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SIGMA BOUNDED SEQUENCE AND SOME MATRIX TRANSFORMATIONS

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Abstract. The object of this paper is to investigate some classes of infinite matrices, *i.e.*, $(l_\infty(p,s), v^\sigma)$ and $(l_\infty(p,s), v^\sigma_\infty)$, where v^σ is the space of all bounded sequences all of whose σ- means are equal, v^σ_∞ is the space of σ-bounded sequence and the space $l_\infty(p,s)$ have been defined and studied by T. Jalal and Z. U. Ahmad [5].

Keywords: Invariant means, Infinite matrices, Matrix transformations.

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Preliminaries, background and Notation: A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let ω denote the space of all sequences (real or complex); l_∞ and c respectively, denotes the space of all bounded sequences, the space of convergent sequences. Also, by cs, l_1 and $l(p)$ we denote the spaces of all convergent, absolutely and p-absolutely convergent series, respectively. Also, by f we denote the set of almost convergent sequences.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional Φ on l_∞ is said to be an invariant mean or a σ-mean if and only if (i) $\Phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n; (ii) $\Phi(e) = 1$, where e={1,1,1,.....}; and (iii) $\Phi(x_{\sigma(n)}) = \Phi(x)$ for all $x \in l_\infty$. Through out this paper, we deal only with mappings σ as one to one and are such that $\sigma^m(n) \neq n$, for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. If σ is the

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translation mapping $n \to n+1$, a σ mean is often called a Banach limit (see, [1, 3-5]). If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown (see[12]) that

$$v^\sigma = \left\{ \ x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) = L \text{ uniformly in } n, \ L = \sigma - \lim x \right\},$$

where,

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} T^j x_n, \ T^j x_n = x_{\sigma^j(n)}, \ t_{-1,n}(x) = 0.$$

We define v_∞^σ the space of $\sigma-$bounded sequences (see [9]) in the following wa:

$$v_\infty^\sigma = \{x \in w : \sup_{m,n} |\Phi_{m,n}(x)| < \infty\},$$

where,

$$\Phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$$

$$= \frac{1}{m(m+1)}\sum_{j=1}^{m} j(T^j x_n - T^{j-1}x_n). \tag{1}$$

If $\sigma(n) = n+1$, then v_∞^σ is the set of almost bounded sequences f_∞ (see, [2, 3, 8, 10-14]). The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., (see, [1, 2, 7-9, 13]). In [2], Jalal and Ahmad [5] have defined the space $l_\infty(p, s)$ and characterized the classes $(l_\infty(p, s), l_\infty)$ and $(l_\infty(p, s), f)$. The object of this paper is to characterize the classes of matrices $(l_\infty(p, s), v^\sigma)$ and $(l_\infty(p, s), v_\infty^\sigma)$, where the space $l_\infty(p, s)$ is defined as follows:

$$l_\infty(p, s) = \left\{ x : \sup_k k^{-s}|x_k|^{p_k} < \infty, \ s \geq 0 \right\}.$$

1. Some matrix transformations

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$ we mean the characterizations of matrices from X to Y *i.e.*, $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

We note that, if Ax is defined, then it follows from (1) that, for all integers $n, m \geq 0$

$$\Phi_{m,n}(Ax) = \sum_k \chi(n,k,m)x_k$$

where

$$\chi(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j\{a(\sigma^j(n),k) - a(\sigma^{j-1}(n),k)\}$$

Theorem 2.1: Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k, then $A \in (l_\infty(p,s), v_\infty^\sigma)$ if and only if there exists an integer $N > 1$ such that

$$\sup_{m,n} \sum_k |\chi(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} < \infty. \tag{2}$$

Proof: Let $A \in (l_\infty(p,s), v_\infty^\sigma)$ and that $x \in l_\infty(p,s)$. Put

$$q_n(x) = \sup_m |\Phi_{mn}(Ax)|.$$

For $n > 0$, q_n is continuous semi-norm on $l_\infty(p,s)$ and (q_n) is pointwise bounded on $l_\infty(p,s)$. Suppose that (2.1) is not true. Then there exists $x \in l_\infty(p,s)$ with

$$\sup_n q_n(x) = \infty.$$

By the principle of condensation of singularities (see[15]), the set

$$\left\{ x \in l_\infty(p,s) : \sup_n q_n(x) = \infty \right\}$$

is of second category in $l_\infty(p,s)$ and hence nonempty i.e.,there is $x \in l_\infty(p,s)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that q_n is pointwise bounded on $l_\infty(p,s)$. Now, by Uniform bounded principle, there is constant M such that

(3) $$q_n(x) \leq Mg(x)$$

Applying (3) to the sequence $x = (x_k)$ defined as in [5] by replacing $a_{nk}(i)$ by $a(n,k,m)$, we then obtain the necessity of (2).

