Available online at http://scik.org
Advances in Fixed Point Theory, 2 (2012), No. 1, 47-57
ISSN: 1927-6303

COINCIDENCE THEOREM AND EXISTENCE THEOREMS OF SOLUTIONS FOR A SYSTEM OF KY FAN TYPE MINIMAX INEQUALITIES IN FC-SPACES

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Abstract. Let I be any index set. By using some existence theorems of maximal elements for a family of set-valued mappings involving a better admissible set-valued mapping under noncompact setting of FC-spaces, we first give a coincidence theorem and a Fan-Browder type fixed point theorem. Next, we obtain some existence theorems of solutions for a system of Ky Fan type minimax inequalities involving a family of $G_{\mathbf{B}}$-majorized mappings defined on the product space of FC-space. Our results improve and generalize some recent results.

Keywords: Maximal element; $G_{\mathbf{B}}$-majorized; System of Ky Fan type minimax inequalities; Coincidence theorem; Product FC-space.

2000 AMS Subject Classification: 47H10; 49J35

1. Introduction

Recently, Ding [9] introduced the concept of FC-space, which includes many classes of topological spaces with various convex structure appearing in nonlinear analysis as special cases, for example, L-convex space (see Ben-EI-Mechaiekh et al. [2]) and G-convex space

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Received February 20, 2012

(see Park and Kim [15]). The concept has become an adequate and important tool for studying various problems in nonlinear analysis, and for the works on the fixed point theorems, coincidence theorems and maximal elements established in spaces with these sorts of abstract convex structure, we refer to Ref. [1], [9], [10]. This paper is a continuum of the preceding paper of the author [12]. For the concepts and notations of FC-space, CFC-space, the class $\mathbf{B}(Y, X)$ of better admissible mappings, generalized $G_{\mathbf{B}}$-mappings, generalized $G_{\mathbf{B}}$-majorant mappings and the related notions, the reader may consult the paper [12].

In this paper, we will continue to study some problems in nonlinear analysis in FC-spaces. By applying the existence theorems of maximal elements obtained by the author [12], we present a component version of coincidence theorem and a Fan-Browder type fixed point theorem. We obtain some existence theorems of solutions for a system of Ky Fan type minimax inequalities involving a family of $G_{\mathbb{B}}$-majorized mappings defined on the product space of FC-space. Our results improve and generalize the corresponding results in Ref. [4], [5], [6], [7], [8], [13], [14].

2. Preliminaries

In order to prove our main results, we need the following Lemmas. The following results are Lemma 2.1, Theorem 3.5 and Theorem 3.6 in [12], respectively.

Lemma 2.1. *Let I be any index set, for each $i \in I$, (Y_i, φ_{N_i}) be an FC-space, $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then (Y, φ_N) is also an FC-space.*

Lemma 2.2. *Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, suppose (Y_i, φ_{N_i}) is an FC-space and $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.1. Let $F \in \mathbf{B}(Y, X)$ such that for each $i \in I$, $A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathbf{B}}$-mapping such that for each $i \in I$ and $N_i \in \langle Y_i \rangle$, there exists a compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X \backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cintA_i(x) \neq \emptyset$. Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.*

Lemma 2.3. *Let X be a topological space and I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an CFC-space and $Y = \prod_{i\in I} Y_i$. Let $F \in \mathbf{B}(Y, X)$ be a compact mapping such that for each $i \in I$,*

(i) *$A_i : X \to 2^{Y_i}$ be a generalized $G_{\mathbf{B}}$-majorized mapping;*

(ii) *$\bigcup_{i\in I}\{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i\in I} cint\{x \in X : A_i(x) \neq \emptyset\}$.*

Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

3. Fixed points and coincidence points in product FC-spaces

In this section, by applying our existence theorems maximal elements, we have the following component version of coincidence theorem and Fan-Browder type fixed point theorem in the product space of FC-spaces.

Theorem 3.1. *Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be an FC-space and $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.1. Let $F \in \mathbf{B}(Y, X)$ and $A_i : X \to 2^{Y_i}$ be a set-valued mapping such that for each $i \in I$ and $x \in X$, $A_i(x)$ is FC-subspace of Y_i. Suppose that*

(i) *for each $i \in I$ and $y_i \in Y_i$, $A_i^{-1}(y_i)$ be transfer compactly open in Y_i;*

(ii) *for each $i \in I$ and $N_i \in \langle Y_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of Y_i containing N_i and for each $x \in X\backslash K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap cintA_i(x) \neq \emptyset$;*

(iii) *for each $x \in K$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$;*

Then there exists $i_0 \in I$ and $(\hat{x}, \hat{y}) \in (X, Y)$ such that $\hat{x} \in F(\hat{y})$ and $\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in A_{i_0}(\hat{x})$. Moreover, if $F = S$ is a single-valued continuous mapping, then we have $\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in A_{i_0}(S(\hat{y}))$.

