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AN APPROXIMATION TO SECOND EXTERIOR DERIVATION OF HIGH ORDER UNIVERSAL MODULES

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Abstract. Let k be a commutative ring and assume that R is a commutative k-algebra. Let $\Omega_2(R)$ be the second order universal module of derivations of R. In this paper, we define the function $\Omega_2(R) \to \Lambda^2(\Omega_2(R))$ of second order exterior derivation and investigate the homological properties of $\Lambda^2(\Omega_2(R))$.

Keywords: exterior derivation; universal module; Kahler differentials.

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1. INTRODUCTION

In order to prove conclusions about algebraic sets and their coordinate rings, one of the methods is to study the universal module of differential operators. This ideas of studying the universal module may decrease questions about algebras to questions of module theory. The idea of using the universal module goes as far back as [6] which was proved some propeties of $\Omega_1(R)$. The universal modules of higher differential operators of an algebra were introduced by [10]. After on the like thought has appeared in [4] and [7]. During the recent years, subject of universal modules of high order differential operators has studied by [1]. In this study, it was

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acquired important outcomes about projective dimension of universal modules and he show that if $R=k[x_1,x_2,...,x_s]$ and I is an ideal of R generated by $f\in R$, then, $pd(\Omega_2(R/I))\leq 1$. Also in [1], the author has speak of exterior powers of universal derivation modules. In [3], it was indicated that, if R is an affine algebra then, there is a split exact sequence of R-modules. In [8], the authors were also remarked that some substantial results on the universal modules and in [9],has studied universal modules of finitely generated algebras. In [2], given any R-module B, there is an R-module $J_i(B)$ and a differential operator $\Delta_i : B\to J_i(B)$ of order i which is universal with regard to that for any R-module T and any differential operator $D : B\to T$ of order $\leq i$, there is a unique R-homomorphism $v : J_i(B)\to T$ such that, $D=v\Delta_i$. In [5], it was stated that a differential operator d of order i if $d(1)=0$. For a diversified consideration of a higher derivation you can see [5].

The purpose of this study is introduced second order exterior derivation on high order universal modules. We will construct the second order exterior derivation on high order universal modules of commutative ring extension R/k and show their fundamental properties. In particular our exact sequence of high order universal modules were not known related to regularity of the commutative rings. Throughout this paper, unless the contrary is stated explicitly, we will let R be a commutative algebra over an algebraically closed field k with charasteristic zero. When R is a k-algebra, $J_n(R/k)$ or $J_n(R)$ denotes the universal module of n-th order differentials of R over k and $\Omega^{(q)}(R/k)$ denotes the module of q-th order universal modules of R over k and $\delta^{(q)}_{R/k}$ or $\delta^{(q)}$ denotes the canonical q-th order k-derivation $R\to\Omega^{(q)}(R/k)$ of R. The pair $\left\{\Omega^{(q)}(R/k),\delta^{(q)}_{R/k}\right\}$ has the universal mapping property with respect to the q-th order k-derivations of R. $I_{R/k}$ or I_R denotes the kernel of the canonical mapping $R\otimes_k R\to R\,(a\otimes b\to ab)$. $\Omega^{(q)}(R/k)$ is identified with I_R/I_R^{q+1}. It ıs very well known that $J_n(R)\approxeq\Omega_n(R)+R$. $\Omega^{(q)}(R/k)$ is generated by the set $\left\{\delta^{(q)}(r) : r\in R\right\}$. Therefore, if R is finitely generated k-algebra, then $\Omega^{(q)}(R/k)$ will be a finitely produced R-module.

2. Preliminaries

Let M and N be R-modules. A bilinear map $\gamma : M\times M\to N$ is named alternating if $\gamma((m,m))=0$ for any $m\in M$. Let $M\otimes_R M$ be the tensor product of M with itself and let U be the submodule of $M\otimes_R M$ generated by the elements of the form $m\otimes m$ where $m\in M$. Bear in mind the following factor module

$$\Lambda^2(M) := (M \otimes_R M)/U$$

Definition 1. *The module $\Lambda^2(M)$ is said to be the second exterior power of M.*[1]

Lemma 1. *Let N be an R-module and $\gamma : M \times M \to N$ be a linear alternating map. Then,there exist an R-module homomorphism $f : \Lambda^2(M) \to N$ such that the following diagram*

$$\begin{array}{ccc} M \times M & \xrightarrow{\gamma} & N \\ \Lambda \searrow & & \nearrow f \\ & \Lambda^2(M) & \end{array}$$

commutes.[5]

Proposition 1. *Let M be an R-module and N be a submodule of M and that L_N is the submodule of $\Lambda^2(M)$ generated by the set*

$$\{m\Lambda n : m \in M \text{ and } n \in N\}.$$

Then,there is an R-module isomorphism

$$\Lambda^2(M)/L_N \approxeq \Lambda^2(M/N).[1]$$

Lemma 2. *Let K be a commutative k-algebra.We assume that, $\Omega_1(K)$ is the universal module of derivations of K with the universal derivation $\Delta : K \to \Omega_1(K)$. Then, the map*

$$D : \Omega_1(K) \to \Lambda^2(\Omega_1(K)),\ D\left(\sum_i a_i \Delta b_i\right) = \sum_i \Delta a_i \Lambda \Delta b_i$$

is a differential operator of order 1 on $\Omega_1(K)$ where $a_i, b_i \in K$. [11]

