Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 3, 355-364
ISSN: 1927-6303

WEAKLY CONTRACTIVE MAPPINGS IN T_0-QUASI-METRIC SPACES

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Abstract. It is the aim of this paper to prove the existence of a fixed point for weakly C-contractive and weakly S-contractive self mappings defined in T_0-quasi-metric spaces.

Keywords: bicomplete di-metric; weakly C-contractive maps; weakly S-contractive maps; fixed point.

2010 AMS Subject Classification: 47H17, 47H05, 47H10.

1. Introduction

There is a growing interest for asymmetric structures, and more specifically for the ”asymmetric distances”. Recently, many results established in metric spaces which have their equivalent formulations in quasi-pseudometric spaces. However, the technicality of the proofs is completely different.

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Received March 31, 2014

In a recent paper, Gaba [1] proved a fixed point result for C-contractive and S-contractive self mappings defined in T_0-quasi-metric spaces. For recent results concerning the theory, we refer the reader to [2]-[5].

2. Preliminaries

In this section, we recall some elementary definitions and terminology from the classical theory as well as for asymmetric topology which are necessary for a good understanding of the work below. For more information about this theory, the reader is referred to [6].

Definition 2.1. Let (X,m) be a metric space. A map $T:X\to X$ is called a *C-contraction* iff there exists $0\leq k<\frac{1}{2}$ such that for all $x,y\in X$, the following inequality holds:

$$m(Tx,Ty)\leq k[m(x,Tx)+m(y,Ty)].$$

Definition 2.2. Let (X,m) be a metric space. A map $T:X\to X$ is called a *weak contraction* or said to be *weakly contractive* iff for all $x,y\in X$, the following inequality holds:

$$m(Tx,Ty)\leq km(x,Tx)-\psi(m(x,y)).$$

where $\psi:[0,\infty)\to[0,\infty)$ is a continuous and non-decreasing mapping such that $\psi(x)=0$ iff $x=0$ and $\lim\limits_{x\to\infty}\psi(x)=\infty$.

Definition 2.3. Let (X,m) be a metric space. A map $T:X\to X$ is called a *weak C-contraction* or said to be *weakly C-contractive* iff for all $x,y\in X$, the following inequality holds:

$$m(Tx,Ty)\leq\frac{1}{2}[m(x,Tx)+m(y,Ty)]-\psi(m(x,Tx),m(y,Ty)).$$

where $\psi:[0,\infty)^2\to[0,\infty)$ is a continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$.

Definition 2.4. Let (X,m) be a metric space. A map $T:X\to X$ is called a *S-contraction* iff there exists $0\leq k<\frac{1}{3}$ such that for all $x,y\in X$, the following inequality holds:

$$m(Tx,Ty)\leq k[m(x,Ty)+m(Tx,y)+m(x,y)].$$

Definition 2.5. Let (X,m) be a metric space. A map $T: X \to X$ is called a *weak S-contraction* or said to be *weakly S-contractive* iff for all $x,y \in X$, the following inequality holds:

$$m(Tx,Ty) \leq \frac{1}{3}[m(x,Ty)+m(Tx,y)+m(x,y)] - \psi(m(x,Ty),m(Tx,y),m(x,y)).$$

where $\psi : [0,\infty)^3 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y,z) = 0$ if and only if $x = y = z = 0$.

Definition 2.6. Let X be a non empty set. A function $d : X \times X \to [0;\infty)$ is called a *quasi-pseudometric* on X iff

i) $d(x,x) = 0 \quad \forall\, x \in X$,

ii) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall\, x,y,z \in X$.

Moreover, if $d(x,y) = 0 = d(y,x) \Longrightarrow x = y$, then d is said to be a T_0*-quasi-pseudometric* or a *di-metric*. The latter condition is referred to as the T_0-condition.

Example 2.7. [7] On $\mathbb{R} \times \mathbb{R}$, we define the real valued map d given by

$$d(a,b) = a \dot{-} b = \max\{a-b,0\}.$$

Then $(\mathbb{R},d)$ is a di-metric space.

Remark 2.8.

