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FIXED POINT THEOREMS IN A SPACE WITH TWO METRICS

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Abstract. In this paper, we prove a generalization of the Banach contraction fixed point theorem in a space with two metrics. The results presented in this paper improve the corresponding results announced by many authors. As an application, we study the existence of the solution for a functional equation arising in dynamic programming.

Keywords: Banach contraction; complete metric space; fixed point theorem; functional equation.

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1. Introduction

Fixed point methods have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, and network. Fixed point problems include many nonlinear problems as special cases, such as variational inequality problems, complementarity problems, and saddle point problems. The most well known result in metric spaces is the Banach contraction principle.

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Theorem 1.1. *Let (E,d) be a complete metric space and let T be a mapping of E into E. If there exists $k \in [0,1[$ such that*

$$d(Tx,Ty) \leq kd(x,y), \quad \forall x,y \in E,$$

then there exists an unique point $x^* \in E$ such that $Tx^* = x^*$.

The result has been improved and generalized by many authors in different directions based on different methods. The purpose of this article is to get a generalization of the Banach contraction fixed point theorem in a space with two metrics. As an application, the existence of the solution for a functional equation arising in dynamic programming is investigated.

2. Preliminaries

Let (X,d) be a metric space. Denote by $CB(X)$ the set of all nonempty closed and bounded subsets of X and by H the Hausdorff distance defined on $CB(X)$ by

$$H(A,B) = \max\{\sup_{a\in A} d(a,B)\ ;\ \sup_{b\in B} d(b,A)\}, \quad \forall A,B \in CB(X).$$

The following results is due to Mizoguchi and Takahashi [1].

Theorem MT-1. *Let (X,d) be a complete metric space. Let T and S be two multi-valued functions from X into $CB(X)$ satisfying*

$$H(T(x),S(y)) \leq \alpha(d(x,y))d(x,y), \quad \forall (x,y) \in X^2.$$

where $\alpha : [0,+\infty[\to [0,1[$ is a function such that $\limsup_{s\to r^+} \alpha(s) < 1$, for all $r \geq 0$. Then there exists $z \in X$ such that $z \in T(z) \cap S(z)$.

Theorem MT-2. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping satisfying*

$$d(T(x),T(y)) \leq \alpha(d(x,y))d(x,y), \quad \forall x,y \in X,$$

where $\alpha : [0,+\infty[\to [0,1[$ is a function such that $\limsup_{s\to r^+} \alpha(s) < 1$, for all $r \geq 0$. Then T has a unique fixed point.

Let ϕ be the function defined from $[0,1[$ into $]0,1]$ by

$$\phi(r) = \begin{cases} 1, & \text{if } \ 0 \leq r \leq \frac{1}{2}, \\ 1-r & \text{if } \ \frac{1}{2} \leq r < 1, \end{cases}$$

and put

$$M(Sx,Ty) = \max\left(d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right).$$

The following results is due to Singh, Mishra, Chugh and Kamal [2].

Theorem SMCK-1. *Let X be a complete metric space and $S,T : X \to CL(X)$. Assume there exists $r \in [0,1[$ such that for every $x,y \in X$,*

$$\phi(r)\min(d(x,Sx), d(y,Ty)) \leq d(x,y) \Rightarrow H(Sx,Ty) \leq rM(Sx,Ty).$$

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$

Theorem SMCK-2. *Let (X,d) be a complete metric space and let T a self-mapping on X. Assume there exists $r \in [0,1[$ such that*

$$\phi(r).min\{d(x,Tx); d(y,Ty)\} \leq d(x,y) \Longrightarrow d(Tx,Ty) \leq r.M(x,y),$$

for all $x,y \in X$. Then T has a unique fixed point.

3. Main results

Let X be a nonempty set and let d,δ be two metrics on X.

