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AFFINE LOCALIZATION FOR FIXED POINTS OF LIPSCHITZ QUOTIENT OPERATORS

XAVIER UDO-UTUN*

Department of Mathematics and Statistics, University of Maiduguri, Maiduguri - Nigeria

Abstract. We modify and apply new property obtained recently in (Udo-utun, Fixed Point Theory and Applications 2014, 2014:65) and results in (Berinde, Carpath. J. Math. 19(1):7-22, 2003; Nonlinear Anal. Forum 9(1):43-53, 2004) on $(\delta,k)-$weak contractions to obtain asymptotic fixed point theorems for bi-Lipschitz mappings and Lipschitz quotient mappings in Banach spaces. Our results complement and improve several fixed point theorems for Lipschitzian mappings.

Keywords: affine approximation; affine localization; $(\delta,k)-$weak contraction; almost contraction; fixed point; Lipschitz quotient maps; metric space; Krasnoselskii's iteration.

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1. Introduction

Studies of Affine localization has been intensive in connection with linear approximations of Lipschitzian mappings [1, 6, 8], differentiability of continuous operators [8] and in the quest for linear isomorphisms on - and linear quotients of Banach spaces concerning Lipschitz quotient mappings [1, 6]. Bates et al [1] initiated the study of nonlinear quotient mappings in the context

*Corresponding author
E-mail address: xavierudoutun@gmail.com
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of existence of linear isomorphisms. In the process they introduced the concept of *approximation by affine property, AAP* for associated classes Lispschitz mappings. This notion is based on the concept of affine localization of Lipschitz quotient mapping, definition of which is, in turn, based on the concept of co-Lipschitz mappings. In this work we digress by introducing and studying the concept of *affine quasi-localization* to formulate asymptotic fixed points of a very wide class of Lipschitz operators and prove corresponding asymptotic fixed point theorems. Our method involves showing that every contraction admits affine localization. Equipped with this, we proceed to show that all contractions satisfy the following modification of a condition derived recently by the author in [11]:

$$\begin{aligned} &\|y-Tx\| \leq M\|x-y\| \\ &\text{whenever } \|x-y\| \leq \|y-Tx\| \end{aligned} \tag{1}$$

for some $M \geq 1$ and for all x and y in the orbit $\mathscr{O}(x_0)$ of certain point x_0 in arbitrary Banach space E, provided $x \neq y$ and $x, y \notin Fix(T) = \{x \in E : Tx = x\}$ called the fixed point set of an operator T.

We have proved that every Lipschitzian mapping which admit affine quasi-localization in the orbits of certain points x_0 in a closed convex set of a Banach space satisfies (1), and hence has a fixed point that depends on convergence of the Krasnoselskii scheme. The significance of our work is by no means trivial in obtaining, for Lipschitzian mappings, existence results which depend on convergence of Krasnoselskii iteration scheme. By this we mean that, in this case, convergence of Krasnoselkii iteration scheme does not assume existence of fixed point rather the existence depends on its convergence. Also, this work is significant in delineating a behavior of contraction mappings through which studies of existence of fixed points for other classes of mappings can be carried out. This is an advantage over prevalent modifications on contractive constants in literature which break down for certain contractive and nonexpansive operators.

Let (K,d) be a metric space (in this case a subset of a real Banach space $(E, \|.\|)$, then $T : K \to K$ is called a $L-$Lipschitzian mapping if there exist the smallest constant $L \geq 0$ such that $d(Tx,Ty) \leq Ld(x,y); x,y \in K$. We recall the following equivalent formulation of Lipschitz condition (see, for example, [1]) - A mapping $T : K \to K$ is called $L-$Lipshitzian if $L \geq 0$ is the smallest constant such that the following condition is satisfied:

$$T(B_r(x)) \subset B_{Lr}(Tx). \tag{2}$$

T is called non-Lipschitzian if there does not exists L such that (2) is satisfied while T is called a contraction if $L < 1$, nonexpansive for $L = 1$ and contractive if $d(Tx,Ty) < d(x,y)$; $x,y \in K$.