Proposition 2. *There is a split exact sequence of R-modules*

$$0 \to \Omega_2 \to J_1(\Omega_1) \to \Lambda^2(\Omega_1) \to 0.[3]$$

Proposition 3. *Assume that $J_m(\Omega_n(R))$ is the universal module of differential operators of order m on $\Omega_n(R)$ with the universal differential Δ_m. Then, there exist a unique R-module* *homomorphism* $\theta : \Omega_{m+n}(R) \to J_m(\Omega_n(R))$ *where* $\sum_i a_i d_{m+n}(b_i) \to \Delta_m(\sum_i a_i d_{m+n}(b_i))$ *such that the following diagram*

$$\begin{array}{ccc} R & \overset{d_n}{\to} & \Omega_n(R) \\ \downarrow d_{m+n} & & \downarrow \Delta_n \\ \Omega_{m+n}(R) & \overset{\theta}{\to} & J_m(\Omega_n(R)) \end{array}$$

commutes.[1]

Remark 1. *Let R be a k-algebra and* $\theta : \Omega_{m+n}(R) \to J_m(\Omega_n(R))$ *be an* $R-$*module homomorphism given in proposition 3. Then, we have an exact esquence of R-modules as follows*

$$0 \to \ker\theta \overset{i}{\to} \Omega_{m+n}(R) \overset{\theta}{\to} J_m(\Omega_n(R)) \overset{p}{\to} co\ker\theta \to 0$$

where i is the inlusion map and p is the natural surjection $J_m(\Omega_n(R)) \to J_m(\Omega_n(R))/\operatorname{Im}\theta$.[1]

Under favour of this test, the which condition $\ker\theta = 0$ and $co\ker\theta = 0$ is found in the following result.

Proposition 4. *Assume that R is a regular local ring with dimension s. Then,* θ *is injective.*[1]

Theorem 1. *Let I is the kernel of R-module homomorphism* $R \otimes_k R \to R$. *Then, there exist an R-module isomorphism* $\Omega_n(R) \simeq I/I^{n+1}$.[4]

Theorem 2. *If R regular and* $m = n = 1$, *then* $\Omega_{m+n}(R) \simeq J_m(\Omega_n(R))$ *with dimension* 1.[1]

Lemma 3. *Let* $R = k[x_1, x_2, ..., x_s]$ *be an affine k-algebra. Then, there is a short exact squence of R-modules*

$$0 \to \ker\theta_1 \to J_2(\Omega_2(R)) \overset{\theta_1}{\to} \Omega_2(R) \to 0.[12]$$

Lemma 4. *Let* $R = k[x_1, x_2, ..., x_s]$ *be an affine k-algebra. Then,*

$$\Omega_2(R) \overset{\Delta_2}{\to} J_2(\Omega_2(R)) \overset{\alpha}{\to} \Lambda^2(\Omega_2(R)) \to 0$$

is an exact sequence of R-modules.[12]

3. Main Conclusions

Throughout this study, R denotes a commutative algebra over an algebrically closed field k with characteristic zero. In this section, we inspired from Lemma 2 and we introduced a generalized version of Lemma 2 for $\Omega_2(R)$. Afterwards, we we define the function $\Omega_2(R) \to \Lambda^2(\Omega_2(R))$ of second order exterior derivation and investigate the homological properties of $\Lambda^2(\Omega_2(R))$.

Lemma 5. *Let R be a commutative k-algebra. Suppose that $\Omega_2(R)$ is the universal module of derivations of R with universal derivations $d_2 : R \to \Omega_2(R)$.Then, the map*

$$D_2 : \Omega_2(R) \to \Lambda^2(\Omega_2(R)), \sum_i a_i d_2(b_i) \to D_2\left(\sum_i a_i d_2(b_i)\right) = \sum_i d_2(a_i) \Lambda d_2(b_i)$$

is a differential operator of order 2 on $\Omega_2(R)$ where $a_i, b_i \in R$.

Proof. Let $R \otimes_k R \overset{\omega}{\to} \Lambda^2(\Omega_2(R))$ denote the k-linear map. $\omega\left(\sum_{i,j} a_i \otimes b_j\right) = \sum_{i,j} d_2(a_i) \Lambda d_2(b_j)$ and assume that $\sum_{i,j} a_i \otimes b_j \in I, \sum_{i,j} a_i^{'} \otimes b_j^{'} \in I, \sum_{i,j} a_i^{''} \otimes b_j^{''} \in I$. Here $\mu : R \otimes_k R \to R, \mu\left(\sum_{i,j} a_i \otimes b_j\right) = \sum_{i,j} a_i b_j$ and $I = \ker \mu$. Then, since $\sum_{i,j} a_i b_j = \sum_{i,j} a_i^{'} b_j^{'} = \sum_{i,j} a_i^{''} b_j^{''} = 0$, for $\ker \mu = I$, there exist a R-module homomorphism $\Psi : I/I^3 \to \Lambda^2(\Omega_2(R))$ such that the following diagram

$$\begin{array}{ccc} R \otimes R & \overset{\omega}{\to} & \Lambda^2(\Omega_2(R)) \\ \searrow & & \nearrow \Psi \\ & I/I^3 & \end{array}$$

commutes. Now, we must show that $\omega\left(I^3\right) = 0$. We know that

$$\sum_{i,j}\left(a_i a_i^{'} a_i^{''} \otimes b_j a_j^{'} b_j^{''}\right) \in I^3.$$

