A FIXED-POINT PRINCIPLE FOR A PAIR OF NON-COMMUTATIVE OPERATORS

PENUMARTHY PARVATEESAM MURTHY\(^1\)*, TANMOY SOM\(^2\), ERDAL KARAPINAR\(^3\)

\(^1\)Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya (A Central University),
Koni, Bilaspur-495009, India
\(^2\)Department of Mathematical Sciences, Indian Institute of Technology(BHU),Varanasi-221005, India
\(^3\)Department of Mathematics, Atilim University 06836, Incek, Ankara, Turkey

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Abstract. In this paper, a fixed point principle for a pair of operators \((f_i, X, d)\), \(i = 1, 2\), where \((X, d)\) is a metric space and \(f_1, f_2 : X \rightarrow X\), is established under the generalized uniform equivalence condition of different orbits generated by the maps \(f_1\) and \(f_2\) separately, which gives another generalization of the fixed point principle of Leader [1] and estimates approximations to the fixed points of both the operators simultaneously.

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1. Introduction

Consider two operators \((f_i, X, d)\), \(i = 1, 2\), where \((X, d)\) is a metric space and \(f_1, f_2 : X \rightarrow X\). From Meir and Keeler [2], an operator is said to have a contractive fixed point if the limit of every orbit generated by the operator is fixed. This can be easily obtained by imposing graph
completeness condition on the operator. The fixed point principle given by Leader [1] needs the uniform equivalence condition of all orbits generated by the operator to have a contractive fixed point. In the present paper, the idea of equivalence condition of orbits by a single operator is further extended to generalized equivalence condition of two different orbits generated separately by two mappings $f_1$ and $f_2$ and a fixed point principle for them is derived.

Let $x, y \in X, \{x, f_1 x, f_1^2 x, ...\}$ and $\{y, f_2 y, f_2^2 y, ...\}$ be the orbits of $x$ and $y$ generated by the repeated application of $f_1$ and $f_2$ separately on $x$ and $y$, respectively. We say that the above two orbits are generalized equivalent if $d(f_{m+1}^i x, f_{m+1}^j y) \to 0$ as $m, n \to \infty$.

2. Main Result

Now we are ready to prove the main Theorem in this section.

**Theorem 2.1.** Let $(f_i, X, d), i = 1, 2$ be a pair of operators on a metric space $(X, d)$. Given $c > 0$, define a sequence of positive real numbers $\{\varepsilon_n\}$ by

$$\varepsilon_n = \sup \{d(f_i^j x, f_i^j y) : i \geq n, d(x, y) \leq c\}. \quad (2.1)$$

If $(m + 1)\varepsilon_n + 2\varepsilon_m \leq c$ and $d(x, y) \leq c$, $d(x, f_2 y) \leq c$, $d(f_1 x, y) \leq c$ then

$$d(f_1^i x, f_1^{i+j} x) \leq (m + 1)\varepsilon_n + 2\varepsilon_m, \quad (2.2)$$

$$d(f_2^j y, f_2^{i+j} y) \leq (m + 1)\varepsilon_n + 2\varepsilon_m, \quad (2.3)$$

for all $i \geq n$ and all $j \in N$. Further if

$$d(f_1^n x, f_2^n y) \to 0 \text{ as } n \to \infty \quad (2.4)$$

uniformly for all $x, y \in X$ with $d(x, y) \leq c$, then the orbits

$$\{f_1^n x\} \text{ and } \{f_2^n y\} \text{ are uniformly Cauchy.} \quad (2.5)$$

If the graphs of both $(f_i, X, d), i = 1, 2$ are complete and (2.4) holds, then $d(x, y) \leq c$, $d(x, f_2 y) \leq c$ and $d(f_1 x, y) \leq c$ imply that the orbits $\{f_1^n x\}$ and $\{f_2^n y\}$ converge to the fixed points $p = f_1 p$.
and \( q = f_2q \), respectively, where \( p \) and \( q \) are the limits of \( \{f_1^n x\} \) and \( \{f_2^n y\} \) respectively. So \( p = q \). Further for \( \varepsilon_n \) as defined in (2.1), we have

\[
d(f_1^n x, p) \leq (m + 1)\varepsilon_n + 2\varepsilon_m \text{ if } (m + 1)\varepsilon_n + 2\varepsilon_m \leq c,  
\]

