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A FIXED POINT THEOREM IN NON ARCHIMEDEAN T_0-QUASI-METRIC SPACES

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Abstract. In this article we prove the existence of unique fixed points for generalized contractive mappings in q-spherically complete T_0-ultra-quasi-metric spaces.

Keywords: q-spherical completeness; T_0-ultra-quasi-metric space; fixed point.

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1. Introduction

In [6], Petalas et al. proved that every contractive mapping on a spherically complete non-Archimedean normed space has a unique fixed point. In this paper we shall prove that every generalized contractive mapping on a q-spherically complete T_0-ultra-quasi-metric space has a unique fixed point. The concept of q-spherical completeness has been studied for T_0-ultra-quasi-metric spaces by Künzi and Otafudu in [2].

For recent results in the area of Asymmetric Topology, the reader is adviced to consult [3, 4, 5].

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2. Preliminaries

In this section we recall some of the basic definitions from asymmetric topology required in order to follow this paper.

Definition 2.1.(Compare [2, page 2]) Let X be a set and $d : X \times X \to [0,\infty)$ be a function mapping into the set $[0,\infty)$ of non-negative reals. Then d is an **ultra-quasi-pseudometric** on X if

(a) $d(x,x) = 0$ for all $x \in X$, and

(b) $d(x,z) \leq max\{d(x,y), d(y,z)\}$ whenever $x,y,z \in X$.

The conjugate d^{-1} of d where $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is also an ultra-quasi-pseudometric on X.

If d also satisfies the following condition (known as the T_0-condition):

(c) for any $x,y \in X$, $d(x,y) = 0 = d(y,x)$ implies that $x = y$, then d is called a **T_0-ultra-quasi-metric** on X. Notice that $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is an **ultra metric** on X.

In the literature, T_0-ultra-quasi-metric spaces are also know as non Archimedean T_0-quasi-metric spaces. The set of open balls $\{\{y \in X : d(x,y) < \varepsilon\} : x \in X, \varepsilon > 0\}$ yields a base for the topology $\tau(d)$ induced by d on X.

Example 2.2.(Compare [7, Example 3]) Let $X = [0,\infty)$. Define for each $x,y \in X$, $n(x,y) = x$ if $x > y$, and $n(x,y) = 0$ if $x \leq y$. It is not difficult to check that (X,n) is a T_0-ultra-quasi-metric space.

Notice also that for $x,y \in [0,\infty)$, we have $n^s(x,y) = \max\{x,y\}$ if $x \neq y$ and $n^s(x,y) = 0$ if $x = y$. The ultra metric n^s is complete on $[0,\infty)$ since n and n^{-1} are complete on $[0,\infty)$ (compare [2, Example 2]).

Furthermore 0 is the only non-isolated point of $\tau(n^s)$. Indeed
$A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is a compact subspace of $([0,\infty), n^s)$.

Definition 2.4.([2, page 3]) A map $f : X \to Y$ between two (ultra-) quasi-pseudometric spaces (X,d_X) and (Y,d_Y) is called **contractive** provided that $d_Y(f(x), f(y)) < d_X(x,y)$ whenever $x,y \in X$.

Definition 2.3.A map $f : X \to Y$ between two (ultra-) quasi-pseudometric spaces (X, d_X) and (Y, d_Y) is said to be a generalized contractive map provided that for each $x, y \in X$ with $d(x, y) > 0$, we have that

$$d_Y(f(x), f(y)) < \max\{d_X(x, y), d_X(f(x), x), d_X(y, f(y)\}.$$

3. q-Spherical Completeness

In this section we shall recall some results about q-spherical completeness belonging mainly to [2].

Let (X, d) be an ultra-quasi-pseudometric space. Let $x \in X$ and
$r \in [0, \infty)$. By $C_d(x, r)$ we mean the closed ball

$$C_d(x, r) = \{y \in X : d(x, y) \leq r\}$$

of radius r around x.

Lemma 3.1.(Compare [2, Lemma 9]) If (X, d) is an ultra-quasi-pseudometric space and $x, y \in X$ and $r, s \in [0, \infty)$, then we have that

$$C_d(x, r) \cap C_{d^{-1}}(y, s) \neq \emptyset$$

if and only if

$$d(x, y) \leq \max\{r, s\}.$$

Definition 3.2.(Compare [2, Definition 2]) Let (X, d) be an ultra-quasi-pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative real numbers. We shall say that the family $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ has the mixed binary intersection property provided that

$$d(x_i, x_j) \leq \max\{r_i, s_j\}$$

whenever $i, j \in I$.

We say that (X,d) is q-spherically complete provided that each family $(C_d(x_i,r_i),C_{d^{-1}}(x_i,s_i))_{i\in I}$ possessing the mixed binary intersection property also satisfies
$\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset$.

For an example of a q-spherically complete ultra-quasi-metric space, the reader is adviced to check [2, Example 2].

Proposition 3.3.(Compare [2, Proposition 2])

(a) Let (X,d) be an ultra-quasi-pseudometric space. Then (X,d) is q-spherically complete if and only if (X,d^{-1}) is q-spherically complete.

(b) Let (X,d) be a T_0-ultra-quasi-metric space. If (X,d) is q-spherically complete, then (X,d^s) is spherically complete.

Definition 3.4. An ultra-quasi-pseudometric space (X,d) is called bicomplete provided that the ultra-pseudometric d^s on X is complete.

Proposition 3.5.(Compare [2, Proposition 3]) Each q-spherically complete T_0-ultra-quasi-metric space (X,d) is bicomplete.