Then, we have

$$\omega\left(\sum aa^{'}a^{''}\otimes bb^{'}b^{''}\right)=\sum d_2\left(aa^{'}a^{''}\right)\Lambda d_2\left(bb^{'}b^{''}\right)$$
$$=\sum\left[\begin{array}{c}\left(ad_2\left(a^{'}a^{''}\right)+a^{'}d_2\left(aa^{''}\right)+a^{''}d_2\left(aa^{'}\right)-aa^{'}d_2\left(a^{''}\right)-aa^{''}d_2\left(a^{'}\right)-a^{'}a^{''}d_2\left(a\right)\right)\\ \Lambda\left(\left(bd_2\left(b^{'}b^{''}\right)+b^{'}d_2\left(bb^{''}\right)+b^{''}d_2\left(bb^{'}\right)-bb^{'}d_2\left(b^{''}\right)-bb^{''}d_2\left(b^{'}\right)-b^{'}b^{''}d_2\left(b\right)\right)\right)\end{array}\right]$$
$$\begin{aligned}&=\sum abd_2\left(a^{'}a^{''}\right)\Lambda d_2\left(b^{'}b^{''}\right)+\sum ab^{'}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(bb^{''}\right)+\sum ab^{''}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(bb^{'}\right)\\&-\sum abb^{'}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(b^{''}\right)-\sum abb^{''}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(b^{'}\right)-\sum ab^{'}b^{''}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(b\right)\\&+\sum a^{'}bd_2\left(aa^{''}\right)\Lambda d_2\left(b^{'}b^{''}\right)+\sum a^{'}b^{'}d_2\left(aa^{''}\right)\Lambda d_2\left(bb^{''}\right)+\sum a^{'}b^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(bb^{'}\right)\\&-\sum a^{'}bb^{'}d_2\left(aa^{''}\right)\Lambda d_2\left(b^{''}\right)-\sum a^{'}bb^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(b^{'}\right)-\sum a^{'}b^{'}b^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(b\right)\\&+\sum a^{''}bd_2\left(aa^{'}\right)\Lambda d_2\left(b^{'}b^{''}\right)+\sum a^{''}b^{'}d_2\left(aa^{'}\right)\Lambda d_2\left(bb^{''}\right)+\sum a^{''}b^{''}d_2\left(aa^{'}\right)\Lambda d_2\left(bb^{'}\right)\\&-\sum a^{''}bb^{'}d_2\left(aa^{'}\right)\Lambda d_2\left(b^{''}\right)-\sum a^{''}bb^{''}d_2\left(aa^{'}\right)\Lambda d_2\left(b^{'}\right)-\sum a^{''}b^{'}b^{''}d_2\left(aa^{'}\right)\Lambda d_2\left(b\right)\\&-\sum aa^{'}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}b^{''}\right)-\sum aa^{'}b^{'}d_2\left(a^{''}\right)\Lambda d_2\left(bb^{''}\right)-\sum aa^{'}b^{''}d_2\left(a^{''}\right)\Lambda d_2\left(bb^{'}\right)\\&+\sum aa^{'}bb^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)+\sum aa^{'}bb^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)+\sum aa^{'}b^{'}b^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)\\&-\sum aa^{''}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{'}b^{''}\right)-\sum aa^{''}b^{'}d_2\left(a^{'}\right)\Lambda d_2\left(bb^{''}\right)-\sum aa^{''}b^{''}d_2\left(a^{'}\right)\Lambda d_2\left(bb^{'}\right)\\&+\sum aa^{''}bb^{'}d_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)+\sum aa^{''}bb^{''}d_2\left(a^{'}\right)\Lambda d_2\left(b^{'}\right)+\sum aa^{''}b^{'}b^{''}d_2\left(a^{'}\right)\Lambda d_2\left(b\right)\\&-\sum a^{'}a^{''}bd_2\left(a\right)\Lambda d_2\left(b^{'}b^{''}\right)-\sum a^{'}a^{''}b^{'}d_2\left(a\right)\Lambda d_2\left(bb^{''}\right)-\sum a^{'}a^{''}b^{''}d_2\left(a\right)\Lambda d_2\left(bb^{'}\right)\\&+\sum a^{'}a^{''}bb^{'}d_2\left(a\right)\Lambda d_2\left(b^{''}\right)+\sum a^{'}a^{''}bb^{''}d_2\left(a\right)\Lambda d_2\left(b^{'}\right)+\sum a^{'}a^{''}b^{'}b^{''}d_2\left(a\right)\Lambda d_2\left(b\right).\end{aligned}$$

