SPLITTING METHODS FOR TREATING STRICTLY PSEUDOCONTRACTIVE AND MONOTONE OPERATORS IN HILBERT SPACES

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Abstract. In this paper, strictly pseudocontractive and monotone operators are investigated based on a viscosity splitting method. Strong convergence theorems for common solutions are established in the framework of Hilbert spaces.

Keywords: splitting methods; zero point; fixed point; variational inclusion; nonexpansive mapping.

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1. Introduction-preliminaries

In what follows, we always assume that $H$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $C$ be a nonempty, closed and convex subset of $H$. Let $S : C \to C$ be a mapping. $F(S)$ denoted by the fixed point set of $S$. $S$ is said to be contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$  

$S$ is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$
$S$ is said to be strictly pseudocontractive iff there exits a constant $\lambda \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \lambda \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$  

The class of strictly pseudocontractive mapping was introduced by Browder and Petryshyn [1]. It is clear that the class strictly pseudocontractive mapping includes the class of nonexpansive mappings as a special case.

Let $A : C \to H$ be a mapping. Recall that $A$ is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$  

Recall that $A$ is said to be inverse-strongly monotone iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \kappa \|Ax - Ay\|^2, \quad \forall x, y \in C.$$  

It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous.

Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be monotone iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g \rangle > 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of $B$. A monotone mapping $B : H \rightrightarrows H$ is maximal iff the graph $\text{Graph}(B)$ of $B$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $B$ is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$, for all $(y, g) \in \text{Graph}(B)$ implies $f \in Bx$. For a maximal monotone operator $B$ on $H$, and $r > 0$, we may define the single-valued resolvent $J_r : H \to \text{Dom}(B)$, where $\text{Dom}(B)$ denote the domain of $B$. It is known that $J_r$ is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

Maximal monotone operators have been extensively studied by many authors; see [2-22] and the references therein. One well-known example of such a mapping is $\partial f$, the subdifferential of a proper closed convex function $f : H \to (-\infty, \infty]$ which is defined by

$$\partial f(x) := \{x^* \in H : f(x) + \langle y - x, x^* \rangle \leq f(y), \forall y \in H\}, \quad \forall x \in H.$$  

Rockafellar [5] proved that $\partial f$ is a maximal monotone operator. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in H} f(x)$. Another example is $M + N_C$, $M$ is a single valued maximal monotone mapping that is continuous on $C$, and $N_C$ is the normal cone mapping

$$N_C(x) := \{x^* \in H : \langle x^*, y - x \rangle \leq 0, \forall y \in C\},$$
for $x \in C$ and is empty otherwise. Then, $0 \in Mx + N_C(x)$ iff $x \in C$ satisfies the variational inequalities of $\langle Mx, y - x \rangle \geq 0$ for all $y \in C$.

For approximating zero points of maximal monotone operator $T$, classical methods for doing this is the proximal point algorithm, proposed by Martinet [19] and generalized by Rockafellar [4-6]. In the case of $T = A + B$, where $A$ and $B$ are monotone operators on $H$. The following splitting method

$$x_{n+1} = J_{r_n}(I - r_n A)x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is real number sequence, was proposed by Lions and Mercier [23], by Passty [24] and, in a dual form for convex programming, by Han and Lou [25].

Since many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two monotone nonlinear operators ($T = A + B$), splitting methods recently have been investigated for treating monotone operators; see [26-31] and the references therein. Splitting methods mean an iterative method for which each iteration involves only with the individual operators $A$ and $B$, but not the sum $A + B$. Indeed, the backward step involves $B$ only, so some portion of $T$ can be put into $A$ to facilitate problem decomposition.

In this paper, we investigate common solutions of fixed point problems and zero points of the sum of two monotone operators based on a viscosity splitting method. Strong convergence theorems for the common solutions of the two problems are established in Hilbert spaces. In order to prove our main results, we also need the following tools.

**Lemma 1.1.** [31] Let $A : C \rightarrow H$ be a mapping, and $B : H \rightharpoonup H$ a maximal monotone operator. Then $F(J_{r}(I - r A)) = (A + B)^{-1}(0)$.