In [2, 3] Berinde introduced and studied the concept of $(\delta,k)-$weak contraction (now referred to as *almost contractions* [4]) defined below:

Definition 1. [2, 3] *Let X be a metric space, $\delta \in (0,1)$ and $k \geq 0$, then a mapping $T : X \to X$ is called $(\delta,k)-$weak contraction (or an almost contraction) if and only if:*

$$d(Tx;Ty) \leq \delta d(x,y) + kd(y,Tx), \text{ for all } x,y \in X \tag{3}$$

It is shown in [2, 3] that a lot of well known contractive conditions in literature are special cases of weak contraction condition (3) as it does not require that $\delta + k$ be less than 1 which is assumed in almost all fixed point theorems based on contractive condition which involves displacements of the forms $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, and $d(y,Tx)$. It is of significance to note that the condition (3) generalizes and extends weak contractive condition of Ciric [5] which in turn generalizes contractive conditions studied by Kannan [7] and Zamfirescu [12]. In [1] Berinde proved the following results:

Theorem 2. [3] *Let (X,d) be a complete metric space and $T : X \to X$ be a $(\delta,k)-$weak contraction. Then*

(1) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{1}$ given by $x_{n+1} = Tx_n$, $n = 0,1,2,...$ converges to some $x^* \in Fix(T)$.

(3) The following estimates:

$$d(x_n,x^*) \leq \frac{\delta^n}{1-\delta} d(x_0,x_1),\ n = 0,1,2,... \tag{4}$$

$$d(x_n,x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1},x_n),\ n = 0,1,2,... \tag{5}$$

hold, where δ is the constant appearing in (3).

(4) Under the additional condition that there exist $\theta \in (0.1)$ and some $k_1 \geq 0$ such that:

$$d(Tx,Ty) \leq \theta . d(x;y) + k_1 . d(x,Tx) \text{ for all } x;y \in X \tag{6}$$

the fixed point x^* is unique and the Picard iteration converges at the rate $d(x_n,x^*) \leq \theta d(x_{n-1},x^*)$; $n \in N$.

Definition 3. *A mapping $T : (X,d) \to (Y,d)$ is called a bi-Lipschitz mapping provided there is a constant L_0 such that for all $x \in X$ and all $r > 0$*

$$\frac{1}{L}\|x-y\| \leq \|Tx-Ty\| \leq L\|x-y\|. \tag{7}$$

Definition 4. *A mapping $T : (X,d) \to (Y,d)$ is called a co-Lipschitz mapping provided there is a constant L_0 such that for all $x \in X$ and all $r > 0$ we have*

$$B_{\frac{r}{L_0}}(Tx) \subset T(B_r(x)) \tag{8}$$

where L_0 denotes the smallest such constant and is called co-Lipschitz constant.

We observe that in a Banach space condition (8) yields the following:

$$\|x-y\| \leq L_0\|Tx-Ty\| \tag{9}$$

A mapping which is both a Lipschitz mapping and a co-Lipschitz mapping is called a *Lipschitz quotient mapping*. Also, a one to one mapping T is a Lipschitz quotient mapping if and only if it is bi-Lipschitz.

Definition 5. [9] *Let K be a convex set, Y a vector space, then a mapping $A : K \to Y$ is called an affine if it satisfies $A[(1-\lambda)x+\lambda y] = (1-\lambda)Ax+\lambda Ay$ for all $x,y \in K$ and $\lambda \in (0,1)$.*

By this definition the identity operator is an example of affines. Also, if $T : K \to K$ is any operator, then a routine exercise verifies that for a fixed $x \in K$, $Az = (1-\lambda)z + \lambda Tx$ defines an affine A for $z \in B_r(Tx)$, $\lambda \in (0,1)$. In this case, if $x \in B_r(Tx)$ then $Ax \in B_r(Tx)$, but if $x \notin B_r(Tx)$ then $A : B_r(x) \to B_{\lambda r}(Tx)$ is also an affine and for all choices of $\lambda \in (0,1)$ we have $Ax \in B_{\lambda r}(Tx)$.

Definition 6. [6] *An operator T is said to admit affine localization if for every $\varepsilon > 0$ and every ball $B \subset X$ there exists a ball $B_r \subset B$ and an affine function $A : K \to Y$ so that*

$$\|Tz - Az\| \leq \varepsilon r \tag{10}$$

whenever $z \in B_r$.

2. Preliminaries

The pair (X,Y) is said to have *approximation by affine property (*AAP) [1] if every Lipschitz map from X into Y admits affine localization. Following Definition 6 we introduce the following definition:

Definition 7. *An operator T is said to admit affine quasi-localization if for every $\varepsilon > 0$ and some open set $B \subset X$ there exists a ball $B_r \subset B$ and an affine mapping $A : K \to Y$ such that*

$$\|Tz - Az\| \leq \varepsilon r \tag{11}$$

whenever $z \in B_r$.