3. Main results

Theorem 3.1.(Compare [1, Theorem 1]) Let (X,d) be a q-spherically complete T_0-ultra-quasi-metric space. If $f:X\to X$ is a generalized contractive mapping, then f has a unique fixed point.

Proof.

Let $a\in X$ and denote by

$$C_a^d=C_d(a,d(f(a),a))\text{ and }C_a^{d^{-1}}=C_{d^{-1}}(a,d(a,f(a)))$$

the closed balls with centers at $a\in X$ and radii $d(f(a),a)$ and $d(a,f(a))$ respectively such that $d(a,f(a))=d(f(a),a)$. Put

$$C_a=C_a^d\cap C_a^{d^{-1}}.$$

Let $\mathscr{A}$ be the collection of all such closed balls C_a such that a runs over X. Define $\preceq$ on $\mathscr{A}$ by

$$C_a \preceq C_b \text{ if and only if } C_b \subseteq C_a.$$

Then $(\mathscr{A}, \preceq)$ is a partially ordered set. We leave the verification of this fact to the reader.

Let $\mathscr{A}_1$ be a nonempty chain in $\mathscr{A}$. Then by q-spherical completeness of (X,d), we have that

$$\bigcap_{C_a \in \mathscr{A}_1} C_a = C \neq \emptyset.$$

Let $b \in C$ and $C_a \in \mathscr{A}_1$. Then we have

$$d(a,b) \leq d(f(a),a) \text{ and } d(b,a) \leq d(a, f(a)).$$

Let now $x \in C_b$. Then

$$d(b,x) \leq d(f(b),b) \text{ and } d(x,b) \leq d(b, f(b)).$$

$$\begin{aligned} d(b,x) &\leq d(f(b),b) \\ &\leq \max\{d(f(b), f(a)), d(f(a),a), d(a,b)\} \\ &= \max\{d(f(b), f(a)), d(f(a),a)\} \end{aligned}$$

If $d(f(b), f(a)) \leq d(f(a),a)$, then we have

$$d(b,x) \leq d(f(a),a).$$

If on the other hand we have $d(f(b), f(a)) > d(f(a),a)$, then

$$d(b,x) < \max\{d(f(b),b), d(a, f(a))\} = d(a, f(a))$$

Thus in both cases, we have

$$d(b,x) \leq d(f(a),a).$$

From the above inequality, we have now that

$$\begin{aligned} d(a,x) &\leq \max\{d(a,b),d(b,x)\} \\ &\leq \max\{d(f(a),a),d(f(a),a)\} \\ &= d(f(a),a) \end{aligned}$$

which means that $x \in C_d(a,d(f(a),a))$. We have thus shown that

$$C_d(b,d(f(b),b)) \subseteq C_d(a,d(f(a),a)). \tag{1}$$

By a similar computation, one can show that

$$C_{d^{-1}}(b,d(b,f(b))) \subseteq C_{d^{-1}}(a,d(a,f(a))). \tag{2}$$

By Equations (1) and (2), we have that for all $C_a \in \mathscr{A}_1$, $C_b \subseteq C_a$. But this just means that $C_a \preceq C_b$ for all $C_a \in \mathscr{A}_1$. Thus C_b is an upper bound in $\mathscr{A}$ for the chain $\mathscr{A}_1$. We therefore appeal to Zorn's lemma to conclude that $\mathscr{A}$ has a maximal element, say, C_u, $u \in X$. We claim that $f(u) = u$.

Suppose on the contrary that $d(u,f(u)) > 0$.
Let $y \in C_{f(u)}$, then

$$d(f(u),y) \leq d(f(f(u)),f(u)) < d(u,f(u))$$

and

$$d(y,f(u)) \leq d(f(u),f(f(u))) < d(f(u),u).$$

$$\begin{aligned} d(y,u) &\leq \max\{d(y,f(u),d(f(u),u)\} \\ &< \max\{d(u,f(u),d(f(u),u)\} \\ &= d(u,f(u)) \end{aligned}$$

Similarly, we can prove that $d(u,y) \leq d(f(u),u)$.

The last two inequalities imply that $y \in C_u$. Therefore $C_{f(u)} \subseteq C_u$.

Indeed, we have that $u \notin C_{f(u)}$. This follows from the following two inequalities:

$$\begin{aligned} d(f(u), f(f(u))) &< \max\{d(f(u),u), d(u, f(u)), d(f(u), f(f(u)))\} \\ &= d(f(u),u) \end{aligned}$$

and

$$\begin{aligned} d(f(f(u)), f(u)) &< \max\{d(f(f(u)), f(u)), d(f(u),u), d(u, f(u))\} \\ &= d(u, f(u)) \end{aligned}$$

This however contradicts the maximality of C_u. Hence we must have that $f(u) = u$.

We shall now prove uniqueness.

Suppose that there is another fixed point, i.e., there exists $z \in X$ such that $f(z) = z$. We shall examine two cases.

Case 1: Suppose $d(z,u) > 0$. Then we have that

$$d(z,u) = d(f(z), f(u)) < \max\{d(f(z),z), d(z,u), d(u, f(u)) = d(z,u),$$

which is a contradiction.

Case 2: Suppose now that $d(u,z) > 0$. Then we get

$$d(u,z) = d(f(u), f(z)) < \max\{d(f(u),u), d(u,z), d(z, f(z)) = d(u,z),$$

which is a contradiction. Thus we must have that $z = u$.

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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