality, we assume that the length of M is also σ . Suppose the statement holds for all lengths $\rho < \sigma$. Now $\frac{N + H_\rho(M)}{H_\rho(M)}$ is isotype in $\frac{M}{H_\rho(M)}$ for all $\rho < \sigma$. Inductively, $\frac{N}{H_\rho(N)} \cong \frac{N + H_\rho(M)}{H_\rho(M)}$ is a direct sum of countably generated modules for all $\rho < \alpha$. If $\sigma = \rho + 1$, then $\frac{N}{H_\rho(N)}$ and $H_\rho(N)$ are the direct sums of countably generated modules and the result follows from Theorem 3.4.

Otherwise, if σ is a limit ordinal, $\frac{N}{H_\rho(N)}$ is a direct sum of countably generated submodules for all $\rho < \sigma$ and N is summable as it is isotype in the summable module M of countable length. Therefore, N is a direct sum of countably generated modules. \square

4. SOME APPLICATIONS

In this section we establish some interesting results using the results of section 3. We start with the following lemma:

Lemma 4.1. *Let N, K be $H_\sigma(M)$ -high submodules of M where $\sigma = \rho + n$, $n < \omega$. If $x + H_\rho(N) \rightarrow y + H_\rho(K)$ if and only if $x - y \in H_\rho(M)$, then this mapping is a height preserving isomorphism from $\text{Soc}(N/H_\rho(N))$ to $\text{Soc}(K/H_\rho(K))$.*

Proof. Let $x \in N$ and $x' \in H_\rho(N)$ such that $d\left(\frac{xR}{x'R}\right) = 1$. Suppose $x \notin K$, then $(K + xR) \cap H_\sigma(M) \neq 0$ and therefore there exists $y' \in K$ and $z \in H_\rho(M)$ such that $y' + x'' = z' \neq 0$. Here $d\left(\frac{xR}{x''R}\right) = i$, $d\left(\frac{zR}{z'R}\right) = n$. Since $K \cap H_\sigma(M) = 0$, $i \leq n$ and there is a $y'' \in K$ such that $d\left(\frac{y''R}{y'R}\right) = i$. By Lemma 3.4, $y'' + x + z'' \in H^i(M) \subseteq H^i(K) + H_\rho(M)$, where $d\left(\frac{z''R}{z'R}\right) = i$. Now $x = (u - y'') + z'' + v$ for some $u \in H^i(K)$ and $v \in H_\rho(M)$. Put $y = u - y''$. Now the map $x + H_\rho(N) \rightarrow y + H_\rho(K)$ is a height preserving isomorphism. \square

Theorem 4.1. *If a $H_\sigma(M)$ -high submodule of a QTAG-module M is a direct sum of countably generated modules, then every $H_\sigma(M)$ -high submodule of M is a direct sum of countably generated submodules.*

Proof. We may assume that σ is countable and $\sigma \geq \omega$. Now $\sigma = \rho + n$, where ρ is a limit ordinal and $n < \omega$. Let N, K be two $H_\sigma(M)$ -high submodules of M and N a direct sum of countably generated submodules. Now $N/H_\rho(N)$ is also a direct sum of countably generated modules and by Lemma 4.1, $K/H_\rho(K)$ is summable. As ρ is a limit ordinal, by Proposition 3.1, we have $N/H_\alpha(N) \cong M/H_\alpha(M) \cong K/H_\alpha(K)$ for all $\alpha < \rho$. Therefore $K/H_\rho(K)$ is a direct sum of countably generated modules. Now $H_\rho(K)$ is bounded, therefore by Theorem 3.4, K is a direct sum of countably generated modules. \square

Proposition 4.1. *For a QTAG-module M , if $M/H_\rho(M)$ is summable for all $\rho \leq \sigma$, where σ is countable then $M/H_\sigma(M)$ is a direct sum of countably generated modules.*

Proof. We assume that the result holds for all ordinals less than σ . If $\sigma = \alpha + 1$, then by induction we may say that $M/H_\alpha(M)$ is a direct sum of countably generated modules. Since $\frac{(M/H_\sigma(M))}{H_\alpha(M/H_\sigma(M))} \cong \frac{M}{H_\alpha(M)}$ and $H_\alpha(M/H_\alpha(M))$ are direct sums of countably generated modules, the result follow from Theorem 3.4. Otherwise, if σ is a limit ordinal, then by assumption

$$\frac{M/H_\sigma(M)}{H_\rho(M/H_\sigma(M))} \cong \frac{M}{H_\rho(M)}$$

is a direct sum of countably generated modules for each $\rho < \sigma$. Since $\frac{M}{H_\sigma(M)}$ is summable, $\frac{M}{H_\sigma(M)}$ is a direct sum of countably generated modules. \square