The concept of affine quasi-localization localization is clearly weaker than the concept of affine localization and if T is differentiable at least at one point of its domain then T admits affine quasi-localization.

Proposition 8. *Every contraction $T : K \to K$ on a convex subset $K \subset X$ of a normed linear space X admits affine quasi-localization.*

PROOF

By definition, $B_{Lr}(Tx) \subset T(B_r(x))$, so we have $\|Tx - Tz\| \leq \|z - Tx\|$ for all $x, z \in B_r(x)$ yielding:

$$\begin{aligned} \|Tz - (1-\lambda)z - \lambda Tx\| &\leq \|z - (1-\lambda)z - \lambda Tx\|, \lambda \in (0,1) \\ \Longrightarrow \|Tz - Az\| &\leq \lambda\|z - Tx\| \leq \varepsilon r. \end{aligned} \tag{12}$$

Here, λ is chosen in such a way that $\lambda \leq \varepsilon$. End of proof. $\square$

Remark 9. Denote by A_x the affine defined by $Az = \lambda z + (1-\lambda)Tx$, then it follows that $\|Tx - A_x y\| = \|Tx - [\lambda y + (1-\lambda)Tx]\|$ can be made as small as we please by choosing $\lambda \in (0,1)$ arbitrarily close to zero. By this we mean that given any $\varepsilon > 0$ and a ball $B_r(Tx)$ we can choose $\lambda \in (0,1)$ so that:

$$\|Tx - A_x y\| = \|Tx - [\lambda y + (1-\lambda)Tx]\| \leq \varepsilon r,\ y \in B_r(Tx). \tag{13}$$

From (13) it follows that if $\|x-y\| \leq \|y - Tx\|$, then given any $\varepsilon > 0$, a ball $B_r(Tx)$ and for any $x, y \in K$ we can find $\lambda_{xy} \in (0,1)$ such that $\lambda_{xy}\|y - Tx\| \leq \|x - y\|$. This yields:

$$\lambda_{xy}\|y - Tx\| = \|Tx - A_x y\| \leq \|x-y\|. \tag{14}$$

Our objective in this work is to prove that every Lipschitz quotient mapping which admits affine quasi-localization in certain subset of a convex subset of a Banach space has a fixed point by showing that admitting affine quasi-localization implies that condition (1) holds. In the sequel we shall make reference to the following modification of an inequality studied in [10]:

Lemma 10. *Let V be a bounded subset of a real normed linear space and let $\alpha, \beta \in \mathbb{R}$ be such that $\|\alpha u - \beta v\| \leq d_1$ for some constant $d_1 > 0$ and for all distinct points $u, v \in V$ such that $\|u - v\| > 0$. Then there exists $\tau \in \mathbb{R}$ such that*

$$\|\alpha u - \beta v\| \leq [2\alpha + \tau\beta]\|u - v\| \tag{15}$$

PROOF

If $\|\alpha u - \beta v\| = 0$, then there is nothing to prove. So, we assume $\|\alpha u - \beta v\| \neq 0$. Given that $d_2 = \text{diam}\{\|u-v\| : u, v \in V; \|\alpha u - \beta v\| \leq d_1\}$, we may write (15) as $d_1 \leq [2\alpha + \tau\beta]d_2$. This yields an estimate for τ as

$$\tau \geq \frac{d_1}{d_2\beta} - \frac{2\alpha}{\beta}. \tag{16}$$

End of proof. □

An application of (16) is readily obtained by rewriting the displacement $\|\lambda_{xy}y - Tx\| \leq d_1$ in the form $\|\lambda_{xy}y + (1-\lambda_{xy})Tx - [2-\lambda_{xy}]Tx\| \leq d_1$ to obtain the following:

$$\|\lambda_{xy}y - Tx\| \leq \frac{d_1}{d_2}\|\lambda_{xy}y + (1-\lambda_{xy})Tx - Tx\|. \tag{17}$$

Here, d_2 is obtained from a working hypothesis involving $0 < \|\lambda_{xy}y + (1-\lambda_{xy})Tx - Tx\| \le d_2$ for some positive constant d_2.