(2.6)

\[
d(f_2^n y, q) \leq (m + 1)\varepsilon_n + 2\varepsilon_m \text{ if } (m + 1)\varepsilon_n + 2\varepsilon_m \leq c.  
\]

(2.7)

**Proof.** Using induction on \( k \), we prove (2.2) and (2.3) for \( j \leq km \) for all \( k \in N \) under the given condition that \( (m + 1)\varepsilon_n + 2\varepsilon_m \leq c \) for a given \( m, n \) and \( d(x, y) \leq c \), \( d(x, f_2y) \leq c \) and \( d(f_1x, y) \leq c, x, y \in X \). Let \( x_i = f_1^i x \) and \( y_i = f_2^i y \). Then for \( k = 1 \), (2.1) implies for all \( i \geq n \) and \( j \leq m \), where \( m \) is even. It follows that

\[
d(x_i, x_{i+j}) \leq d(x_i, y_{i+1}) + d(y_{i+1}, x_{i+2}) + \ldots + d(y_{i+j-1}, x_{i+j})  
\]

\[
\leq d((x)_i, (y_1)i) + d((y)i_{i+1}, (x_1)i_{i+1}) + \ldots + d((y)i_{i+j-1}, (x_1)i_{i+j-1})  
\]

\[
\leq j\varepsilon_n \leq m\varepsilon_n.  
\]

If \( m \) is odd, we get

\[
d(x_i, x_{i+j}) \leq d(x_i, y_{i+1}) + d(y_{i+1}, x_{i+2}) + \ldots + d(x_{i+j-1}, y_{i+j}) + d(y_{i+j}, x_{i+j})  
\]

\[
\leq d((x)_i, (y_1)i) + d((y)i_{i+1}, (x_1)i_{i+1}) + \ldots + d((x)i_{i+j-1}, (y_1)i_{i+j-1}) + d((y)i_{i+j}, (x)i_{i+j})  
\]

\[
\leq (j + 1)\varepsilon_n \leq (m + 1)\varepsilon_n.  
\]

Thus, we have

\[
d(x_i, x_{i+j}) \leq (m + 1)\varepsilon_n \forall i \geq n \text{ and } j \leq m  
\]

(2.8)

independent of \( m \) even or odd, that is, (2.2) holds for all \( j \leq m \). Similarly, we find that (2.3) holds for all \( j \leq m \). Now, suppose for a given \( k \in N \) that (2.2), (2.3) hold for all \( j \leq km \), we prove it for \( j \leq (k + 1)m \). Taking \( km < j \leq (k + 1)m \), we find \( 0 < j - m \leq (k + 1)m \) and so the induction process gives that

\[
d(x_i, x_{i+j-m}) \leq (m + 1)\varepsilon_n + 2\varepsilon_m \leq c  
\]

for all \( i \geq n \). Then iterating \( x_i \) and \( x_{i+j-m} \) by \( f_1 \) and \( f_2 \) respectively \( m \) times, we get

\[
d(x_{i+m}, x'_{p+m}) \leq \varepsilon_m \text{ for all } i \geq n \text{ where } x'_{p} = x_{i+j-m}.  
\]
Note that
\[ d(x_{i+m}, x_{i+j}) \leq d(x_i, x_{i+j}) + d(f_m(x_{i+m}), f_m(x_{i+j})) \]
\[ = d(x_i, x_{i+j}) + d(f_m(x_{i+m}), f_m(x_{i+j})) \]
\[ \leq \epsilon_m + \epsilon_m = 2\epsilon_m \text{ for all } i \geq n. \]  

Therefore, from (2.8) with \( j = m \) and (2.9), we get
\[ d(x_i, x_{i+j}) \leq (m+1)\epsilon_n + 2\epsilon_m \text{ for all } i \geq n. \]

Thus (2.2) holds for all \( j \leq (k+1)m \) and hence for all \( j \in N \). (2.3) can be proved in a similar way. So (2.2) and (2.3) holds for all \( i \geq n \) and \( j \in N \). Now (2.1) and (2.4) gives \( \epsilon_n \downarrow 0 \).