If we rearrange the terms for above identity, the expression is reduced to as below:

$$\omega\left(\sum aa^{'}a^{''}\otimes bb^{'}b^{''}\right)=\sum d_2\left(aa^{'}a^{''}\right)\Lambda d_2\left(bb^{'}b^{''}\right)$$
$$\begin{aligned}&=\sum ab^{'}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(bb^{''}\right)+\sum ab^{''}d_2\left(a^{'}a^{''}\right)\Lambda d_2\left(bb^{'}\right)-\sum a^{'}b^{'}b^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(b\right)\\&+\sum a^{'}bd_2\left(aa^{''}\right)\Lambda d_2\left(b^{'}b^{''}\right)+\sum a^{'}b^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(bb^{'}\right)-\sum a^{'}bb^{''}d_2\left(aa^{''}\right)\Lambda d_2\left(b^{'}\right)\\&+\sum a^{''}bd_2\left(aa^{'}\right)\Lambda d_2\left(b^{'}b^{''}\right)+\sum a^{''}b^{'}d_2\left(aa^{'}\right)\Lambda d_2\left(bb^{''}\right)-\sum a^{''}bb^{'}d_2\left(aa^{'}\right)\Lambda d_2\left(b^{''}\right)\\&-\sum aa^{'}b^{''}d_2\left(a^{''}\right)\Lambda d_2\left(bb^{'}\right)-\sum aa^{''}b^{'}d_2\left(a^{'}\right)\Lambda d_2\left(bb^{''}\right)-\sum a^{'}a^{''}bd_2\left(a\right)\Lambda d_2\left(b^{'}b^{''}\right).\end{aligned}$$

According to definition of exterir derivation, we construct all terms for previous equation as noted below

$$\begin{gathered}
\omega\left(\sum aa^{'}a^{''}\otimes bb^{'}b''\right)=\sum d_2\left(aa^{'}a^{''}\right)\Lambda d_2\left(bb^{'}b''\right)\\
=ab^{'}\left[\left(a^{'}d_2\left(a^{''}\right)-a^{''}d_2\left(a^{'}\right)\right)\Lambda\left(bd_2\left(b^{''}\right)-b^{''}d_2(b)\right)\right]\\
+ab^{''}\left[\left(a^{'}d_2\left(a^{''}\right)-a^{''}d_2\left(a^{'}\right)\right)\Lambda\left(bd_2\left(b^{'}\right)-b^{'}d_2(b)\right)\right]\\
-a^{'}b^{'}b^{''}\left[\left(ad_2\left(a^{''}\right)-a^{''}d_2(a)\right)\Lambda d_2(b)\right]\\
+a^{'}b\left[\left(ad_2\left(a^{''}\right)-a^{''}d_2(a)\right)\Lambda\left(b^{'}d_2\left(b^{''}\right)-b^{''}d_2\left(b^{'}\right)\right)\right]\\
+a^{'}b^{''}\left[\left(ad_2\left(a^{''}\right)-a^{''}d_2(a)\right)\Lambda\left(bd_2\left(b^{'}\right)-b^{'}d_2(b)\right)\right]\\
-a^{'}bb^{''}\left[\left(ad_2\left(a^{''}\right)-a^{''}d_2(a)\right)\Lambda d_2\left(b^{'}\right)\right]\\
+a^{''}b\left[\left(ad_2\left(a^{'}\right)-a^{'}d_2(a)\right)\Lambda\left(b^{'}d_2\left(b^{''}\right)-b^{''}d_2\left(b^{'}\right)\right)\right]\\
+a^{''}b^{'}\left[\left(ad_2\left(a^{'}\right)-a^{'}d_2(a)\right)\Lambda\left(bd_2\left(b^{''}\right)-b^{''}d_2(b)\right)\right]\\
-a^{''}bb^{'}\left[\left(ad_2\left(a^{'}\right)-a^{'}d_2(a)\right)\Lambda d_2\left(b^{''}\right)\right]\\
-aa^{'}b^{''}\left[d_2\left(a^{''}\right)\Lambda\left(bd_2\left(b^{'}\right)-b^{'}d_2(b)\right)\right]\\
-aa^{''}b^{'}\left[^{'}d_2\left(a^{'}\right)\Lambda\left(bd_2\left(b^{''}\right)-b^{''}d_2(b)\right)\right]\\
-a^{'}a^{''}b\left[d_2(a)\Lambda\left(b^{'}d_2\left(b^{''}\right)-b^{''}d_2\left(b^{'}\right)\right)\right].
\end{gathered}$$

Now, we will product the terms considering the exterior product, hence we have,

$$\begin{gathered}
\omega\left(\sum aa^{'}a^{''}\otimes bb^{'}b''\right)=\sum d_2\left(aa^{'}a^{''}\right)\Lambda d_2\left(bb^{'}b''\right)\\
=ab^{'}\left[a^{'}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)-a^{'}b^{''}d_2\left(a^{''}\right)\Lambda d_2(b)-a^{''}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-a^{''}b^{''}d_2\left(a^{'}\right)\Lambda d_2(b)\right]\\
+ab^{''}\left[a^{'}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{'}b^{'}d_2\left(a^{''}\right)\Lambda d_2(b)-a^{''}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{'}\right)+a^{''}b^{'}d_2\left(a^{'}\right)\Lambda d_2(b)\right]\\
-a^{'}b^{'}b^{''}\left[ad_2\left(a^{''}\right)\Lambda d_2(b)-a^{''}d_2(a)\Lambda d_2(b)\right]\\
+a^{'}b\left[ab^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)-ab^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{''}b^{'}d_2(a)\Lambda d_2\left(b^{''}\right)+a^{''}b^{''}d_2(a)\Lambda d_2\left(b^{'}\right)\right]\\
+a^{'}b^{''}\left[abd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-ab^{'}d_2\left(a^{''}\right)\Lambda d_2(b)-a^{''}bd_2(a)\Lambda d_2\left(b^{'}\right)+a^{''}b^{'}d_2(a)\Lambda d_2(b)\right]\\
-a^{'}bb^{''}\left[ad_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{''}d_2(a)\Lambda d_2\left(b^{'}\right)\right]\\
+a^{''}b\left[ab^{'}d_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-ab^{''}d_2\left(a^{'}\right)\Lambda d_2\left(b^{'}\right)-a^{'}b^{'}d_2(a)\Lambda d_2\left(b^{''}\right)+a^{'}b^{''}d_2(a)\Lambda d_2\left(b^{'}\right)\right]\\
+a^{''}b^{'}\left[abd_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-ab^{''}d_2\left(a^{'}\right)\Lambda d_2(b)-a^{'}bd_2(a)\Lambda d_2\left(b^{''}\right)+a^{'}b^{''}d_2(a)\Lambda d_2(b)\right]\\
-a^{''}bb^{'}\left[ad_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-a^{'}d_2(a)\Lambda d_2\left(b^{''}\right)\right]\\
-aa^{'}b^{''}\left[bd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-b^{'}d_2\left(a^{''}\right)\Lambda d_2(b)\right]\\
-aa^{''}b^{'}\left[bd_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-b^{''}d_2\left(a^{'}\right)\Lambda d_2(b)\right]\\
-a^{'}a^{''}b\left[b^{'}d_2(a)\Lambda d_2\left(b^{''}\right)-b^{''}d_2(a)\Lambda d_2\left(b^{'}\right)\right].
\end{gathered}$$