Proof. By condition (iii), Lemma 2.2 does not hold. By Lemma 2.2, there exists $i_0 \in I$ such that $A_{i_0} : X \to 2^{Y_{i_0}}$ does not satisfy the condition (a) in the definition of a generalized $G_{\mathbf{B}}$- mapping, i.e., there exists $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$ such that $F(\varphi_N(\Delta_k))\bigcap(\bigcup_{y\in N_1} cintA_{i_0}^{-1}(\pi_{i_0}(y)))\neq\emptyset$. It follows that $(\hat{x},\hat{y})\in(X,Y)$ such that $\hat{y}\in\varphi_N(\Delta_k)$, $\hat{x}\in F(\hat{y})$, and $\hat{x}\in\bigcup_{y\in N_1} cintA_{i_0}^{-1}(\pi_{i_0}(y))$. Hence we have $\pi_{i_0}(N_1)\subset cintA_{i_0}(\hat{x})\subset A_{i_0}(\hat{x})$. Since $A_i(\hat{x})$ is FC-subspace of Y_i, we have

$$\hat{y}_{i_0}=\pi_{i_0}(\hat{y})\in\pi_{i_0}(\varphi_N(\Delta_k))=\varphi_{N_{i_0}}(\Delta_k)\subset A_{i_0}(\hat{x}).$$

Furthermore, if $F=S$ is a single-valued continuous mapping, then $S\in\mathbf{B}(Y,X)$, and so we have $\hat{y}_{i_0}\in A_{i_0}(S(\hat{y}))$.

Remark 3.1. (1) Theorem 3.1 improves and generalizes Theorem 4 in [8] in several aspects: (1.1) from G-convex space to FC-space without linear structure; (1.2) from compactly open to transfer compactly open. (1.3) condition (ii) of Theorem 3.1 is weaker than condition (ii) of Theorem 4 in [8].

(2) Theorem 3.1 generalizes Theorem 6 and Theorem 9 in [5] from the product space of a nonempty convex subsets of topological vector spaces to the product space of FC-spaces and from S is a single-valued continuous mapping to $F\in\mathbf{B}(Y,X)$.

(3) Theorem 3.1, in turn, generalizes Theorem 5.1 of Deguire and Lassonde in [4], and the corresponding results of Lassonde [13] and Ding [6, 7] in several aspects.

Theorem 3.2. *Let I be any index set. For each $i\in I$, suppose $(X_i,\varphi_{N_i'})$ and (Y_i,φ_{N_i}) are FC-spaces, $X=\prod_{i\in I}X_i$ and $Y=\prod_{i\in I}Y_i$ such that both $(X,\varphi_{N'})$ and (Y,φ_N) are FC-spaces defined as in Lemma 2.1. For each $i\in I$, $A_i:X\to 2^{Y_i}$ and $B_i:Y\to 2^{X_i}$ are set-valued mappings such that for each $x\in X, y\in Y$, $A_i(x)$ is FC-subspace of Y_i and $B_i(y)$ is FC-subspace of X_i. Suppose that there exist a nonempty compact subset K of X and a nonempty compact subset L of Y such that*

(i) *for each $i\in I$ and $(x_i,y_i)\in X_i\times Y_i$, A_i^{-1} and B_i^{-1} are both transfer compactly open in Y_i and X_i, respectively;*

(ii) *for each $i\in I, M_i\in\langle X_i\rangle$ and $N_i\in\langle Y_i\rangle$, there exist nonempty compact FC-subspace L_{M_i} of X_i containing M_i and L_{N_i} of Y_i containing N_i, and for each $(x,y)\in(X\times Y)\backslash(K\times L)$, there exists $i\in I$ such that $L_{M_i}\bigcap cintB_i(y)\neq\emptyset$ and $L_{N_i}\bigcap cintA_i(x)\neq\emptyset$;*

(iii) *for each $(x,y)\in K\times L$, there exists $i\in I$ such that $A_i(x)\neq\emptyset$ and $B_i(y)\neq\emptyset$;*

Then there exists $i_0 \in I$ and $(\hat{x}, \hat{y}) \in X \times Y$ such that $\hat{y}_{i_0} \in A_{i_0}(\hat{x})$ and $\hat{x}_{i_0} \in B_{i_0}(\hat{y})$.