- Let d be a quasi-pseudometric on X. Then the map d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ whenever $x,y \in X$ is also a quasi-pseudometric on X, called the conjugate of d. (In the literature, it is also denoted by d^t or $\bar{d}$).
- It is easy to verify that the function d^s defined by $d^s := d \vee d^{-1}$, *i.e.*,

$$d^s(x,y) = \max\{d(x,y),d(y,x)\}$$

  defines a metric on X whenever d is a T_0-quasi-pseudometric.

Definition 2.9. The di-metric space (X,d) is said to be *bicomplete* if the metric space (X,d^s) is complete.

Example 2.10. Let $X = [0;\infty)$. Define for each $x,y \in X$, $n(x,y) = x$ if $x > y$, and $n(x,y) = 0$ if $x \leq y$. It is not difficult to check that (X,n) is a T_0-quasi-pseudometric space. Notice also that for $x,y \in [0;\infty)$, we have $n^s(x,y) = \max\{x,y\}$ if $x \neq y$ and $n^s(x,y) = 0$ if $x = y$. The metric n^s is complete on $[0,\infty)$.

Definition 2.11. Let (X,d) be a quasi-pseudometric space. For $x \in X$ and $\varepsilon > 0$,

$$B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$$

denotes the open ε-ball at x. The collection of all such balls is a base for a topology $\tau(d)$ induced by d on X. Similarly, for $x \in X$ and $\varepsilon \geq 0$,

$$C_d(x,\varepsilon) = \{y \in X : d(x,y) \leq \varepsilon\}$$

denotes the closed ε-ball at x.

In the case where (X,d) is a T_0 quasi-pseudometric space, we know that d^s defined by $d^s := d \vee d^{-1}$, i.e. $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ defines a metric on X. Hence, we shall say that a subset $E \subset X$ is *join closed* if it is d^s-closed, *i.e.*, closed with respect to the topology generated by d^s.

Definition 2.12. Let X be a nonempty set. Two self mappings $F,G : X \to X$ are said to be *weakly compatible* iff for all $x \in X$ the equality $Fx = Gx$ implies $FGx = GFx$.

Next, we recall the following interesting results established in Chatterja *et al.* [8], Shukla *et al.* [9] and Vahid [10].

Theorem 2.13. *A weak C-contraction on a complete metric space has a unique fixed point. A weak S-contraction on a complete metric space has a unique fixed point.*

Theorem 2.14. *Let (X,d) be a complete metric space and let E be a nonempty closed subset of X. Let $T,S : E \to E$ be such that*

$$d(Tx,Sy) \leq \frac{1}{2}[d(Rx,Sy) + d(Tx,Ry)] - \psi(d(Rx,Sy), d(Tx,Ry))$$

for every pair $(x,y) \in X^2$, where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$ and $R : E \to X$ satisfying the following hypothesis:

(i) *$TE \subseteq RE$ and $SE \subseteq RE$,*

(ii) *the pairs (T,R) and (S,R) are weakly compatible.*

In addition, we assume that RE is a closed subset of X. Then T, R and S have a unique common fixed point.

The following results generalize the above theorems to the setting of a bicomplete di-metric space.

Definition 2.15. Let (X,d) be a quasi-pseudometric space. A map $T : X \to X$ is called a *weak C-pseudocontraction* or said to be *weakly C-pseudocontractive* iff for all $x,y \in X$, the following inequality holds:

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx)+d(y,Ty)] - \psi(d(x,Tx),d(y,Ty)).$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

Definition 2.16. Let (X,d) be a quasi-pseudometric space. A map $T : X \to X$ is called a *weak S-pseudocontraction* or said to be *weakly S-pseudocontractive* iff for all $x,y \in X$, the following inequality holds:

$$d(Tx,Ty) \leq \frac{1}{3}[d(x,Ty)+d(Tx,y)+d(x,y)] - \psi(d(x,Ty),d(Tx,y),d(x,y)).$$

where $\psi : [0,\infty)^3 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y,z) = 0$ if and only if $x = y = z$.

Definition 2.17. Let $E_1,\cdots,E_n,F$ be totally ordered spaces with respective orders $\leq_{E_1},\cdots,\leq_{E_n}$ and $\leq_F$. A map $f : E_1 \times E_2 \times \cdots \times E_n \to F$ is said to be *component non-increasing* if

$$f(x_1,\cdots,x_n) \leq_F f(a_1,\cdots,a_n)$$

whenever $a_i \leq_{E_i} x_i$ for any $i = 1,\cdots,n$.