If we use some basic mathematical operations, we obtain

$$
\begin{gathered}
\omega\left(\sum aa^{'}a^{''}\otimes bb^{'}b^{''}\right)=\sum d_2\left(aa^{'}a^{''}\right)\Lambda d_2\left(bb^{'}b^{''}\right)\\
ab^{'}a^{'}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)-ab^{'}a^{'}b^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)-ab^{'}a^{''}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)-ab^{'}a^{''}b^{''}d_2\left(a^{'}\right)\Lambda d_2\left(b\right)\\
+ab^{''}a^{'}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-ab^{''}a^{'}b^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)-ab^{''}a^{''}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{'}\right)+ab^{''}a^{''}b^{'}d_2\left(a^{'}\right)\Lambda d_2\left(b\right)\\
-a^{'}b^{'}b^{''}ad_2\left(a^{''}\right)\Lambda d_2\left(b\right)+a^{'}b^{'}b^{''}a^{''}d_2\left(a\right)\Lambda d_2\left(b\right)\\
a^{'}bab^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)-a^{'}bab^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{'}ba^{''}b^{'}d_2\left(a\right)\Lambda d_2\left(b^{''}\right)+a^{'}ba^{''}b^{''}d_2\left(a\right)\Lambda d_2\left(b^{'}\right)\\
+a^{'}b^{''}abd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{'}b^{''}ab^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)-a^{'}b^{''}a^{''}bd_2\left(a\right)\Lambda d_2\left(b^{'}\right)+a^{'}b^{''}a^{''}b^{'}d_2\left(a\right)\Lambda d_2\left(b\right)\\
-a^{'}b^{'}b^{''}ad_2\left(a^{''}\right)\Lambda d_2\left(b\right)+a^{'}b^{'}b^{''}a^{''}d_2\left(a\right)\Lambda d_2\left(b\right)\\
+a^{'}bab^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b^{''}\right)-a^{'}bab^{''}d_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{'}ba^{''}b^{'}d_2\left(a\right)\Lambda d_2\left(b^{''}\right)+a^{'}ba^{''}b^{''}d_2\left(a\right)\Lambda d_2\left(b^{'}\right)\\
+a^{'}b^{''}abd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)-a^{'}b^{''}ab^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)-a^{'}b^{''}a^{''}bd_2\left(a\right)\Lambda d_2\left(b^{'}\right)+a^{'}b^{''}a^{''}b^{'}d_2\left(a\right)\Lambda d_2\left(b\right)\\
-a^{''}bb^{'}ad_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)+a^{''}bb^{'}a^{''}d_2\left(a\right)\Lambda d_2\left(b^{'}\right)\\
-aa^{'}b^{''}bd_2\left(a^{''}\right)\Lambda d_2\left(b^{'}\right)+aa^{'}b^{''}b^{'}d_2\left(a^{''}\right)\Lambda d_2\left(b\right)\\
-aa^{''}b^{'}bd_2\left(a^{'}\right)\Lambda d_2\left(b^{''}\right)+aa^{''}b^{'}b^{''}d_2\left(a^{'}\right)\Lambda d_2\left(b\right)\\
-a^{'}a^{''}bb^{'}d_2\left(a\right)\Lambda d_2\left(b^{''}\right)+a^{'}a^{''}bb^{''}d_2\left(a\right)\Lambda d_2\left(b^{'}\right)\\
=0.
\end{gathered}
$$

Therefore, we construct our main aim such that $\omega\left(I^3\right)=0$. □

Let $R\otimes_k R\overset{\omega}{\to}\Lambda^2\left(\Omega_2\left(R\right)\right)$ denote the k-linear map

$$\omega\left(\sum_i b_i\otimes a_i\right)=\sum_i d_2\left(b_i\right)\Lambda d_2\left(a_i\right)$$

and assume that $\sum_i b_i\otimes a_i\in J$ and $\sum_j b_j^{'}\otimes a_j^{'}\in J$. Then, since $\sum_i b_ia_i=\sum_j b_j^{'}a_j^{'}=0$, it follows that

$$\omega\left(\sum_{i,j}b_ib_j^{'}\otimes a_ia_j^{'}\right)=\left(\sum_{i,j}b_ia_j^{'}d_2\left(b_j^{'}\right)\Lambda d_2\left(a_i\right)+b_j^{'}a_id_2\left(b_i\right)\Lambda d_2\left(a_j^{'}\right)\right).$$