Definition 3.1. *(X,d,δ) is called an (M)-space if for all Cauchy sequence $(x_n)_n$ in (X,d) and (X,δ), there exist $x^*, y^* \in X$ such that*

$$\lim_n d(x_n,x^*) = \lim_n \delta(x_n,y^*) = 0.$$

Example 3.1. *If (X,d) and (X,δ) are complete metric spaces, then (X,d,δ) is an (M)-space.*

Example 3.2. *Let X be the set of all C^1 functions u from $[0,1]$ into $\mathbb{R}$ with $u(0)=0$. We define two metrics on X by*

$$d(u,v) = \sup_{x\in[0,1]} |u(x)-v(x)| \quad \textit{and} \quad \delta(u,v) = \sup_{x\in[0,1]} |u'(x)-v'(x)|,$$

for all $u, v \in X$. It is well known that the sequence of the polynomial functions defined by:

$$u_1(x) = 0 \quad \text{and} \quad u_{n+1}(x) = u_n(x) + \frac{1}{2}(x - u_n^2(x))$$

are in X and converges uniformly to $\sqrt{x}$ which is not in X. Hence, (X,d) is non complete. If $(v_n)_n$ is a Cauchy sequence in (X,d) and (X,δ), there exist two continuous functions u and v such that $(v_n)_n$ and $(v'_n)_n$ converges uniformly to u et v, respectively. Then u is derivable and $u' = v$. Thus, $u \in X$ and $\lim_n d(v_n,u) = \lim_n \delta(v_n,u) = 0$. It follows that (X,d,δ) is an (M)-space.

We define mapping α as before. Now, we are in a position to state our main results.

Theorem 3.1. *Let X be a non-empty set, d and δ two metrics on X; and $T : X \to X$ a mapping such that*

(1) (X,d,δ) is an (M)-space.

(2) For all $x,y \in X$, one of the following two conditions:

(i) $d(x,Ty) \le \delta(x,y)$,

(ii) $\delta(x,Ty) \le d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \le \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) \le \alpha(d(x,y))d(x,y). \end{cases}$$

Then T has a unique fixed point in X.

Proof. We divide the proof into four steps.

Setp 1. Letting $x_o \in X$, we define the sequence $(x_n)_n$ by $x_{n+1} = Tx_n$. For each $n \in \mathbb{N}$, we have

$$\delta(x_{n+1},Tx_n) = 0 \le d(x_n,x_{n+1}).$$

It follows that

$$\begin{cases} \delta(Tx_n,Tx_{n+1}) = \delta(x_{n+1},x_{n+2}) \le \alpha(d(x_n,x_{n+1}))d(x_n,x_{n+1}), \\ d(Tx_n,Tx_{n+1}) = d(x_{n+1},x_{n+2}) \le \alpha(\delta(x_n,x_{n+1}))\delta(x_n,x_{n+1}). \end{cases}$$

For any $n \in I\!N^*$, we have

$$\begin{cases} \delta(x_{n+1},x_{n+2}) \leq \alpha(d(x_n,x_{n+1}))\alpha(\delta(x_{n-1},x_n))\delta(x_{n-1},x_n) \leq \delta(x_{n-1},x_n), \\ d(x_{n+1},x_{n+2}) \leq \alpha(\delta(x_n,x_{n+1}))\alpha(d(x_{n-1},x_n))d(x_{n-1},x_n) \leq d(x_{n-1},x_n). \end{cases}$$

It follows that $(d(x_{2p},x_{2p+1}))_p$, $(d(x_{2p+1},x_{2p+2}))_p$, $(\delta(x_{2p},x_{2p+1}))_p$ and $(\delta(x_{2p+1},x_{2p+2}))_p$ converges to d_1, d_2, δ_1 and δ_2, respectively. Since $\limsup_{t\to d_1^+}\alpha(t) < 1$ and $\limsup_{t\to \delta_2^+}\alpha(t) < 1$, there exist $r_1 \in [0,1[$ and an integer p_1 such that

$$\delta(x_{2p+1},x_{2p+2}) \leq r_1\delta(x_{2p-1},x_{2p}), \quad \forall p \geq p_1.$$

Since $\limsup_{t\to d_2^+}\alpha(t) < 1$ and $\limsup_{t\to \delta_1^+}\alpha(t) < 1$, there exist $r_2 \in [0,1[$ and an integer p_2 such that

$$\delta(x_{2p+2},x_{2p+3}) \leq r_2\delta(x_{2p},x_{2p+1}), \quad \forall p \geq p_2.$$