**Lemma 1.2** [32] Let $E$ be a Banach space and let $A$ be an $m$-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_{\lambda x} = J_{\mu}\left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_{\lambda} x\right)$, where $J_{\lambda} = (I + \lambda A)^{-1}$ and $J_{\mu} = (I + \mu A)^{-1}$.

**Lemma 1.3** [33] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$, and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 -$
\[ \beta_n y_n + \beta_n x_n, \forall n \geq 1 \text{ and} \]
\[ \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \]

Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0. \)

**Lemma 1.4** [34] Let \( \{a_n\} \) be a sequence of nonnegative numbers satisfying the condition \( a_{n+1} \leq (1 - t_n) a_n + t_n b_n + e_n, \forall n \geq 0, \) where \( \{t_n\} \) is a number sequence in \((0, 1)\) such that \( \lim_{n \to \infty} t_n = 0 \) and \( \sum_{n=0}^{\infty} t_n = \infty, \) \( \{b_n\} \) is a number sequence such that \( \limsup_{n \to \infty} b_n \leq 0 \) and \( \{e_n\} \) is a number sequence such that \( \sum_{n=0}^{\infty} e_n < \infty. \) Then \( \lim_{n \to \infty} a_n = 0. \)

**Lemma 1.5** [34] Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H. \) Let \( T : C \to C \) be a \( \lambda \)-strictly pseudocontractive mapping. Define \( S_t \) by \( S_t = tx + (1 - t) T x, \) where \( t \in [\lambda, 1). \) Then \( S_t \) is nonexpansive with \( F(S_t) = F(T) \) and \( I - T \) is also demiclosed.

**2. Main results**

Now, we are in a position to give our main results.

**Theorem 2.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H. \) Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and let \( B \) be a maximal monotone operator on \( H. \) Let \( f : C \to C \) be a \( \kappa \)-contractive mapping and let \( T : C \to C \) be a \( \lambda \)-strictly pseudocontractive mapping with fixed points. Assume that \( \text{Dom}(B) \subset C \) and \( F(T) \cap (A + B)^{-1}(0) \) is not empty. Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be real number sequences in \((0, 1)\) and \( \{r_n\} \) be a positive real number sequence in \((0, 2\alpha). \) Let \( \{x_n\} \) be a sequence generated in the following process: \( x_1 \in C \) and

\[
\begin{cases}
  y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\
  x_{n+1} = \beta_n x_n + (1 - \beta_n) S_{\lambda r_n} (y_n - r_n Ay_n + e_n), \quad \forall n \geq 1,
\end{cases}
\]

where \( S_{\lambda x} = \lambda x + (1 - \lambda) T x, \) \( J_{r_n} = (I + r_n B)^{-1} \) and \( \{e_n\} \) is a sequence in \( H \) such that \( \sum_{n=1}^{\infty} \|e_n\| < \infty. \) Assume that the above sequences satisfy the following restrictions:

(a) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty; \)

(b) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1; \)

(c) \( 0 < a \leq r_n \leq b < 2\alpha \) and \( \sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty, \)
where \( a \) and \( b \) are two real numbers. Then \( \{x_n\} \) converges strongly to \( q = P_{F(T)\cap(A+B)^{-1}(0)}f(q) \).

**Proof.** For any \( x, y \in C \), we find that
\[
\| (I - r_nA)x - (I - r_nA)y \|^2 \\
= \| x - y \|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \| Ax - Ay \|^2 \\
\leq \| x - y \|^2 - r_n(2\alpha - r_n) \| Ax - Ay \|^2.
\]

Using the restriction (c), we obtain that \( I - r_nA \) is nonexpansive. Fixing \( p \in (A + B)^{-1}(0) \cap F(S) \), we see that
\[
\| y_n - p \| \leq \alpha_n \| f(x_n) - p \| + (1 - \alpha_n) \| x_n - p \| \\
\leq \alpha_n \| f(x_n) - f(p) \| + \alpha_n \| f(p) - p \| + (1 - \alpha_n) \| x_n - p \| \\
\leq \left( 1 - \alpha_n(1 - \kappa) \right) \| x_n - p \| + \alpha_n(1 - \kappa) \frac{\| f(p) - p \|}{1 - \kappa}.
\]