We know that,

$$\sum_i\left(b_id_2\left(a_i\right)+a_id_2\left(b_i\right)\right)=\sum_j b_j^{'}d_2\left(a_j^{'}\right)+a_j^{'}d_2\left(b_j^{'}\right)=0$$

the latter two terms cancel each other. Hence, $\omega J^2=0$ and one verifies directly that induced map

$$J/J^2\to\Lambda^2\left(\Omega_2\left(R\right)\right)$$

is the restriction to degree 2 of an exterior derivation of $\Lambda^2(\Omega_2(R))$.Let R be a k-algebra and $\Omega_2(R)$ be the universal module of second order derivation of R. Let $J_2(\Omega_2(R))$ be the universal module of differantial operators of order less than or equal to 2 on $\Omega_2(R)$ with the universal differantial operator $\Delta_2 : \Omega_2(R) \to J_2(\Omega_2(R))$. Recall from [1], there exist $\theta : \Omega_3(R) \to J_2(\Omega_2(R))$ such that the following diagram

$$\begin{array}{ccc} R & \overset{d_2}{\to} & \Omega_2(R) \\ \downarrow d_3 & & \downarrow \Delta_2 \\ \Omega_3(R) & \underset{\theta}{\to} & J_2(\Omega_2(R)) \end{array}$$

commutes. Let $D_2 : \Omega_2(R) \to \Lambda^2(\Omega_2(R))$ where $D_2(\sum_i a_i d_2(b_i)) = \sum_i d_2(a_i) \Lambda d_2(b_i)$. By the universal property of $J_2(\Omega_2(R))$, there is a R-module homomorphism

$$\beta : J_2(\Omega_2(R)) \to \Lambda^2(\Omega_2(R))$$

such that the following diagram

$$\begin{array}{ccccc} \Omega_2(R) & & \overset{D_2}{\to} & & \Lambda^2(\Omega_2(R)) \\ & \Delta_2 \searrow & & \nearrow \beta & \\ & & J_2(\Omega_2(R)) & & \end{array}$$

commutes.

Proposition 5. *Let S be an affine algebra presented by $S = R/I$ and $R = k[x_1, x_2, ..., x_s]$. Then, there is a split exact sequence*

$$\Omega_3(S) \overset{\theta}{\to} J_2(\Omega_2(S)) \overset{\beta}{\to} \Lambda^2(\Omega_2(S)) \to 0.$$

Proof. First of all we must show that β is surjective. Let R be a commutative algebra with unity and $\Omega_2(R)$ be the universal module of second order derivation of R. For $x_i, x_j, x_k, x_l \in R$, $m, n, t, r = 0, 1, 2$ and $1 \le i, j, k, l \le s$ the map

$$\tilde{A}_2 : \Omega_2(R) \to \Lambda^2(\Omega_2(R)), \text{ where } \tilde{A}_2\left(\sum_{i,j,k,l} x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \sum_{i,j,k,l} d_2\left(x_i^m x_j^n\right) \Lambda d_2\left(x_k^t x_l^r\right)$$

is a second order differential operator identify on $\Omega_2(R)$. Since universal property of $J_2(\Omega_2(R))$, there exist a unique an R-module homomorphism $\beta : J_2(\Omega_2(R)) \to \Lambda^2(\Omega_2(R))$ such that $\beta\Delta_2 =$

$\tilde{A}_2$ and the following diagram

$$\begin{array}{ccc} \Omega_2(R) & \stackrel{\widetilde{A_2}}{\rightarrow} & \Lambda^2(\Omega_2(R)) \\ \Delta_2 \searrow & & \nearrow \beta \\ & J_2(\Omega_2(R)) & \end{array}$$

commutes. Since

$$\begin{aligned} \beta\Delta_2 &= \left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) = \tilde{A}_2\left(x_i^m x_j^n d_2\left(x_k^t x_l^r\right)\right) \\ &= d_2\left(x_i^m x_j^n\right)\Lambda d_2\left(x_k^t x_l^r\right), \end{aligned}$$

β is surjective. Now, we shall show that $\operatorname{Im}\theta = \ker\beta$. We know that $\operatorname{Im}\theta$ generated by the set

$$\theta\left(d_3\left(x_i\right)\right), \theta\left(d_3\left(x_i x_j\right)\right), \theta\left(d_3\left(x_i x_j x_k\right)\right)$$

for $i, j, k = 1, 2, ..., s$. Therefore, we can write the following equality

$$\begin{gathered} d_2\left(x_i\right)\Lambda d_2\left(x_j x_k\right) + d_2\left(x_j\right)\Lambda d_2\left(x_i x_k\right) + d_2\left(x_k\right)\Lambda d_2\left(x_i x_j\right) + d_2\left(x_i x_j\right)\Lambda d_2\left(x_k\right) \\ + d_2\left(x_i x_k\right)\Lambda d_2\left(x_j\right) + d_2\left(x_j x_k\right)\Lambda d_2\left(x_i\right) = 0. \end{gathered}$$

