Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 1, 41-47
ISSN: 1927-6303

GENERALIZATION OF SOME FIXED POINT THEOREMS IN ULTRAMETRIC SPACES

S. N. MISHRA[1] AND RAJENDRA PANT[2,*]

[1]Department of Mathematics, Walter Sisulu University, Mthatha 5117, South Africa

[2]Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur 440010, India

Abstract. In this note, we obtain a fixed point theorem for generalized contractive mappings in an untrametric space which generalizes some known results.

2000 AMS Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

A metric space (X,d) is said to be an ultrametric space if the triangle inequality is replaced by the strong triangle inequality, i.e.,

$$d(x,y) \leq \max\{d(x,z), d(y,z)\}$$

for all $x,y,z \in X$.

Example 1. [1]. *Every discrete metric space is an ultrametric space.*

An ultrametric space (X,d) is said to be spherically complete if every descending collection of closed balls in X has a nonempty intersection. For details we refer to Khamsi and Kirk [3].

*Corresponding author

Received July 17, 2011

Definition 1. *A self-mapping T of a metric (resp. an ultrametric) space X is said to be* contractive (or strictly contractive) mapping *if*

$$d(Tx,Ty) < d(x,y)$$

for all $x,y \in X$ with $x \neq y$.

It is well-known that a contractive mapping of a complete metric space need not have a fixed point.

Example 2. [6]. *Let $X = (-\infty, -\infty)$ endowed with the usual metric and $T : X \to X$ defined by*

$$Tx = x + \frac{1}{1+e^x}$$

for all $x \in X$. Notice that X is complete and T is a contractive mapping but T does not have a fixed point.

Definition 2. *A self-mapping T of a metric (resp. an ultrametric) space X is said to be* generalized contractive mapping *if*

$$d(Tx,Ty) < M(x,y) \tag{1.1}$$

for all $x,y \in X$ with $x \neq y$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

We remark that the condition (1.1) is considered as one of the most general contractive conditions listed in Rhoades [7].

In [6], Petalas and Vidalis obtained the following fixed point theorem:

Theorem 1. *Let (X,d) be a spherically complete ultrametric space and $T : X \to X$ a contractive mapping. Then T has a unique fixed point.*

In [1], Gajić obtained the following generalization of the above theorem:

Theorem 2. *Let (X,d) be a spherically complete ultrametric space and $T : X \to X$ a mapping such that for all $x, y \in X, x \neq y$,*

$$d(Tx,Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{1.2}$$

Then T has a unique fixed point.

In this note we obtain a generalization of Theorem 2 and which, in turn, generalizes Theorem 1.

2. Results

Our main result, Theorem 3, is prefaced by the following Lemma.

Lemma 1. *Let X be an ultrametric space and $T : X \to X$ a generalized contractive mapping. Then for all $a, b \in X$,*

$$d(Ta,Tb) < \max\{d(a,b), d(a,Ta), d(b,Tb)\}.$$

Proof. Since T is a generalized contractive mapping, we have

$$d(Ta,Tb) < \max\{d(a,b), d(a,Ta), d(b,Tb), d(a,Tb), d(b,Ta)\}. \tag{2.1}$$

Now by the strong triangle inequality, we have

$$d(a,Tb) \leq \max\{d(a,b), d(b,Tb)\}. \tag{2.2}$$

Similarly, we have

$$d(b,Ta) \leq \max\{d(b,a), d(a,Ta)\}. \tag{2.3}$$

By (2.1)-(2.3), we conclude that

$$d(Ta,Tb) < \max\{d(a,b), d(a,Ta), d(b,Tb)\}.$$

□

Now we present our main result.

Theorem 3. *Let (X,d) be a spherically complete ultrametric space and $T : X \to X$ a generalized contractive mapping. Then T has a unique fixed point.*

Proof. Let $B_a =: B(a, d(a,Ta))$ denote the closed sphere centered at a with radius $d(a,Ta)$, and let $\mathscr{A}$ be the collection of these spheres for all $a \in X$. Then the relation

$$B_a \leq B_b \text{ iff } B_b \subseteq B_a$$

is a partial order. Let $\mathscr{A}_1$ be a totally ordered subfamily of $\mathscr{A}$. From the spherical completeness of X, we have

$$\bigcap_{B_a \in \mathscr{A}_1} B_a =: B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathscr{A}_1$. Then if $x \in B_b$,

$$\begin{aligned} d(x,b) \leq d(b,Tb) &\leq \max\{d(b,a), d(a,Ta), d(Ta,Tb)\} \\ &= \max\{d(a,Ta), d(Ta,Tb)\}. \end{aligned}$$

Since $d(a,b) \leq d(a,Ta)$, the above inequality reduces to

$$d(x,b) \leq \max\{d(a,Ta), d(Ta,Tb)\}.$$

Now two cases arise.