It follows that
\[
\| x_{n+1} - p \| \leq \beta_n \| x_n - p \| + (1 - \beta_n) \| S_{\lambda}J_{r_n}(y_n - r_nAy_n + e_n) - p \| \\
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| J_{r_n}(y_n - r_nAy_n + e_n) - p \| \\
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| (y_n - r_nAy_n + e_n) - p \| \\
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| y_n - p \| + \| e_n \| \\
\leq \beta_n \| x_n - p \| + (1 - \beta_n)(1 - \alpha_n(1 - \kappa)) \| x_n - p \| \\
\quad + \alpha_n(1 - \beta_n)(1 - \kappa) \frac{\| f(p) - p \|}{1 - \kappa} + \| e_n \| \\
\leq \left( 1 - \alpha_n(1 - \beta_n)(1 - \kappa) \right) \| x_n - p \| + \alpha_n(1 - \beta_n)(1 - \kappa) \frac{\| f(p) - p \|}{1 - \kappa} + \| e_n \|.
\]

This implies that the sequence \( \{x_n\} \) is bounded. Note that
\[
\| y_n - y_{n-1} \| \leq \alpha_n \| f(x_n) - f(x_{n-1}) \| + \| f(x_{n-1}) - x_{n-1} \| \| \alpha_n - \alpha_{n-1} \| \\
+ (1 - \alpha_n) \| x_n - x_{n-1} \| \\
\leq \left( 1 - \alpha_n(1 - \kappa) \right) \| x_n - x_{n-1} \| + \| f(x_{n-1}) - x_{n-1} \| \| \alpha_n - \alpha_{n-1} \| \\
(2.1)
\]
Set $z_n = y_n - r_nAy_n + e_n$. Using Lemma 1.2, we find that

$$
\| J_{r_n}z_n - J_{r_{n-1}}z_{n-1} \| \\
\leq \| \frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + (1 - \frac{r_{n-1}}{r_n})(J_{r_n}z_n - z_{n-1}) \| \\
\leq \| z_n - z_{n-1} \| + \frac{r_n - r_{n-1}}{a} \| J_{r_n}z_n - z_n \| \\
\leq \| y_n - y_{n-1} \| + [r_{n-1} - r_n]([\| Ay_{n-1} \| + \frac{\| J_{r_n}z_n - z_n \|}{a}) + \| e_n \| + \| e_{n-1} \|].
$$

(2.2)

Substituting (2.1) into (2.2) yields that

$$
\| J_{r_n}z_n - J_{r_{n-1}}z_{n-1} \| \leq (1 - \alpha_n(1 - \kappa)) \| x_n - x_{n-1} \| + \| f(x_{n-1}) - x_{n-1} \| (\alpha_n - \alpha_{n-1}) \\
+ [r_{n-1} - r_n]([\| Ay_{n-1} \| + \frac{\| J_{r_n}z_n - z_n \|}{a}) + \| e_n \| + \| e_{n-1} \|].
$$

It follows that

$$
\| S_{\lambda}J_{r_n}z_n - S_{\lambda}J_{r_{n-1}}z_{n-1} \| \\
\leq \| J_{r_n}z_n - J_{r_{n-1}}z_{n-1} \| \\
\leq (1 - \alpha_n(1 - \kappa)) \| x_n - x_{n-1} \| + \| f(x_{n-1}) - x_{n-1} \| (\alpha_n - \alpha_{n-1}) \\
+ [r_{n-1} - r_n]([\| Ay_{n-1} \| + \frac{\| J_{r_n}z_n - z_n \|}{a}) + \| e_n \| + \| e_{n-1} \|].
$$

Using the restrictions (a) and (c), we find that

$$
\limsup_{n \to \infty} \left( \| S_{\lambda}J_{r_n}z_n - S_{\lambda}J_{r_{n-1}}z_{n-1} \| - \| x_n - x_{n-1} \| \right) \leq 0.
$$

Using Lemma 1.3, we see that $\lim_{n \to \infty} \| S_{\lambda}J_{r_n}z_n - x_n \| = 0$. Since $x_{n+1} - x_n = (1 - \beta_n)(S_{\lambda}J_{r_n}z_n - x_n)$, we find that

$$
\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0.
$$

(2.3)

Since $y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n$, we find from the restriction (a) that

$$
\lim_{n \to \infty} \| y_n - x_n \| = 0.
$$

(2.4)