Sufficiency. Let (2) holds and $x \in l_\infty(p,s)$. Using the following inequality

$$|ab| \leq C(|a|^q C^{-q} + |b|^p)$$

for $C > 0$ and a, b two complex numbers ($p > 1$ and $p^{-1} + q^{-1} = 1$) (see [7, 15]),we have

$$\begin{aligned} |\Phi_{m,n}(Ax)| &= \left|\sum_k \chi(n,k,m)x_k\right| \\ &\leq \sum_k |\chi(n,k,m)x_k| \\ &\leq \sum_k N[|\chi(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} + |x_k|^{p_k} k^{\frac{-s}{p_k}}]. \end{aligned}$$

Taking the supremum over m, n and using (2.2) we get $Ax \in v_\infty^\sigma$ for $x \in l_\infty(p,s)$. i.e, $A \in (l_\infty(p,s), v_\infty^\sigma)$. This completes the proof of the theorem.

Theorem 2.2: Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k, then $A \in (l_\infty(p,s), v^\sigma)$ if and only if there exists an integer $N > 1$ such that

$$(i)\ \sup_{m,n} \sum_k |t(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} < \infty,$$

$$(ii)\ \lim_m t(n,k,m) = a_k \text{ uniformly in } n, \text{ for every } k.$$

Necessity: Let $A \in (l_\infty(p,s), v^\sigma)$ and that $x \in l_\infty(p,s)$. Write $q_n(x) = \sup_m |t_{mn}(Ax)|$. It is easy to see that for $n \geq 0,\ q_n$ is continuous semi-norm on $l_\infty(p,s)$ and q_n is pointwise bounded on $l_\infty(p,s)$. Suppose that (i) is not true. Then there exists $x \in l_\infty(p,s)$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities [15], the set

$$\left\{ x \in l(p,s) : \sup_n q_n(x) = \infty \right\}$$

is of second category in $l_\infty(p,s)$ and hence non empty i.e, there is $x \in l_\infty(p,s)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that (q_n) is pointwise bounded on $l_\infty(p,s)$. Now by Banach-Steinhauss theorem, there is constant M such that

$$q_n(x) \leq Mg(x) \tag{4}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} (sgn\ t(n,k,m))k^{\frac{s}{p_k}} N^{\frac{-1}{p_k}}, & 1 \leq k \leq k_0 \\ 0 & ,\ k > k_0 \end{cases}$$

Then it is easy to see that $x \in l(p,s)$. Applying this sequence to (4) we get the condition (i). Since $e_k \in l_\infty(p,s)$, condition (ii) follows immediately by taking $x = e_k$.

Sufficiency. Let (i) and (ii) hold and $x \in l_\infty(p,s)$. For $j \geq 1$

$$\sum_{k=1}^{j} |t(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} \leq \sup_m \sum_k |t(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} < \infty \text{ for every } n.$$

Therefore,

$$\begin{aligned} \sum_k |\alpha_k|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} &= \lim_j \lim_m \sum_{k=1}^{j} |t(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} \\ &\leq \sup_m \sum_k |t(n,k,m)|^{q_k} k^{\frac{s}{p_k}} N^{\frac{1}{p_k}} < \infty. \end{aligned}$$

Consequently the series $\sum_k t(n,k,m)x_k$ and $\sum_k \alpha_k x_k$ converges for every n, m and for every $x \in l_\infty(p,s)$.

Now for $\epsilon > 0$ and $x \in l_\infty(p, s)$. Choose $k_0 \in N$ such that

$$\sum_{k \geq k_0+1} |x_k|^{p_k} k^{\frac{-s}{p_k}} < 1.$$

By condition (ii), there exits m_0 such that

$$\left| \sum_{k=1}^{k_0} [t(n,k,m) - \alpha_k] \right| < \infty$$

for every $m > m_0$. By condition (i), it follows that

$$\left| \sum_{k \geq k_0+1} [t(n,k,m) - \alpha_k] \right|$$

is arbitrarily small. Therefore

$$\lim_m \sum_k t(n,k,m)x_k = \sum_k \alpha_k x_k \text{ uniformly in } n.$$

Hence $A \in (l_\infty(p, s), v^\sigma)$

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