Example 2.18. Let $E_1 = E_2 = F = [0,\infty)$ with the natural order and define the function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $f(x,y) = -(x^2+y^2)$. Clearly, if $a \leq b$ and $c \leq d$, we have $f(b,d) \leq f(a,c)$.

More generally, by setting $E_1 = E_2 = \cdots = E_n = F = [0,\infty)$ with the natural order and defining the function $f : [0,\infty) \times \cdots \times [0,\infty) \to [0,\infty)$ by $f(x_1,x_2,\cdots,x_n) = -(x_1^2+x_2^2+\cdots+x_n^2)$, f is component non-increasing.

3. Main results

We are in a position to state our first fixed point result.

Theorem 3.1. *Let (X,d) be a totally ordered bicomplete di-metric space and let $T : X \to X$ be a weak C-pseudocontraction. Moreover, we assume that ψ is component non-increasing. Then T has a unique fixed point.*

Proof. Since $T : X \to X$ is a a weak C-pseudocontraction, for all $x,y \in X$, the following inequality holds

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(Tx,y)] - \psi(d(x,Ty),d(Tx,y)).$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$. For any $x,y \in X$, we have

$$\begin{aligned} d^{-1}(Tx,Ty) = d(Ty,Tx) &\leq \frac{1}{2}[d(y,Tx)+d(Ty,x)] - \psi(d(y,Tx),d(Ty,x)). \\ &\leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)] - \psi(d^{-1}(Tx,y),d^{-1}(x,Ty), \end{aligned}$$

that is,

$$d^{-1}(Tx,Ty) \leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)] - \psi(d^{-1}(Tx,y),d^{-1}(x,Ty),$$

and we see that $T : (X,d^{-1}) \to (X,d^{-1})$ is a weak C-pseudocontraction. Therefore, since ψ is component non-increasing, we have

$$\begin{aligned} d(Tx,Ty) &\leq \frac{1}{2}[d(x,Ty)+d(Tx,y)] - \psi(d(x,Ty),d(Tx,y)) \\ &\leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)] - \psi(d^s(x,Ty),d^s(Tx,y)), \end{aligned}$$

and

$$\begin{aligned} d^{-1}(Tx,Ty) &\leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)] - \psi(d^{-1}(Tx,y),d^{-1}(x,Ty), \\ &\leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)] - \psi(d^s(x,Ty),d^s(Tx,y)), \end{aligned}$$

for all $x,y \in X$. Hence, we have

$$d^s(Tx,Ty) \leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)] - \psi(d^s(x,Ty),d^s(Tx,y)),$$

for all $x,y \in X$ and so, $T : (X,d^s) \to (X,d^s)$ is a weak C-contraction. By assumption, (X,d) is bicomplete, hence (X,d^s) is complete. Therefore by Theorem 2.13, T has a unique fixed point. This completes the proof.

Theorem 3.2. *Let (X,d) be a totally ordered bicomplete di-metric space and let E be a nonempty join closed subset of X. Let $T,S : E \to E$ be such that*

$$d(Tx,Sy) \leq \frac{1}{2}[d(Rx,Sy)+d(Tx,Ry)] - \psi(d(Rx,Sy),d(Tx,Ry)),$$

and

$$d(Sx,Ty) \leq \frac{1}{2}[d(Sx,Ry)+d(Rx,Ty)] - \psi(d(Sx,Ry),d(Rx,Ty)),$$

for every pair $(x,y) \in X^2$, where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$ and $R : E \to X$ satisfy the following hypothesis:

(i) *$TE \subseteq RE$ and $SE \subseteq RE$,*

(ii) *the pairs (T,R) and (S,R) are weakly compatible.*

In addition, we assume that RE is a join closed subset of X. Then T, R and S have a unique common fixed point.