An immediate consequence of this proposition may be stated as follows:

Corollary 4.1. *Let σ be a countable limit ordinal and let M be a QTAG-module of length σ .*

Then the following conditions are equivalent:

- (i) *M is a direct sum of countably generated modules;*
- (ii) *M is summable and $\frac{M}{H_\rho(M)}$ is a direct sum of countably generated modules for all $\rho < \sigma$;*
- (iii) *$\frac{M}{H_\rho(M)}$ is summable for all $\rho \leq \sigma$;*
- (iv) *M is summable and for each $\rho < \sigma$, the $H_\rho(M)$ -high submodules of M are direct sums of countably generated modules.*

Ulm factors play a very important role in the study of QTAG-modules. In order to establish our next result involving *Ulm* factors we need the following lemma:

Lemma 4.2. *Let σ be a countable ordinal. Then for a QTAG-module M , $M/H_\rho(M)$ is a direct sum of countably generated modules for all $\rho < \omega\sigma$ if and only if $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules for all $\alpha < \sigma$. Here M^α is the α^{th} -Ulm submodule of M .*

Proof. If $\rho < \omega\sigma$, then $\rho = \omega\alpha + n$, $n < \omega$, $\alpha < \sigma$. Now we have

$$\frac{M}{M^\alpha} = \frac{M}{H_{\omega\alpha}(M)} \cong \frac{(M/H_\rho(M))}{H_{\omega\alpha}(M)/H_\rho(M)} = \frac{M/H_\rho(M)}{H_{\omega\alpha}(M/H_\rho(M))}.$$

Therefore, $H_{\omega\alpha}\left(\frac{M}{H_\rho(M)}\right)$ is bounded and if $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules, then by Theorem 3.4, $\frac{M}{H_\rho(M)}$ is a direct sum of countably generated modules. \square

Theorem 4.2. *Let M be a QTAG-module of countable length. Then M is a direct sum of countably generated modules if and only if*

- (i) *all Ulm factors of M are direct sums of uniserial modules and*
- (ii) *M/M^ρ is summable, for all limit ordinals ρ .*

Proof. The second condition implies that M is summable. Now, M will be a direct sum of countably generated modules if $M/H_\tau(M)$ is a direct sum of countably generated modules for all τ less than σ , the length of M . Assume that $\rho < \sigma$ and $\frac{M}{M^\tau}$ is a direct sum of countably generated modules for all $\tau < \rho$. In the light of Lemma 4.2, we have to show that $\frac{M}{M^\rho}$ is a direct sum of countably generated modules for all $\rho < \sigma$. If $\rho = \alpha + 1$, then

$$\frac{(M/M^\rho)}{H_{\omega\alpha}(M/M^\rho)} = \frac{(M/M^\rho)}{M^\alpha/M^\rho} \cong \frac{M}{M^\alpha}$$

is a direct sum of countably generated modules by induction. Now

$$H_{\omega\alpha} \left(\frac{M}{M^\rho} \right) = \frac{M^\alpha}{M^\rho} = \frac{M^\alpha}{M^{\alpha+1}}$$

is a direct sum of uniserial modules by (i). Therefore by applying Theorem 3.4, $\frac{M}{M^\rho}$ is a direct sum of countably generated modules. Otherwise, if ρ is a limit ordinal, then $\frac{M}{M^\rho} = \frac{M}{H_{\omega\rho}(M)}$ is summable by (ii). Inductively, $\frac{M}{M^\alpha}$ is a direct sum of countably generated modules for all $\alpha < \rho$. By Lemma 4.2, $M/H_\tau(M)$ is a direct sum of countably generated modules for all $\tau < \omega\rho$. Thus $\frac{M}{H_{\omega\rho}(M)} = \frac{M}{M^\rho}$ is a direct sum of countably generated modules. \square

As an immediate consequence of this result, we can say that a summable *QTAG*-module M of length ω is a direct sum of countably generated modules if and only if all of its *Ulm* factors are direct sums of uniserial modules.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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