Note that

$$
\| y_n - p \|^2 \leq \alpha_n \| f(x_n) - p \|^2 + (1 - \alpha_n) \| x_n - p \|^2.
$$
Hence, we have

\[ \|x_{n+1} - p\|^2 \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|S_{\lambda_n}J_{r_n}z_n - p\|^2 \]

\[ \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|J_{r_n}(I - r_nA)y_n + e_n - p\|^2 \]

\[ \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|(I - r_nA)y_n + e_n - (I - r_nA)p\| \leq \|e_n\|^2 \]

\[ \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|(I - r_nA)y_n - (I - r_nA)p\|^2 \]

\[ + \|e_n\|\|(\|e_n\| + 2\|y_n - p\|)\) \]

\[ \leq (1 - \alpha_n(1 - \beta_n)) \|x_n - p\|^2 + (1 - \beta_n) \alpha_n \|f(x_n) - p\|^2 \]

\[ - r_n(2\alpha - r_n)(1 - \beta_n) \|Ay_n - Ap\|^2 + \|e_n\|\|(\|e_n\| + 2\|y_n - p\|)\) \]

\[ \leq \|x_n - p\|^2 + \alpha_n \|f(x_n) - p\|^2 - r_n(2\alpha - r_n)(1 - \beta_n) \|Ay_n - Ap\|^2 \]

\[ + \|e_n\|\|(\|e_n\| + 2\|y_n - p\|). \]

Therefore, we have

\[ r_n(1 - \beta_n)(2\alpha - r_n) \|Ay_n - Ap\|^2 \]

\[ \leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - p\|^2 + \|e_n\|\|(\|e_n\| + 2\|y_n - p\|)\) \]

Using the restrictions (a), (b) and (c), we find from (2.3) that

\[ \lim_{n \to \infty} \|Ay_n - Ap\| = 0. \quad (2.5) \]

Since \(J_{r_n}\) is firmly nonexpansive, we see that

\[ \|J_{r_n}z_n - p\|^2 \leq \langle J_{r_n}z_n - p, (y_n - r_nAy_n) - (p - r_nAp) \rangle \]

\[ = \frac{1}{2} \left( \|J_{r_n}z_n - p\|^2 + \|(y_n - r_nAy_n) - (p - r_nAp)\|^2 \right. \]

\[ - \left. \|(J_{r_n}z_n - p) - (y_n - r_nAy_n) - (p - r_nAp)\| \right) \]

\[ \leq \frac{1}{2} \left( \|J_{r_n}z_n - p\|^2 + \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2 \right. \]

\[ - \left. \|r_nAy_n - r_nAp\|^2 + 2r_n \|Ay_n - Ap\| \|J_{r_n}z_n - y_n\| \right). \]
It follows that
\[
\|J_{r_n}z_n - p\|^2 \leq \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2
\]
\[
- \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|
\]
\[
\leq \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2
\]
\[
- \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|
\]
Using the convexness of \(\| \cdot \|^2\), we find that
\[
\|x_{n+1} - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|S_{\lambda}J_{r_n}z_n - p\|^2
\]
\[
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|J_{r_n}z_n - p\|^2
\]
\[
\leq \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - (1 - \beta_n)\|J_{r_n}z_n - y_n\|^2
\]
\[
+ 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\]
It follows that
\[
(1 - \beta_n)\|J_{r_n}z_n - y_n\|^2 \leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - p\|^2
\]
\[
+ 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\]
Using the restrictions (a) and (b), we from (2.3) and (2.5) see that
\[
\lim_{n \to \infty} \|J_{r_n}z_n - y_n\| = 0. \quad (2.6)
\]
Since \(P_{F(T) \cap (A+B)^{-1}(0)}f\) is contractive, we see that there exits a unique fixed point. Next, we use \(q\) to denote the unique fixed point. Now, we are in a position to show that \(\lim sup_{n \to \infty} \langle f(q) - q, y_n - q \rangle \leq 0\). To show it, we can choose a subsequence \(\{y_{n_i}\}\) of \(\{y_n\}\) such that
\[
\lim sup_{n \to \infty} \langle f(q) - q, y_n - q \rangle = \lim_{i \to \infty} \langle f(q) - q, y_{n_i} - q \rangle.
\]
Since \(\{y_{n_i}\}\) is bounded, we can choose a subsequence \(\{y_{n_{i_j}}\}\) of \(\{y_{n_i}\}\) which converges weakly some point \(x\). We may assume, without loss of generality, that \(y_{n_{i_j}}\) converges weakly to \(x\).