Case I: $d(Ta,Tb) \leq d(a,Ta)$. Then

$$d(x,b) \leq d(a,Ta).$$

Case II: $d(Ta,Tb) > d(a,Ta)$. Then

$$d(x,b) \leq d(b,Tb) \leq d(Ta,Tb).$$

By Lemma 1, the above inequality will lead to

$$\begin{aligned} d(x,b) \leq d(b,Tb) &\leq d(Ta,Tb) \\ &< \max\{d(b,a), d(a,Ta), d(b,Tb)\} \\ &= \max\{d(a,Ta), d(b,Tb)\} \\ &= d(a,Ta), \end{aligned}$$

otherwise we have the contradiction $d(b,Tb) < d(b,Tb)$. Therefore in both the cases, we have

$$d(x,b) \leq d(a,Ta). \tag{2.4}$$

Now

$$d(x,a) \leq \max\{d(a,b), d(b,x)\}.$$

By the fact that $d(b,a) \leq d(a,Ta)$ and (2.4), we get

$$d(x,a) \leq \max\{d(a,b), d(b,x)\} \leq d(a,Ta).$$

So, $x \in B_a$ and $B_b \subseteq B_a$ for every $B_a \in \mathscr{A}_1$. Thus B_b is an upper bound in $\mathscr{A}$ for the family $\mathscr{A}_1$. By Zorn's lemma, $\mathscr{A}$ has a maximal element, say B_z, for some $z \in X$. We claim that $z = Tz$.

Since T is a generalized contractive mapping, we have

$$d(Tz, T^2z) < \max\{d(z,Tz), d(z,Tz), d(Tz,T^2z), d(z,T^2z), d(Tz,Tz)\}.$$

Using Lemma 1, we get

$$d(Tz,T^2z) < \max\{d(z,Tz), d(Tz,T^2z)\} = d(z,Tz), \tag{2.5}$$

and

$$Tz \in B\left(Tz, d(Tz,T^2z)\right) \cap B\left(z, d(z,Tz)\right).$$

Hence B_{Tz} is not a subset of B_z. And this contradicts the maximality of B_z. Therefore T has a fixed point. Uniqueness of fixed point is obvious. □

Example 3. *Let $X = [0,1]$ endowed with the discrete metric and $T : X \to X$ defined by $Tx = 1/2$ for all $x \in X$. Then T satisfies Theorem 3.*

Example 4. *(Compare Kirk and Shahzad [4]). Let $X = \{a,b,c,d\}$ with $d(a,b) = d(c,d) = 1/2$; $d(a,c) = d(a,d) = d(b,c) = d(b,d) = 1$. Then (X,d) is a spherically complete ultrametric space. Define $T : X \to X$ by $Ta = a; Tb = a; Tc = a; Td = b$. Then*

$$d(Tc,Td) = d(a,b) = \frac{1}{2} = d(c,d),$$

and the mapping T does not satisfy the contractive condition of Theorem 1. However, the mapping T satisfies the conditions of Theorem 3 and a is the unique fixed point of T in X.

Corollary 1. *Theorem 2.*

Proof. It comes from Theorem 3, when $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$. □

Corollary 2. *Theorem 1.*

Proof. It comes from Theorem 3, when $M(x,y) = d(x,y)$. □

Finally, we present a multi-valued version of Theorem 3.

Let (X,d) be an ultrametric space and $C(X)$ the collection of all compact subsets of X. Then the Hausdorff metric induced by d is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}$$

for all $A,B \subseteq C(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

Theorem 4. *Let (X,d) be a spherically complete ultrametric space and $T : X \to C(X)$ a multi-valued mapping such that for all $x,y \in X$, $x \neq y$,*

$$H(Tx,Ty) < M(x,y). \tag{2.6}$$

Then T has a fixed point.

Proof. Since T satisfies (2.6), we have

$$H(Tx,Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{2.7}$$

By the strong triangle inequality, we have

$$d(x,Ty) \leq \max\{d(x,y), d(y,Ty)\} \tag{2.8}$$

Similarly, we have

$$d(y,Tx) \leq \max\{d(y,x), d(x,Tx)\}. \tag{2.9}$$

By (2.7)-(2.9), we conclude that

$$H(Tx,Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Now, rest of the proof can be completed as in [2] (see, also [5]). □

We remark that Theorem 2.1 [5] and the main result in [2] are the spacial cases of Theorem 4.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] L. Gajic, On ultrametric space, Novi. Sad. J. Math. 31 (2001), 69-71.

[2] L. Gajic, A multi valued fixed point theorem in ultrametric spaces, Math. Vesnik 54 (2002), 89-91.

[3] M.A. Khamsi, W.A. Kirk, An introduction to metric spaces and fixed point theory, John Wiley & Sons, Inc., 2001.

[4] W.A. Kirk, N. Shahzad, Some fixed point results in ultrametric spaces, Topology Appl. 159 (2012), 3327-3334.

[5] J. Kubiaczyk, N.M. Ali, A multivalued fixed point theorem in non-Archimedean vector spaces, Novi Sad. J. Math. 26 (1996) 111-116.

[6] C. Petalas, T. Vidalis, A fixed point theorem in non-Archimedean vector spaces, Proc. Amer. Math. Soc 118 (1993) 819-821.

[7] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257-290.