Proof. We prove that $T : (E,d^s) \to (E,d^s)$ satisfies the assumptions of Theorem 2.14. For every pair $(x,y) \in X^2$, we have

$$d(Tx,Sy) \leq \frac{1}{2}[d(Rx,Sy)+d(Tx,Ry)] - \psi(d(Rx,Sy),d(Tx,Ry)).$$

It is also very clear that

$$d^{-1}(Tx,Sy) = d(Sy,Tx) \leq \frac{1}{2}[d^{-1}(Rx,Sy)+d^{-1}(Tx,Ry)] - \psi(d^{-1}(Rx,Sy),d^{-1}(Tx,Ry)).$$

Since ψ is component non-increasing, we have

$$\begin{aligned} d(Tx,Sy) &\leq \frac{1}{2}[d(Rx,Sy)+d(Tx,Ry)] - \psi(d(Rx,Sy),d(Tx,Ry)) \\ &\leq \frac{1}{2}[d^s(Rx,Sy)+d^s(Tx,Ry)] - \psi(d^s(Rx,Sy),d^s(Tx,Ry)) \end{aligned}$$

and

$$d^{-1}(Tx,Sy) \leq \frac{1}{2}[d^{-1}(Rx,Sy)+d^{-1}(Tx,Ry)] - \psi(d^{-1}(Rx,Sy),d^{-1}(Tx,Ry))$$
$$\frac{1}{2}[d^{s}(Rx,Sy)+d^{s}(Tx,Ry)] - \psi(d^{s}(Rx,Sy),d^{s}(Tx,Ry)).$$

Hence, we see that

$$d^{s}(Tx,Sy) \leq \frac{1}{2}[d^{s}(Rx,Sy)+d^{s}(Tx,Ry)] - \psi(d^{s}(Rx,Sy),d^{s}(Tx,Ry)).$$

By assumption, (X,d) is bicomplete, hence (X,d^s) is complete. Moreover, since E and RE are join closed, we conclude by Theorem 2.14 that T,R and S have a unique common fixed point.

Theorem 3.3. *Let (X,d) be a totally ordered bicomplete di-metric space and let $T:X\to X$ be a weak S-pseudocontraction. Moreover, we assume that ψ is component non-increasing. Then T has a unique fixed point.*

Proof. It is enough to prove that $T:(X,d^s)\to(X,d^s)$ is a weak S-contraction. Since $T:X\to X$ is a weak S-pseudocontraction, for all $x,y\in X$ the following inequality holds

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(Tx,y)+d(x,y)] - \psi(d(x,Ty),d(Tx,y),d(x,y)).$$

where $\psi:[0,\infty)^3\to[0,\infty)$ is a continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=z=0$. For any $x,y\in X$, we have

$$\begin{aligned}
&d^{-1}(Tx,Ty) = d(Ty,Tx)\\
&\leq \frac{1}{2}[d(y,Tx)+d(Ty,x)+d(x,y)] - \psi(d(y,Tx),d(Ty,x),d(x,y))\\
&\leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)d^{-1}(x,y)] - \psi(d^{-1}(Tx,y),d^{-1}(x,Ty),d^{-1}(y,x)),
\end{aligned}$$

that is,

$$d^{-1}(Tx,Ty) \leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)+d^{-1}(x,y)] - \psi(d^{-1}(Tx,y),d^{-1}(x,Ty),d^{-1}(y,x)),$$

and we see that $T:(X,d^{-1})\to(X,d^{-1})$ is a weak S-pseudocontraction. Since ψ is component non-increasing, we have

$$\begin{aligned} d(Tx,Ty) &\leq \frac{1}{2}[d(x,Ty)+d(Tx,y)+d(x,y)]-\psi(d(x,Ty),d(Tx,y),d(x,y)),\\ &\leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)+d^s(x,y)]-\psi(d^s(x,Ty),d^s(Tx,y),d^s(x,y)), \end{aligned}$$

and

$$\begin{aligned} d^{-1}(Tx,Ty) &\leq \frac{1}{2}[d^{-1}(Tx,y)+d^{-1}(x,Ty)]-\psi(d^{-1}(Tx,y),d^{-1}(x,Ty),\\ &\leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)+d^s(x,y)]-\psi(d^s(x,Ty),d^s(Tx,y),d^s(x,y)), \end{aligned}$$

for all $x,y\in X$. Hence, we have

$$d^s(Tx,Ty)\leq \frac{1}{2}[d^s(x,Ty)+d^s(Tx,y)+d^s(x,y)]-\psi(d^s(x,Ty),d^s(Tx,y),d^s(x,y)),$$

for all $x,y\in X$ and so, $T:(X,d^s)\to(X,d^s)$ is a weak S-contraction. By assumption, (X,d) is bicomplete, hence (X,d^s) is complete. Therefore by Theorem 2.13, T has a unique fixed point. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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