Now, we are in a position to prove that \(x \in F(T)\). Setting \(w_n = J_{r_n}z_n\), we find that
\[
\|S_{\lambda}w_n - y_n\| \leq \frac{1}{1 - \beta_n} \|x_{n+1} - y_n\| + \frac{\beta_n}{1 - \beta_n} \|y_n - x_n\|.
\]
Using (2.3) and (2.4), we find that this implies that \( \lim_{n \to \infty} \| S_{\lambda}w_n - y_n \| = 0 \). Note that

\[
\| S_{\lambda}w_n - w_n \| \leq \| S_{\lambda}w_n - y_n \| + \| y_n - w_n \|.
\]

In view of (2.6), we find that \( \| S_{\lambda}w_n - w_n \| \to 0 \). In view of demiclosed of the mapping, we find that \( x \in F(S_{\lambda}) = F(T) \).

Now, we are in a position to show that \( x \in (A + B)^{-1}(0) \). It follows that

\[
y_n - r_nAy_n + e_n \in (I + r_nB)w_n
\]

That is, \( \frac{y_n - w_n}{r_n} - Ay_n + e_n \in Bw_n \). Since \( B \) is monotone, we get, for any \((\mu, v) \in B\), that

\[
\langle w_n - \mu, \frac{y_n - w_n}{r_n} - Ay_n + e_n - v \rangle \geq 0.
\]

It follows from (2.6) that

\[
\langle x - \mu, -Ax - v \rangle \geq 0.
\]

This gives that \(-Ax \in Bx\), that is, \( 0 \in (A + B)(x) \). This proves that \( x \in (A + B)^{-1}(0) \). This complete the proof that \( x \in F(T) \cap (A + B)^{-1}(0) \). Hence

\[
\limsup_{n \to \infty} \langle f(q) - q, y_n - q \rangle \leq 0.
\]

Finally, we show that \( x_n \to q \). Notice that

\[
\| y_n - q \|^2 \leq \alpha_n \langle f(x_n) - q, y_n - q \rangle + (1 - \alpha_n) \| x_n - q \| \| y_n - q \|
\leq (1 - \alpha_n(1 - \kappa)) \| x_n - q \| \| y_n - q \| + \alpha_n \langle f(q) - q, y_n - q \rangle
\]

This implies that

\[
\| y_n - q \|^2 \leq (1 - \alpha_n(1 - \kappa)) \| x_n - q \|^2 + 2\alpha_n \langle f(q) - q, y_n - q \rangle
\]

It follows that

\[
\| x_{n+1} - q \|^2 \leq \beta_n \| x_n - q \|^2 + (1 - \beta_n)\| S_{\lambda}J_{r_n}(y - r_nAy_n + e_n) - q \|^2
\leq \beta_n \| x_n - q \|^2 + (1 - \beta_n) \| y_n - q \|^2 + \| e_n \| (\| e_n \| + 2\| y_n - q \|)
\leq (1 - \alpha_n(1 - \beta_n)(1 - \kappa)) \| x_n - q \|^2 + 2\alpha_n(1 - \beta_n) \langle f(q) - q, y_n - q \rangle
+ \| e_n \| (\| e_n \| + 2\| y_n - q \|).
\]
Using the restrictions (a) and (b), we find from Lemma 1.4 that \( x_n \to q \). This completes the proof.

If \( T \) is nonexpansive, then we have the following.