3. Main Results

Lemma 11. *Let $K \subset E$ be a closed convex subset of a real Banach space E and $T : K \to K$ a contraction mapping (i.e $\|Tx - Ty\| \le \delta\|x - y\|$; $\delta \in (0,1)$). Then there exist $n_0 \in \mathbb{N}$ and a number $M \ge 1$ such that: $\|y - Tx\| \le M\|x - y\|$ whenever $x, y \in \{T^n x_0\}_{n=0}^{\infty}$ are such that $x = T^n x_0$, and $y = T^m x_0$ with $n, m \ge n_0$ for all $x_0 \in K$.*

PROOF

Given that $\|x - y\| = \|T^n x_0 - T^m x_0\|$ and $\|y - Tx\| = \|T^m x_0 - T^{n+1}x_0\|$. Assuming that $n > m \ge n_0$, we proceed as follows:

$$\begin{aligned}
\|y - Tx\| &\le \|x - y\| + \|x - Tx\| \\
&\le \|x - y\| + \|x - \lambda_{xy}y\| + \|\lambda_{xy}y - Tx\|
\end{aligned}$$

If there exists $j, k \ge n_0$ and $M_1, M_2 \ge 1$ such that $\|T^j x - \lambda_{xy}y\| \le M_1\|x - y\|$ and $\|\lambda_{xy}y - T^k x\| \le M_2\|x - y\|$, then the proof is done. On the other hand, suppose either $\|T^j x_0 - \lambda_{xy}y\| > \|x - y\|$ or $\|\lambda_{xy}y - T^k x_0\| > \|x - y\|$ for all $j, k \ge n_0$. Then putting $d_2 = \max_{k \ge n_0} \|T^k x_0 - \lambda_{xy}y\|$ with $k > n$, applications of (14) and (17), to above, yield the following:

$$\begin{aligned}
\|y - Tx\| &\le \|x - y\| + \|x - T^j x_0\| + \|T^j x_0 - \lambda_{xy}y\| + \|x - T^k x_0\| + \|T^k x_0 - \lambda_{xy}y\| \\
&\le 3\|x - y\| + \|\lambda_{xy}y - TT^{j-1}x_0\| + \|\lambda_{xy}y - TT^{k-1}x_0\| \\
&\le 4\|x - y\| + 2\|\lambda_{xy}y - TT^{k-1}x_0\| \\
&\le 4\|x - y\| + \frac{2d_1}{d_2}\|\lambda_{xy}y + (1-\lambda_{xy})TT^{k-1}x_0 - TT^{k-1}x_0\| \\
&\le 4\|x - y\| + \frac{2d_1}{d_2}\|T(T^{k-1}x_0) - A_{xy}y\| \\
&\le 4\|x - y\| + \frac{2d_1}{d_2}\varepsilon r;\ \forall\, y \in B_r(T^k x_0)\ (\text{by } (13)).
\end{aligned}$$

Since the last inequality holds for all choices of ε and r and since ε and the ball $B_r(T^k x_0)$ are arbitrary we conclude that for $M \ge 4$ we have $\|y - Tx\| \le M\|x - y\|$. End of proof. □

Theorem 12. *Let $K \subset E$ be a closed convex subset of a real Banach space E and $T : K \to K$ a Lipschitz quotient mapping which admits an affine quasi-localization. Then T has a fixed point x^* and its Krasnoselskii iterations converges to x^*.*

PROOF

The proof constitutes an application Theorem 2. We are only required to verify that if a Lipschitz quotient mapping T admits affine quasi-localization then the averaged operator $S_\lambda = \lambda I + (1-\lambda)T$ is a $(\delta, k)-$weak contraction mapping (i.e almost contraction) so that Theorem 2 applies. Suppose $\|y - Tx\| \leq \|x - y\|$ for all $x, y \in \{T^n x_0\}_{n=0}^{\infty}$ where $x_0 \in K$ and T is Lipschitz quotient mapping with co-Lipschitz constant L_0. We consider the Krasnoselskii scheme defined by the averaged operator S_λ, $\lambda \in [0,1)$ given by $S_\lambda x = \lambda x + (1-\lambda)Tx$. We obtain:

$$\begin{aligned} \|S_\lambda x - S_\lambda y\| &= \|\lambda x + (1-\lambda)Tx - [\lambda y + (1-\lambda)Ty]\| \\ &\leq \|y - [\lambda x + (1-\lambda)Tx]\| + (1-\lambda)\|y - Ty\| \\ &\leq \|y - S_\lambda x\| + (1-\lambda)\|y - Tx\| + (1-\lambda)\|Tx - Ty\| \end{aligned} \tag{18}$$

Since $\|y - Tx\| \leq \|x - y\|$ then (18) yields $\|S_\lambda x - S_\lambda y\| \leq (1-\lambda)(1 + L^2 L_0)\|x - y\| + \|y - S_\lambda x\|$. Clearly, choosing λ in such a manner that $\lambda \in (\frac{L^2 L_0}{L^2 L_0 + 1}, 1)$ then (18) becomes: $S_\lambda x - S_\lambda y\| \leq \delta\|x - y\| + \|y - S_\lambda x\|$, where λ and δ satisfy $0 < \frac{\delta - (1-\lambda)}{1-\lambda} < L^2 L_0$. So in this case S_λ is $(\delta, k)-$weak contraction with $k = 1$ while δ is constrained by λ, L and L_0 as shown above.