Proof. Let $C = K \times L$. Then C is a nonempty compact subset of $X \times Y$. Clearly, for each $i \in I$, $Y_i \times X_i$ and $Y \times X = \prod_{i \in I}(Y_i \times X_i)$ are also FC-spaces. By condition (ii), for each $i \in I$ and $N_i \times M_i \in \langle Y_i \times X_i \rangle$, there exists a nonempty compact FC-subspace $L_{N_i} \times L_{M_i}$ of $Y \times X$ containing $N_i \times M_i$. Define a mapping $F : Y \times X \to 2^{X \times Y}$ by $F(y, x) = \{(x, y)\}$, then we have $F \in \mathbf{B}(Y \times X, X \times Y)$. Define a mapping $W_i : X \times Y \to 2^{Y_i \times X_i}$ by

$$W_i(x, y) = A_i(x) \times B_i(y), \quad \forall (x, y) \in X \times Y.$$

We first show that W_i^{-1} is transfer compactly open in $Y_i \times X_i$. Indeed, for any nonempty compact subset $D \times E$ of $X \times Y$, if $(x, y) \in (W_i^{-1}(y_i, x_i) \bigcap (D \times E))$, then $x \in A_i^{-1}(y_i) \bigcap D$ and $y \in B_i^{-1}(x_i) \bigcap E$. Since both A_i^{-1} and B_i^{-1} are transfer compactly open in Y_i and X_i, respectively. There exist $(\hat{y}_i, \hat{x}_i) \in Y_i \times X_i$. $x \in int_D(A_i^{-1}(\hat{y}_i) \bigcap D)$ and $y \in int_E(B_i^{-1}(\hat{x}_i) \bigcap E)$. So we have $(x, y) \in (int_D(A_i^{-1}(\hat{y}_i) \bigcap D)) \times (int_E(B_i^{-1}(\hat{x}_i) \bigcap E)) \subset int_{D \times E}(A_i^{-1}(\hat{y}_i) \times B_i^{-1}(\hat{x}_i)) \bigcap (D \times E)$. Now

$$\begin{aligned} (u, v) \in W_i^{-1}(\hat{y}_i, \hat{x}_i) &\Leftrightarrow (\hat{y}_i, \hat{x}_i) \in W_i(u, v) = A_i(u) \times B_i(v) \\ &\Leftrightarrow \hat{y}_i \in A_i(u) \text{ and } \hat{x}_i \in B_i(v) \\ &\Leftrightarrow u \in A_i^{-1}(\hat{y}_i) \text{ and } v \in B_i^{-1}(\hat{x}_i) \\ &\Leftrightarrow (u, v) \in A_i^{-1}(\hat{y}_i) \times B_i^{-1}(\hat{x}_i). \end{aligned}$$

This shows that $W_i^{-1}(\hat{y}_i, \hat{x}_i) = A_i^{-1}(\hat{y}_i) \times B_i^{-1}(\hat{x}_i)$. Therefore, we have

$$(x, y) \in int_{D \times E}(W_i^{-1}(\hat{y}_i, \hat{x}_i) \bigcap (D \times E)) = cint W_i^{-1}(\hat{y}_i, \hat{x}_i) \bigcap (D \times E).$$

Hence, we infer that W_i^{-1} is transfer compactly open in $Y_i \times X_i$. Clearly, the conditions (ii) and (iii) of Theorem 3.2 imply conditions (ii) and (iii) of Theorem 3.1. By Theorem 3.1, there exist $i_0 \in I$ and $(\hat{x}, \hat{y}) \in (X, Y)$ such that

$$(\hat{y}_{i_0}, \hat{x}_{i_0}) \in W_{i_0}(F(\hat{y}, \hat{x})) = W_{i_0}(\hat{x}, \hat{y}) = A_{i_0}(\hat{x}) \times B_{i_0}(\hat{y}),$$

and hence, we have $\hat{y}_{i_0} \in A_{i_0}(\hat{x})$ and $\hat{x}_{i_0} \in B_{i_0}(\hat{y})$.

Remark 3.2. Theorem 3.2 improves and generalizes Theorem 10 in [5] and Theorem 4.3 in [4] from convex subsets of topological vector spaces to the product space of FC-spaces. Theorem 3.2 generalizes Theorem 5 in [8].

4. System of minimax inequalities

In this section, some existence theorems of solutions for a system of Ky Fan type minimax inequalities (see Ky Fan in [11]) will be proved under much weaker assumptions.

Definition 4.1 ([3]). Let X and Y be two topological spaces, $f : X \times Y \to R \cup \{-\infty, +\infty\}$ be a function. Then f is said to be transfer compactly lower semicontinuous (in short, transfer compactly l.s.c) in y, if for each $y \in Y$, for any nonempty compact subset K of Y and $\gamma \in R$ with $y \in \{u \in Y : f(x, u) > \gamma\} \bigcap K$, there exists $\bar{x} \in X$ such that $y \in cint\{u \in Y : f(\bar{x}, u) > \gamma\} \bigcap K$. f is said to be transfer compactly u.s.c in y if and only if $-f$ is transfer compactly l.s.c in y.