Hence, we have

$$\beta\theta\left(d_3\left(x_i\right)\right) = \beta\left(\Delta_2\left(d_2\left(x_i\right)\right)\right) = D_2\left(d_2\left(x_i\right)\right) = d_2(1)\Lambda d_2\left(x_i\right) = 0$$

$$\beta\theta\left(d_3\left(x_i x_j\right)\right) = \beta\left(\Delta_2\left(d_2\left(x_i x_j\right)\right)\right) = D_2\left(d_2\left(x_i x_j\right)\right) = d_2(1)\Lambda d_2\left(x_i x_j\right) = 0$$

$$\begin{gathered} \beta\theta\left(d_3\left(x_i x_j x_k\right)\right) = \beta\left(\Delta_2\left(d_2\left(x_i x_j x_k\right)\right)\right) = D_2\left(d_2\left(x_i x_j x_k\right)\right) \\ = D_2\left[x_i d_2\left(x_j x_k\right) + x_j d_2\left(x_i x_k\right) + x_k d_2\left(x_i x_j\right) - x_i x_j d_2\left(x_k\right) - x_i x_k d_2\left(x_j\right) - x_j x_k d_2\left(x_i\right)\right] \end{gathered}$$

$$\begin{gathered} \beta\theta\left(d_3\left(x_i x_j x_k\right)\right) = \beta\left(\Delta_2\left(d_2\left(x_i x_j x_k\right)\right)\right) = D_2\left(d_2\left(x_i x_j x_k\right)\right) \\ = D_2\left[x_i d_2\left(x_j x_k\right) + x_j d_2\left(x_i x_k\right) + x_k d_2\left(x_i x_j\right) - x_i x_j d_2\left(x_k\right) - x_i x_k d_2\left(x_j\right) - x_j x_k d_2\left(x_i\right)\right] \\ = d_2\left(x_i\right)\Lambda d_2\left(x_j x_k\right) + d_2\left(x_j\right)\Lambda d_2\left(x_i x_k\right) + d_2\left(x_k\right)\Lambda d_2\left(x_i x_j\right) \\ - d_2\left(x_i x_j\right)\Lambda d_2\left(x_k\right) - d_2\left(x_i x_k\right)\Lambda d_2\left(x_j\right) - d_2\left(x_j x_k\right)\Lambda d_2\left(x_i\right) = 0. \end{gathered}$$

We have an induced map $p : J_2\left(\Omega_2(S)\right) / \operatorname{Im}\theta \rightarrow \Lambda^2\left(\Omega_2(S)\right)$ defined by

$$p\left(\Delta_2\left(x_i d_2\left(x_j x_k\right)\right)\right) = d_2\left(x_i\right)\Lambda d_2\left(x_j x_k\right).$$

Now suppose that $\Lambda^2 F$ and L_N are defined as [1] and $\Lambda^2(\Omega_2(S)) = S = L_N$. Here, $J_2(\Omega_2(S))/\operatorname{Im}\theta$ is generated by the set

$$\left\{\Delta_2\left(\overline{x_i d_2(x_j x_k)}\right) : 1 \le i < j < k \le s\right\}.$$

We can define a map

$$q : \Lambda^2 F \to J_2(\Omega_2(S))/\operatorname{Im}\theta,\ q\left(d_2(x^\alpha)\Lambda d_2\left(x^\beta\right)\right) = \Delta_2\left(x^\alpha d_2\left(x^\beta\right)\right).$$

Thus, if $\{f_k\}$ is generating set for I , then we have

$$\begin{aligned} q(d_2(f_k)\Lambda d_2(x^\alpha)) &= q\left(\sum_i \frac{\partial f_k}{\partial x_i} d_2(x_i)\Lambda d_2(x^\alpha)\right) \\ &= \sum_i \frac{\partial f_k}{\partial x_i}\Delta_2(x_i d_2(x^\alpha)) \\ &= \Delta_2(f_k d_2(x^\alpha)) \\ &= 0. \end{aligned}$$

Hence, $q(L_N) = 0$. Therefore, q induces an S-module homomorphism and

$$\overline{q} : \Lambda^2(F/L_N) \to J_2(\Omega_2(S))/\operatorname{Im}\theta, \overline{q}\left(\overline{d_2(x^\alpha)\Lambda d_2\left(x^\beta\right)}\right) = \overline{\Delta_2\left(x^\alpha d_2\left(x^\beta\right)\right)}.$$

It is evident that, $\overline{q}p$ and $p\overline{q}$ are the identitier. So, $\ker p = \ker\beta/\operatorname{Im}\theta$ and thus, $\ker\beta = \operatorname{Im}\theta$. □