Then for a given \( 0 < \epsilon < c \), we take \( m \) so large that \( 2\epsilon_m < \epsilon \). Further choose \( n \) so large that \( \epsilon_n < (m+1)^{-1}(\epsilon - 2\epsilon_m) \) giving \( (m+1)\epsilon_n + 2\epsilon_m < \epsilon < c \) and therefore \( d(f^i_1x, f^{i+j}x) < \epsilon \) and \( d(f^i_2y, f^{i+j}y) < \epsilon \) for all \( i \geq n \) and all \( j \in N \). Hence (2.5) holds. Further considering graph completeness of both the maps it can be easily obtained that \( f_ip_i = p_i, i = 1, 2 \) and that \( p_1 = p_2 \) by (2.4). Finally in (2.2) and (2.3) taking \( i = n \) and letting \( j \to \infty \) we obtain (2.6) and (2.7). The theorem is completed.

### 3. A Fixed-point principle

In this section, we extend Theorem 3 of Som and Mukherjee [3] to three and four mappings under some weaker condition than the condition of commutativity of the mappings, used in Theorem 3 of Som and Mukherjee [3]. Next, we give here the definition of a weakly commutative pair of mappings with an example [4].

**Definition 3.1.** Let \( S \) and \( T \) be a pair of self mappings of a metric space \((X, d)\). Then \( \{S, T\} \) is said to be a weakly commutative pair if
\[ d(STx, TSx) \leq d(Tx, Sx), \quad \forall x \in X. \]

Clearly every commutative pair of mappings is weakly commutative but the converse is not true in general.
Example 3.2. Let $X = [0, 1]$ with the usual metric. Let $T, S : X \to X$ be defined by $T x = \frac{3x}{5}$, $S x = \frac{x}{x+3}$ for every $x \in X$. Then for all $x \in X$, we have

$$d(ST x, T S x) = \frac{3x}{3x+15} - \frac{3x}{5x+15} = \frac{12x}{15x+75} - \frac{2x}{15x+75} = \frac{10x}{15x+75} = \frac{x}{x+3} = d(T x, S x)$$

So, $S$ and $T$ are commute weakly. However $S$ and $T$ are not a commuting pair for

$$ST x = \frac{3x}{3x+15} > \frac{3x}{5x+15} = TS x, \forall x (x \neq 0) \in X.$$

Theorem 3.3. Let $(X, d)$ be a metric space and $f, g$ and $h$ be three self mappings of $X$ with $f$ continuous and $g(X) \subset f(X), h(X) \subset f(X)$. Let for some $x_0 \in X$, $\{y_n\}$ be a sequence defined by

$$y_1 = f(x_1) = g(x_0), y_2 = f(x_2) = h(x_1),$$

and in general,

$$y_{2n+1} = f(x_{2n+1}) = g(x_{2n}), y_{2n+2} = f(x_{2n+2}) = h(x_{2n+1}), n = 0, 1, \ldots$$

Similarly, for some $u_0 \in X$, we have a sequence $\{z_n\}$, that is, for $n = 0, 1, \ldots$,

$$z_{2n+1} = f(u_{2n+1}) = g(u_{2n}), z_{2n+2} = f(u_{2n+2}) = h(u_{2n+1}).$$

For some $c > 0$, define

$$\varepsilon_{n+1} = \sup \{d(y_{p+i}, z_q+i) : i \geq n, d(y_p, z_q) \leq c \text{ for some } p, q \in N \}. \quad (3.1)$$

If $m \varepsilon_n + \varepsilon_{m+1} \leq c$ and $d(f(x), g(x)) \leq c, d(f(x), h(x)) \leq c$, then for all $i \geq n$ and all $j \in N$,

$$(y_i, y_{i+j}) \leq m \varepsilon_n + \varepsilon_{m+1}. \quad (3.2)$$

Hence if $d(y_n, z_n) \to 0$ uniformly for all $x_0, u_0 \in X$ with $d(y_p, z_q) \leq c$ for some $p, q \in N$ then the sequence $\{y_n\}$ is uniformly Cauchy. Further if $g, h$ satisfy

$$d(g(x), h(y)) \leq d(f(x), f(y)) \text{ for all } x \neq y \in X \quad (3.3)$$
and either \(\{f, g\}\) or \(\{f, h\}\) is a weakly commutative pair then \(f, g\) and \(h\) have a coincidence point. Moreover if

\[
d(f, y) \leq d(x, y), x \neq y \in X, \tag{3.4}
\]

then \(f, g\) and \(h\) have a common fixed point in \(X\).