**Corollary 2.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and let \( B \) be a maximal monotone operator on \( H \). Let \( f : C \to C \) be a \( \kappa \)-contractive mapping and let \( T : C \to C \) be a nonexpansive mapping with fixed points. Assume that \( \text{Dom}(B) \subset C \) and \( F(T) \cap (A+B)^{-1}(0) \) is not empty. Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be real number sequences in \((0, 1)\) and \( \{r_n\} \) be a positive real number sequence in \((0, 2\alpha)\). Let \( \{x_n\} \) be a sequence generated in the following process: \( x_1 \in C \) and

\[
\begin{align*}
  y_n &= \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\
  x_{n+1} &= \beta_n x_n + (1 - \beta_n)TJ_{r_n}(y_n - r_nAy_n + e_n), \quad \forall n \geq 1,
\end{align*}
\]

where \( J_{r_n} = (I + r_nB)^{-1} \) and \( \{e_n\} \) is a sequence in \( H \) such that \( \sum_{n=1}^{\infty} \|e_n\| < \infty \). Assume that the above sequences satisfy the following restrictions:

(a) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty \);

(b) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);

(c) \( 0 < a \leq r_n \leq b < 2\alpha \) and \( \sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty \),

where \( a \) and \( b \) are two real numbers. Then \( \{x_n\} \) converges strongly to \( q = P_{F(T) \cap (A+B)^{-1}(0)}f(q) \).

If \( T \) is the identity mapping, then we have the following.

**Corollary 2.3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and let \( B \) be a maximal monotone operator on \( H \). Let \( f : C \to C \) be a \( \kappa \)-contractive mapping. Assume that \( \text{Dom}(B) \subset C \) and \( (A+B)^{-1}(0) \) is not empty. Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be real number sequences in \((0, 1)\) and \( \{r_n\} \) be a positive real number sequence in \((0, 2\alpha)\). Let \( \{x_n\} \) be a sequence generated in the following process: \( x_1 \in C \) and

\[
\begin{align*}
  y_n &= \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\
  x_{n+1} &= \beta_n x_n + (1 - \beta_n)J_{r_n}(y_n - r_nAy_n + e_n), \quad \forall n \geq 1,
\end{align*}
\]
where $J_{r_n} = (I + r_nB)^{-1}$ and $\{e_n\}$ is a sequence in $H$ such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ and $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where $a$ and $b$ are two real numbers. Then $\{x_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)}f(q)$.

3. Applications

In this section, we give applications of the main results. Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

Find $x \in C$ such that $F(x,y) \geq 0$, $\forall y \in C$. \hspace{1cm} (3.1)

In this paper, we use $EP(F)$ to denote the solution set of the equilibrium problem (3.1).

To study the equilibrium problems (3.1), we may assume that $F$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

**Lemma 3.1.** [30] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $F$ a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) and $A_F$ a multivalued mapping of $H$ into itself defined by

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y-x,z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \hspace{1cm} (3.2)$$

Then $A_F$ is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1} x, \hspace{0.5cm} \forall x \in H, r > 0,$$
where \( T_r \) is defined as
\[
T_r x = \{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \}
\]

**Theorem 3.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and let \( F_B \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) which satisfies (A1)-(A4). Let \( T : C \to C \) be a \( \lambda \)-strictly pseudocontractive mapping with fixed points. Assume that \( F(T) \cap \text{EP}(F) \) is not empty. Let \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) be real number sequences in \((0, 1)\) and \( \{ r_n \} \) be a positive real number sequence in \((0, 2\alpha)\). Let \( \{ x_n \} \) be a sequence generated in the following process: \( x_1 \in C \) and
\[
\begin{align*}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n) x_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n) S_\lambda T_r \left( y_n - r_n Ay_n + e_n \right), \quad \forall n \geq 1,
\end{align*}
\]
where \( S_\lambda = \lambda x + (1 - \lambda) T x, \quad T_r = (I + r A)^{-1} \) and \( \{ e_n \} \) is a sequence in \( H \) such that \( \sum_{n=1}^{\infty} \| e_n \| < \infty \). Assume that the above sequences satisfy the following restrictions:

(a) \( \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty \);

(b) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);

(c) \( 0 < a \leq r_n \leq b < 2\alpha \) and \( \sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty \),

where \( a \) and \( b \) are two real numbers. Then \( \{ x_n \} \) converges strongly to \( q = P_{F(T) \cap \text{EP}(F)} f(q) \).

If \( T = I \), the identity mapping, we have the following result.