It follows that $\sum_{p\geq 0}\delta(x_{2p},x_{2p+1})$ and $\sum_{p\geq 1}\delta(x_{2p-1},x_{2p})$ are convergent. Then

$$\sum_{n\geq 0}\delta(x_n,x_{n+1}) = \sum_{p\geq 0}\delta(x_{2p},x_{2p+1}) + \sum_{p\geq 1}\delta(x_{2p-1},x_{2p})$$

is convergent. In the same way, we find $\sum_{n\geq 0} d(x_n,x_{n+1})$ is convergent. Hence, $(x_n)_n$ is a Cauchy sequence in (X,d) and in (X,δ). Since, (X,d,δ) is an (M)-space, there exist $x^*, y^* \in X$ such that

$$\lim_n d(x_n,x^*) = \lim_n \delta(x_n,y^*) = 0.$$

Step 2. Assume $x^* \neq y^*$. Since $\lim_n d(Tx_n,x^*) = 0$ and $\lim_n \delta(x_n,x^*) = \delta(y^*,x^*) > 0$, we obtain $d(Tx_n,x^*) \leq \delta(x_n,x^*)$, for large integers, which gives

$$\begin{cases} d(x_{n+1},Tx^*) = d(Tx_n,Tx^*) \leq \alpha(\delta(x_n,x^*))\delta(x_n,x^*), \\ \delta(x_{n+1},Tx^*) = \delta(Tx_n,Tx^*) \leq \alpha(d(x_n,x^*))d(x_n,x^*). \end{cases}$$

The second inequality in turn implies $y^* = Tx^*$. On the other hand, we have $\limsup_n \alpha(\delta(x_n,x^*)) < 1$ implies that there exists $k_1 \in [0,1[$ such that $d(Tx_n,Tx^*) \leq k_1\delta(x_n,x^*)$, for large integers. It follows that $d(x^*,y^*) \leq k_1\delta(y^*,x^*)$. Similarly, we obtain, for some $k_2 \in [0,1[$, $\delta(x^*,y^*) \leq k_2 d(y^*,x^*)$. Hence, we find that $x^* = y^*$, which is contradiction. Thus $x^* = y^*$.

Step 3. To prove that $Tx^* = x^*$, we consider the sets A and B defined by

$$A = \{n \in I\!N / d(Tx_n,x^*) \leq \delta(x_n,x^*)\},$$

and

$$B = \{n \in I\!N / \delta(Tx_n, x^*) \leq d(x_n, x^*)\}.$$

We asserts that A or B is infinite. If A and B are finite, there exists an integer N such that, for all integers $n \geq N$,

$$\begin{cases} d(Tx_n, x^*) > \delta(x_n, x^*), \\ \delta(Tx_n, x^*) > d(x_n, x^*). \end{cases}$$

Hence, we have $d(x_n, x^*) < d(x_{n+2}, x^*)$, for all integers $n \geq N$. Thus, the sequence $d(x_{2n}, x^*)_n$ is strictly increasing to 0, which is a false assertion. If we assume that A is infinite, there exists some subsequence $(x_{\sigma(n)})_n$ such that $d(Tx_{\sigma(n)}, x^*) \leq \delta(x_{\sigma(n)}, x^*)$. Then

$$\begin{cases} d(Tx_{\sigma(n)}, Tx^*) \leq \alpha(\delta(x_{\sigma(n)}, x^*))\delta(x_{\sigma(n)}, x^*) \\ \delta(Tx_{\sigma(n)}, Tx^*) \leq \alpha(d(x_{\sigma(n)}, x^*))d(x_{\sigma(n)}, x^*), \end{cases}$$

which implies that $d(x_{\sigma(n)+1}, Tx^*) \leq \delta(x_{\sigma(n)}, x^*)$. Thus, $d(x^*, Tx^*) = 0$. Hence, x^* is a fixed point of T.