**Proof.** The proofs of (3.2) and that \(\{y_n\}\) is Cauchy follows in the lines of Theorem 3 of Som and Mukherjee [4]. So we omit the proof here. Let \(y_n \to t \in X\). Since \(f\) is continuous, we have \(f(y_n) \to f(t)\). From (3.3), we have

\[
d(g(y_n), h(t)) \leq d(f(y_n), f(t)),
\]

which in the limiting case implies that \(g(y_n) \to h(t)\). Similarly it can be shown that \(h(y_n) \to g(t)\). Further putting \(x = y_n, y = y_{n+1}\) in (3.3) and taking the limits we get \(g(t) = h(t)\). Let \(\{f, g\}\) be weakly commutative. then we have

\[
d(fg(x_{2n}), gf(x_{2n})) \leq d(g(x_{2n}), f(x_{2n})),
\]

which in the limiting case gives that \(d(f(t), h(t)) \leq d(t, t)\) and therefore \(g(t) = h(t) = f(t)\). Similarly, we have the same result if \(\{f, g\}\) is weakly commutative. Thus we conclude that \(t\) is a coincidence point of \(f, g\) and \(h\). Finally, putting \(x = t, y = y_n\) in (3.4) and taking the limit, we obtain a common fixed point for \(f, g\) and \(h\). This completes the proof of the theorem.

**Remark 3.4.** If \(g = h\) in theorem 3.3, then our theorem improves theorem 3 of Som and Mukherjee [4]. Moreover from (3.4), we observe that \(f\) is not necessarily an identity mapping to have a common fixed point result.

**Theorem 3.5.** Let \((X, d)\) be a metric space and \(g_k, f_k, k = 1, 2\), be four self mappings of \(X\) with each \(f_k\) continuous for each \(k = 1, 2\) and \(g_k(X) \subset f_k(X)\). Let for some \(x_0 \in X\), \(\{y_n\}\) be a sequence defined by

\[
y_1 = f_1(x_1) = g_1(x_0), y_2 = f_2(x_2) = g_2(x_1), ...
\]

and in general

\[
y_{2n+1} = f_1(x_{2n+1}) = g_1(x_{2n}) \text{ and } y_{2n+2} = f_2(x_{2n+2}) = g_2(x_{2n+1}), n = 0, 1, ...
\]
Similarly, for some \( u_0 \in X \), define a sequence \( \{ z_n \} \), that is, for \( n = 0, 1, \ldots \), \( z_{2n+1} = f_1(u_{2n+1}) = g_1(u_{2n}) \) and \( z_{2n+2} = f_2(u_{2n+2}) = g_2(u_{2n+1}) \). For some \( c > 0 \), we define
\[
epsilon_{n+1} = \sup \{ d(y_{p+i}, z_{q+i}) : i \geq n, d(y_p, z_q) \leq c \text{ for some } p, q \in N \}.
\]
If \( m\epsilon_n + \epsilon_{m+1} \leq c \) and \( d(f_k(x), g_l(x)) \leq c, k \neq l \ (k, l = 1, 2) \), then for all \( i \geq n \) and all \( j \in N \),
\[
d(y_i, y_{i+j}) \leq m\epsilon_n + \epsilon_{m+1}.
\]
Hence, if \( d(y_n, z_n) \rightarrow 0 \) uniformly for all \( x_0, u_0 \in X \) with \( d(y_p, z_q) \leq c \) for some \( p, q \in N \), then the sequence \( \{ y_n \} \) is uniformly Cauchy. Further if \( g_1; g_2 \) satisfy
\[
d(g_1(x), g_2(y)) \leq d(x, y) \ \forall \ x, y \in X
\]
and \( \{ f_1, g_2 \}, \{ f_2, g_1 \} \) are weakly commutative pairs, then \( f_k, g_k, k = 1, 2 \) have a coincidence point. Moreover if
\[
d(f_k x, y) \leq d(x, y) \ \forall \ x \neq y \in X \text{ for } k = 1, 2,
\]
then \( f_k, g_k \) have a common fixed point in \( X \).

**Proof.** From Theorem 4 of Som and Mukherjee [3], we find the desired conclusion immediately.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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