In the case where $\|x - y\| < \|y - Tx\|$, the remaining part of the proof is based on the argument that if an operator T admits affine quasi-localization A then it admits affine localization A_x given by $A_x z = \lambda z + (1-\lambda)Tx$ for all $z \in B_r(Tx)$, for some $\lambda \in (0,1)$ and some radius r. This follows from the fact that if, given an $\varepsilon > 0$ and certain open ball B, there exists a ball $B_r \subset B$ such that $\|Tz - Az\| \leq \frac{\varepsilon}{2r}$, for all $z \in B_r$, then for $Tx \in B_r$ there exists $\lambda \in (0,1)$ such that $A_x z = \lambda z + (1-\lambda)Tx \in B_{\frac{r}{2}}$ so that $\|Tz - A_x z\| \leq \frac{\varepsilon}{2r}$. The above argument yields $\|Az - A_x z\| \leq \varepsilon r$ since $\|Az - A_x z\| \leq \|Tz - Azk\| + \|Tz - A_x z\|$.

Now, it is clear that T maps $B_r(Tx)$ into itself given that A_x maps $B_r(Tx)$ into itself and $\|Tz - A_x z\| \leq \frac{\varepsilon}{2} r$. So, we can find $n_0 \in \mathbb{N}$ and complete the proof by repeating the steps in the proof of Lemma 11 to arrive at the same conclusion that for $M \geq 4$ the condition (1) holds (i.e $\|y = Tx\| \leq M\|x - y\|$ whenever x and y are appropriate points in the orbit $\{T^n x_0\}$ of any point

x_0 of K. To this end we proceed by putting $\|y-Tx\| \leq 4\|x-y\| + \frac{2d_1}{d_2}\varepsilon r$ in (18) to obtain:

$$\begin{aligned} \|S_\lambda x - S_\lambda y\| &\leq \|y - S_\lambda x\| + 4(1-\lambda)\|x-y\| + (1-\lambda)L^2L_0\|x-y\| + \frac{2(1-\lambda)d_1}{d_2}\varepsilon r \\ &= \|y - S_\lambda x\| + (1-\lambda)[4+L^2L_0]\|x-y\| + \frac{2(1-\lambda)d_1}{d_2}\varepsilon r \end{aligned}$$

This yields $\|S_\lambda x - S_\lambda y\| \leq (1-\lambda)[4+L^2L_0]\|x-y\| + \|y-S_\lambda x\|$ Since ε and r are arbitrary. This means that the averaged operator S_λ is an almost contraction (i.e $(\delta,k)-$weak contraction mapping) with $k=1$ and $\delta = (1-\lambda)[4+L^2L_0]$ where $\lambda \in \left(\frac{L^2L_0}{L^2L_0+1}, 1\right)$ with λ chosen such that δ satisfies $0 < \frac{\delta-(1-\lambda)}{1-\lambda} < L^2L_0$. By Theorem 2 S_λ has a fixed point x^* and therefore T has the same fixed point and by the same Theorem 2 the Krasnoselskii scheme converges to this fixed point. End of proof $\square$

4. CONCLUSION

Uniqueness of fixed points is not guaranteed except in the cases where condition (6) of Theorem 2 is satisfied by S_λ i.e; $S_\lambda x - S_\lambda y\| \leq \theta\|x-y\| + k_1\|x - S_\lambda x\|$, for all $x,y \in \{T^n x_0\}_{n=0}^{\infty}$ and for any $x_0 \in K$ where $\theta \in (0,1)$ and $k_1 \geq 0$. It should be observed that condition (1) is equivalent to existence of affine quasi-localization since both imply each other. It will be interesting to extend this result to other classes of continuous and discontinuous operators especially in the context of multi-valued mappings and unification of fixed point results using a property that cuts across contractive mappings, continuous mappings and certain discontinuous operators. Finally, our results confirm that Lipschitz quotient mappings $Lip(X,Y)$ that have (AAP), studied by Bates et al [1], have nonempty fixed point sets. This class, however, is a subclass of those investigated in this work.

Conflict of Interests

The authors declare that there is no conflict of interests.

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