**Corollary 3.3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \to H \) be an \( \alpha \)-inverse-strongly monotone mapping and let \( F_B \) be a bifunction from \( C \times C \) to \( \mathbb{R} \) which satisfies (A1)-(A4). Assume that \( \text{EP}(F) \) is not empty. Let \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) be real number sequences in \((0, 1)\) and \( \{ r_n \} \) be a positive real number sequence in \((0, 2\alpha)\). Let \( \{ x_n \} \) be a sequence generated in the following process: \( x_1 \in C \) and
\[
\begin{align*}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n) x_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n) T_r \left( y_n - r_n Ay_n + e_n \right), \quad \forall n \geq 1,
\end{align*}
\]
where $T_r = (I + rA)^{-1}$ and $\{e_n\}$ is a sequence in $H$ such that $\sum_{n=1}^{\infty} ||e_n|| < \infty$. Assume that the above sequences satisfy the following restrictions:

(a) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;
(c) $0 < a \leq r_n \leq b < 2\alpha$ and $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where $a$ and $b$ are two real numbers. Then $\{x_n\}$ converges strongly to $q = P_{EP(F)}f(q)$.

Recall the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$ 

The solution set of the inequality is denoted by $VI(C,A)$ in this section. Let $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential $\partial f$ of $f$ is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y \rangle, \quad z \in H\}, \quad \forall x \in H.$$ 

From Rockafellar [5], we know that $\partial f$ is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$. Let $I_C$ be the indicator function of $C$, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.3}$$

Since $I_C$ is a proper lower semicontinuous convex function on $H$, we see that the subdifferential $\partial I_C$ of $I_C$ is a maximal monotone operator.

**Lemma 3.4** [5] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $\text{Proj}_C$ the metric projection from $H$ onto $C$, $\partial I_C$ the subdifferential of $I_C$, where $I_C$ is as defined in (3.2) and $J_\lambda = (I + \lambda \partial I_C)^{-1}$. Then $y = J_\lambda x \iff y = \text{Proj}_C x$, $\forall x \in H, y \in C$.

**Theorem 3.5.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A : C \to H$ be an $\alpha$-inverse-strongly monotone mapping and let $T : C \to C$ be a nonexpansive mapping with fixed points. Assume that $\text{Fix}(T) \cap \text{VI}(C,A)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$ and $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Let $\{x_n\}$ be a
sequence generated in the following process: \(x_1 \in C\) and
\[
\begin{align*}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n)TP_C(y_n - r_nAy_n + e_n), \quad \forall n \geq 1,
\end{align*}
\]
where \(\{e_n\}\) is a sequence in \(H\) such that \(\sum_{n=1}^{\infty} \|e_n\| < \infty\). Assume that the above sequences satisfy the following restrictions:

(a) \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty\);

(b) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\);

(c) \(0 < a \leq r_n \leq b < 2\alpha\) and \(\sum_{n=1}^{\infty} |r_n - r_n - 1| < \infty\),

where \(a\) and \(b\) are two real numbers. Then \(\{x_n\}\) converges strongly to \(q = PF(T) \cap VI(C,A)f(q)\).

**Proof.** Putting \(Bx = \partial I_C\), we find from Lemma 3.4 the desired conclusion immediately.

If \(T\) is the identity mapping, we find from Lemma 3.4 the desired conclusion immediately.

**Corollary 3.6.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\). Let \(A : C \to H\) be an \(\alpha\)-inverse-strongly monotone mapping. Assume that \(VI(C,A)\) is not empty. Let \(\{\alpha_n\}\) and \(\{\beta_n\}\) be real number sequences in \((0, 1)\) and \(\{r_n\}\) be a positive real number sequence in \((0, 2\alpha)\). Let \(\{x_n\}\) be a sequence generated in the following process: \(x_1 \in C\) and
\[
\begin{align*}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n)P_C(y_n - r_nAy_n + e_n), \quad \forall n \geq 1,
\end{align*}
\]
where \(\{e_n\}\) is a sequence in \(H\) such that \(\sum_{n=1}^{\infty} \|e_n\| < \infty\). Assume that the above sequences satisfy the following restrictions:

(a) \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty\);

(b) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\);

(c) \(0 < a \leq r_n \leq b < 2\alpha\) and \(\sum_{n=1}^{\infty} |r_n - r_n - 1| < \infty\),

where \(a\) and \(b\) are two real numbers. Then \(\{x_n\}\) converges strongly to \(q = PV_{VI(C,A)}f(q)\).

**Conflict of Interests**

The author declares that there is no conflict of interests.

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