Step 4. For the uniqueness of the fixed point, we assume that $\bar{x}$ and $\bar{y}$ are two different fixed points of T. We have $d(\bar{x}, \bar{y}) \leq \delta(\bar{x}, \bar{y})$ or $\delta(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{y})$. For the first case, we obtain $d(\bar{x}, T\bar{y})) = d(\bar{x}, \bar{y}) \leq \delta(\bar{x}, \bar{y})$ and then

$$\begin{cases} d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \leq \alpha(\delta(\bar{x}, \bar{y}))\delta(\bar{x}, \bar{y}) < \delta(\bar{x}, \bar{y}) \\ \delta(\bar{x}, \bar{y}) = \delta(T\bar{x}, T\bar{y}) \leq \alpha(d(\bar{x}, \bar{y}))d(\bar{x}, \bar{y}) < d(\bar{x}, \bar{y}) \end{cases}$$

which is a contradiction. Thus, T has a unique fixed point in X.

If $d = \delta$, we obtain the following results.

Corollary 3.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that, for all $x, y \in X$, we have*

$$d(x, Ty) \leq d(x, y) \textit{ implies } d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y).$$

Then here exists an unique element $x^* \in X$ such that $Tx^* = x^*$.

Corollary 3.2. *Let X a non-empty set and let d, δ be two metrics on X. Let $T : X \to X$ a mapping such that*

(1) (X,d,δ) *is an (M)-space.*

(2) There exists $r \in [0,1[$ *such that for all* $x,y \in X$*, one of the following assertions*

`(i)`**:** $d(x,Ty) \leq \delta(x,y)$,

`(ii)`**:** $\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq r\,\delta(x,y), \\ \delta(Tx,Ty) \leq r\,d(x,y). \end{cases}$$

Then, there exists an unique element $x^* \in X$ *such that* $Tx^* = x^*$.

If $d = \delta$, we find the following.

Corollary 3.3. *Let* (X,d) *a complete metric space and let* $T : X \to X$ *be a mapping. Assume there exists* $r \in [0,1[$*, such that for all* $x,y \in X$*,* $d(x,Ty) \leq d(x,y)$ *implies* $d(Tx,Ty) \leq rd(x,y)$*. Then, there exists a unique element* $x^* \in X$ *such that* $Tx^* = x^*$.

Let ψ the function from $[0,1[$ into $]0,1]$. We have the following result.

Theorem 3.2. *Let* X *be a non-empty set and let* d *and* δ *be two metrics of* X*. Let* $T : X \to X$ *be a mapping such that*

(1) (X,d,δ) *is an (M)-space.*

(2) For all $x,y \in X$*, one of the conditions:*

`(i)`**:** $\psi(\alpha(d(x,Ty)))d(x,Ty) \leq \delta(x,y)$,

`(ii)`**:** $\psi(\alpha(\delta(x,Ty)))\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) \leq \alpha(d(x,y))d(x,y). \end{cases}$$

Then T *has a unique fixed point in* X.

Corollary 3.4. *Let* (X,d) *an complete metric space and let* $T : X \to X$ *be a mapping such that, for all* $(x,y) \in X^2$,

$$\psi(\alpha(d(x,Ty)))d(x,Ty) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha(d(x,y))\,d(x,y).$$

Then, there exists an unique element $x^* \in X$ *such that* $Tx^* = x^*$.

Corollary 3.5. *Let X a non-empty set and let d, δ be two metrics on X. Let $T : X \to X$ a mapping such that:*

(1) (X,d,δ) is an (M)-space.

(2) There exists $r \in [0,1[$ such that for all $x,y \in X$, one of the following assertions

(i) : $\psi(r)d(x,Ty) \leq \delta(x,y)$,

(ii) : $\psi(r)\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq r.\delta(x,y), \\ \delta(Tx,Ty) \leq r.d(x,y). \end{cases}.$$

Then, there exists an unique element $x^* \in X$ such that $Tx^* = x^*$.