Example 1. *Let $R = k[x,y]$ polynomial algebra and I be an ideal of R such that, generating by elements $f = y^2 - x^3$. Let $S = k[x,y]/(f)$. Here $\Omega_2(S) = F/L_N$ and F is a free S-module with basis*

$$\left\{d_2(x), d_2(y), d_2(xy), d_2\left(x^2\right), d_2\left(y^2\right)\right\}$$

and L_N is a submodule of $\Lambda^2(F)$ generated by the set

$$\{d_2(f), d_2(xf), d_2(yf)\}.$$

Since

$$d_2(f) = d_2\left(y^2\right) - 3x d_2\left(x^2\right) + 3x^2 d_2(x)$$

$$d_2(xf) = x d_2\left(y^2\right) - 6x^2 d_2\left(x^2\right) + 2y d_2(xy) + 7x^3 d_2(x) - 2xy d_2(y)$$

$$d_2(yf) = 3y d_2\left(y^2\right) - 3xy d_2\left(x^2\right) - 3x^2 d_2(xy) + 6x^2 d_2(x) - y^2 d_2(y)$$

and $rank\Omega_2(S) = \begin{pmatrix} 2+1 \\ 1 \end{pmatrix} - 1 = 2,$

$$rankL_N = rankF - rank\Omega_2(S) = 5 - 2 = 3.$$

Therefore, L_N *is a free S-module. Now,* $\Lambda^2(F)$ *is a free module with the basis*

$$\left\{ \begin{array}{c} d_2(x)\Lambda d_2(y), d_2(x)\Lambda d_2(xy), d_2(x)\Lambda d_2(x^2), d_2(x)\Lambda d_2(y^2), d_2(y)\Lambda d_2(xy), d_2(y)\Lambda d_2(x^2) \\ , d_2(y)\Lambda d_2(y^2), d_2(xy)\Lambda d_2(x^2), d_2(xy)\Lambda d_2(y^2), d_2(x^2)\Lambda d_2(y^2) \end{array} \right\}$$

and L_N *is a submodule of* $\Lambda^2(F)$ *generated by the set,*

$$\begin{aligned}
d_2(f)\Lambda d_2(x) &= d_2(y^2)\Lambda d_2(x) - 3xd_2(x^2)\Lambda d_2(x) \\
d_2(f)\Lambda d_2(y) &= d_2(y^2)\Lambda d_2(y) - 3xd_2(x^2)\Lambda d_2(y) + 3x^2 d_2(x)\Lambda d_2(y) \\
d_2(f)\Lambda d_2(xy) &= d_2(y^2)\Lambda d_2(xy) - 3xd_2(x^2)\Lambda d_2(xy) + 3x^2 d_2(x)\Lambda d_2(xy) \\
d_2(f)\Lambda d_2(x^2) &= d_2(y^2)\Lambda d_2(x^2) + 3x^2 d_2(x)\Lambda d_2(x^2) \\
d_2(f)\Lambda d_2(y^2) &= -3xd_2(x^2)\Lambda d_2(y^2) + 3x^2 d_2(x)\Lambda d_2(y^2) \\
d_2(xf)\Lambda d_2(x) &= xd_2(y^2)\Lambda d_2(x) - 6x^2 d_2(x^2)\Lambda d_2(x) + 2yd_2(xy)\Lambda d_2(x) - 2xyd_2(y)\Lambda d_2(x) \\
d_2(xf)\Lambda d_2(y) &= xd_2(y^2)\Lambda d_2(y) - 6x^2 d_2(x^2)\Lambda d_2(y) + 2yd_2(xy)\Lambda d_2(y) + 7x^3 d_2(x)\Lambda d_2(y) \\
d_2(xf)\Lambda d_2(xy) &= xd_2(y^2)\Lambda d_2(xy) - 6x^2 d_2(x^2)\Lambda d_2(xy) + 7x^3 d_2(x)\Lambda d_2(xy) - 2xyd_2(y)\Lambda d_2(xy) \\
d_2(xf)\Lambda d_2(x^2) &= xd_2(y^2)\Lambda d_2(x^2) + 2yd_2(xy)\Lambda d_2(x^2) + 7x^3 d_2(x)\Lambda d_2(x^2) - 2xyd_2(y)\Lambda d_2(x^2) \\
d_2(xf)\Lambda d_2(y^2) &= -6x^2 d_2(x^2)\Lambda d_2(y^2) + 2yd_2(xy)\Lambda d_2(y^2) + 7x^3 d_2(x)\Lambda d_2(y^2) - 2xyd_2(y)\Lambda d_2(y^2) \\
d_2(yf)\Lambda d_2(x) &= 3yd_2(y^2)\Lambda d_2(x) - 3xyd_2(x^2)\Lambda d_2(x) - 3x^2 d_2(xy)\Lambda d_2(x) - y^2 d_2(y)\Lambda d_2(x) \\
d_2(yf)\Lambda d_2(y) &= 3yd_2(y^2)\Lambda d_2(y) - 3xyd_2(x^2)\Lambda d_2(y) - 3x^2 d_2(xy)\Lambda d_2(y) + 6x^2 d_2(x)\Lambda d_2(y) \\
d_2(yf)\Lambda d_2(xy) &= 3yd_2(y^2)\Lambda d_2(xy) - 3xyd_2(x^2)\Lambda d_2(xy) + 6x^2 d_2(x)\Lambda d_2(xy) - y^2 d_2(y)\Lambda d_2(xy) \\
d_2(yf)\Lambda d_2(x^2) &= 3yd_2(y^2)\Lambda d_2(x^2) - 3x^2 d_2(xy)\Lambda d_2(x^2) + 6x^2 d_2(x)\Lambda d_2(x^2) - y^2 d_2(y)\Lambda d_2(x^2) \\
d_2(yf)\Lambda d_2(y^2) &= -3xyd_2(x^2)\Lambda d_2(y^2) - 3x^2 d_2(xy)\Lambda d_2(y^2) + 6x^2 d_2(x)\Lambda d_2(y^2) - y^2 d_2(y)\Lambda d_2(y^2).
\end{aligned}$$

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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