Example 3.3. *Let $X = [0,1] \cup \{2,3,4,5\}$ endowed with the usual distance d and the distance δ defined by*

$$\delta(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1], \\ x+y & \text{if } x \text{ or } y \text{ is not in [0,1] and } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(X,d) and (X,δ) are complete metric spaces. We define α from $[0,+\infty[$ into $[0,1[$ by $\alpha(t) = \frac{2}{3}e^{-t}$, and consider the mapping defined on X by

$$Tx = \begin{cases} kx & \text{if } x \in [0,1[, \\ 0 & \text{if } x \geq 1, \end{cases}$$

where $k \in]0, \frac{4}{3e^6}[$. We asserts that

$$\begin{cases} d(Tx,Ty) & \leq \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) & \leq \alpha(d(x,y))d(x,y), \end{cases}$$

for all $x,y \in X - \{1\}$. This assertion is obviously satisfied if $x,y \in [0,1[$ or $x,y \in \{2,3,4,5\}$. Assume $x \in [0,1[$ and $y = 1$. If $d(Tx,Ty) \leq \alpha(\delta(x,y))\delta(x,y)$, we obtain $kx \leq \frac{2}{3}e^{-(1-x)}(1-x)$, which is not true for x near 1. If $d(x,Ty) \leq \delta(x,y)$ or $\delta(x,Ty) \leq d(x,y)$, we obtain $x \in [0, \frac{1}{2}]$.

Then

$$\begin{cases} d(Tx,Ty) & = kx \leq \frac{2}{3e} \leq \frac{2}{3}e^{-(1-x)}(1-x) = \alpha(\delta(x,y))\delta(x,y), \\ \delta(Tx,Ty) & = kx \leq \frac{2}{3e} \leq \frac{2}{3}e^{-(1-x)}(1-x) = \alpha(d(x,y))d(x,y). \end{cases}$$

If $d(Tx,y) \leq \delta(x,y)$ *or* $\delta(Tx,y) \leq d(x,y)$, *we obtain* $x = 0$. *Thus the assertion is satisfied. Note that* 0 *is the unique fixed point of* T

Example 3.4 *Let* $X = [0,1]$ *and* d, δ *two distances on* X *defined by*

$$d(x,y) = |x-y| \text{ and } \delta(x,y) = 2|x-y|, \quad \forall (x,y) \in X^2.$$

Considering $T : X \to X$ *such that*

$$T(x) = \begin{cases} \frac{x}{8} \text{ if } x \in [0,1[, \\ 0 \text{ if } x = 1. \end{cases} \quad .$$

T *satisfies the hypotheses of corollary 3.2, but does not meet the assumption: for all* $(x,y) \in X^2$

$$\begin{cases} d(Tx,Ty) \leq r\, \delta(x,y), \\ \delta(Tx,Ty) \leq r\, d(x,y), \end{cases}$$

where $r = \frac{2}{3}$.

Example 3.5 *Let* $X = [0,1]$ *and let* d, δ *be two distances on* X *defined by*

$$d(x,y) = |x-y| \text{ and } \delta(x,y) = 2|x-y|, \forall (x,y) \in X^2.$$

Considering $T : X \to X$ *such that*

$$T(x) = \begin{cases} \frac{x}{20} \text{ if } x \in [0,1[, \\ 0 \text{ if } x = 1. \end{cases}$$

T *satisfies the hypotheses of corollary 3.2*

(i) $\phi(r)d(x,Ty) \leq \delta(x,y)$,

(ii) $\phi(r)\delta(x,Ty) \leq d(x,y)$,

implies

$$\begin{cases} d(Tx,Ty) \leq r.\delta(x,y). \\ \delta(Tx,Ty) \leq r.d(x,y), \end{cases}$$

where ϕ is the function defined from $[0,1[$ into $]0,1]$ by

$$\phi(r)=\begin{cases} 1 & \text{if} \quad 0\le r\le \frac{1}{2}, \\ 1-r & \text{if} \quad \frac{1}{2}\le r<1. \end{cases}$$

But it does not meet the assumption for all $(x,y)\in X^2$

$$\begin{cases} d(Tx,Ty) & \le r.\delta(x,y), \\ \delta(Tx,Ty) & \le r.d(x,y), \end{cases}$$

where $r=\frac{2}{3}$.

4. Application

We assume that E and F are Banach spaces, $X\subset E$ and $Y\subset F$. Let $g: X\times Y\to I\!R$ and $G: X\times Y\times I\!R\to I\!R$ two mappings. A problem arising in dynamic programming reduces to the problem of solving the functional equation:

$$\text{(E)} \qquad p(x)=\sup_{y\in Y}\{g(x,y)+G(x,y,p(x)\},\quad \forall x\in X.$$

Denote by $\mathscr{B}(X)$ the space of all real bounded functions on X and define the metric of uniform convergence d_∞ by

$$d_\infty(h,k)=\sup_{x\in X}|h(x)-k(x)|,\quad \forall h,k\in\mathscr{B}(X).$$

Define the functional A on $\mathscr{B}(X)$ by:

$$Ah(x)=\sup_{y\in Y}\{g(x,y)+G(x,y,h(x))\},\quad \forall h\in\mathscr{B}(X).$$

And consider the following conditions:

(C1) **:** g and G are bounded functions,

(C2) **:** For every $(x,y,z)\in X\times Y\times Y$, $t\in X$ and $(h,k)\in\mathscr{B}(X)\times\mathscr{B}(X)$

$$\phi(\alpha(d_\infty(h,Ak)))|h(t)-Ak(t)|\le|h(t)-k(t)|$$

implies

$$|(g(x,y)-g(x,z))+(G(x,y,h(t))-G(x,z,k(t))|\le\alpha(d_\infty(h,k))|h(t)-k(t)|,$$

where ϕ and α is defined as before.

Theorem 4.1. *Under the conditions* `(C1)` *and* `(C2)` *the functional equation* (E) *has a unique solution in* $\mathscr{B}(X)$.

Proof. For any strictly positive real ε and for all $h,k \in \mathscr{B}(X)$ and $x \in X$, there exists $y_h, y_k \in Y$ such that

$$\begin{cases} Ah(x) - \varepsilon < g(x,y_h) + G(x,y_h,h(x)) \le Ah(x), \\ Ak(x) - \varepsilon < g(x,y_k) + G(x,y_k,k(x)) \le Ak(x), \end{cases}$$

which gives

$$\begin{cases} g(x,y_h) - g(x,y_k) + G(x,y_h,h(x)) - G(x,y_k,k(x)) - \varepsilon < Ah(x) - Ak(x), \\ Ah(x) - Ak(x) < g(x,y_h) - g(x,y_k) + G(x,y_h,h(x)) - G(x,y_k,k(x)) + \varepsilon. \end{cases}$$

It follows that

$$|Ah(x) - Ak(x)| < |(g(x,y_h) - g(x,y_k)) + (G(x,y_h,h(x)) - G(x,y_k,k(x)))| + \varepsilon.$$

Hence conditions `(C2)` becomes

$$\phi(\alpha(d_\infty(h,Ak))).|h(t) - Ak(t)| \le |h(t) - k(t)|,$$

which implies

$$|(g(x,y) - g(x,z)) + (G(x,y,h(t)) - G(x,z,k(t))| \le \alpha(d_\infty(h,k)).|h(t) - k(t)|.$$

Then,

$$|Ah(x) - Ak(x)| \le \alpha(d_\infty(h,k)).|h(t) - k(t)| + \varepsilon \le \alpha(d_\infty(h,k)).d_\infty(h,k) + \varepsilon.$$

Since this inequality is true for any $x \in W$, and $\varepsilon > 0$ is arbitrary, we obtain

$$\phi(\alpha(d_\infty(h,Ak)))d_\infty(h,Ak) \le d_\infty(h,k)$$

implies

$$d_\infty(Ah,Ak) \le \alpha(d_\infty(h,k)).d_\infty(h,k).$$

Since $(\mathscr{B}(X), d_\infty)$ is a complete metric space, the corollary prove that there exists an unique element $h^* \in \mathscr{B}(X)$ such that $Ah^* = h^*$ and then h^* is the unique bounded solution of the functional equation (E).

Conflict of Interests

The authors declare that there is no conflict of interests.

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