Let X be a topological space and I be any index set. For each $i \in I$, suppose (Y_i, φ_{N_i}) is an FC-space and $Y = \prod_{i \in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.1. Let $F \in \mathbf{B}(Y, X)$ and for each $i \in I$, $f_i : X \times Y_i \to R$ be a real valued function.

(1) for each $x \in X$, the function $f_i(x, y_i)$ is said to be FC-quasiconvex (resp., FC-quasiconcave) in y_i, if for each $\lambda \in R$, the set $\{y_i \in Y_i : f_i(x, y_i) < \lambda\}$ (resp., $\{y_i \in Y_i : f_i(x, y_i) > \lambda\}$) is an FC-subspace of Y_i.

(2) $f_i(x, y_i)$ is said to be generalized $G_{\mathbf{B}}$-majorized if the following conditions are satisfied: for each $\lambda \in R$, if there exists $(x, y_i) \in X \times Y_i$ such that $f_i(x, y_i) > \lambda$, then there exist a nonempty open neighborhood $N(x)$ of x in X and a real valued function $f_{i,x} : X \times Y_i \to R$ such that

(a) $f_i(z, y_i) \le f_{i,x}(z, y_i)$ for all $(z, y_i) \in N(x) \times Y_i$;

(b) for each $y_i \in Y_i$, $f_{i,x}(z, y_i)$ be transfer compactly lower semicontinuous in y_i;

(c) for each $N = \{y_0, ..., y_n\} \in \langle Y \rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$, and for each $y \in \varphi_N(\Delta_k)$ and $z \in F(y)$, $f_i(z, \pi_i(y)) \le \lambda$ implies that there exists $y' \in N_1$ such that $f_{i,x}(z, \pi_i(y')) \le \lambda$.

Theorem 4.1. *Let X be a topological space and I be any index set. For each $i \in I$, suppose (Y_i, φ_{N_i}) is a CFC-space and $Y = \prod_{i\in I} Y_i$. Let $F \in \mathbf{B}(Y, X)$ be a compact mapping and for each $i \in I$, the function $f_i : X \times Y_i \to R$ satisfy that for each $x \in X$, $f_i(x, y_i)$ be transfer compactly lower semicontinuous in x. Then at least one of the following statements holds:*

(1) *For each $\lambda \in R$, there exists $\hat{x} \in X$ such that $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leq \lambda$;*

(2) *There exist $i \in I$, $\lambda \in R$, $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$ such that $F(\varphi_N(\Delta_k)) \bigcap(\bigcap_{y\in N_1} cint\{x \in X : f_i(x, \pi_i(y)) > \lambda\}) \neq \emptyset$.*

Proof. By using similar argument as in the proof of Theorem 3.5 in [12], we can show that (Y, φ_N) is also a CFC-space. If the statement (2) is false, then for any $\lambda \in R$, $i \in I$, $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$, we have

$$F(\varphi_N(\Delta_k)) \bigcap(\bigcap_{y\in N_1} cint\{x \in X : f_i(x, \pi_i(y)) > \lambda\}) = \emptyset.$$

For each $i \in I$, define a mapping $A_i : X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\},\ \forall\, x \in X.$$

Hence we have

$$F(\varphi_N(\Delta_k)) \bigcap(\bigcap_{y\in N_1} cint A_i^{-1}(\pi_i(y))) = \emptyset.$$

Since $f_i(x, y_i)$ is transfer compactly lower semicontinuous in x, we have that for each $y_i \in Y_i$, $A_i^{-1}(y_i) = \{x \in X : f_i(x, y_i) > \lambda\}$ is transfer compactly open-valued on Y_i. Hence, for each $i \in I$, A_i is a generalized $G_{\mathbf{B}}$-mapping, and the condition (ii) of Lemma 2.3 is also satisfied. By Lemma 2.3, there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for all $i \in I$. Hence, we have

$$\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leq \lambda,$$

i.e., the statement (1) holds.

Remark 4.1. Theorem 4.1 generalizes Theorem 6 in [8] and Theorem 11 in [5].

Theorem 4.2. *Let X be a topological space and I be any index set. For each $i \in I$, suppose (Y_i, φ_{N_i}) is a CFC-space and $Y = \prod_{i\in I} Y_i$. Let $F \in \mathbf{B}(Y, X)$ be a compact mapping, for each $i \in I$. The function $f_i : X \times Y_i \to R$ satisfies*

(i) *for each $x \in X$, $f_i(x, y_i)$ is FC-quasiconcave in the first variable;*

(ii) *for each $x \in X$, $f_i(x, y_i)$ be transfer compactly lower semicontinuous in x.*

Then we have

(A) *For any $\lambda \in R$, at least one of the following statements holds:*

(1) *there exists $\hat{x} \in X$ such that $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leq \lambda$;*

(2) *there exist $i \in I$, $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$ such that $\hat{y} \in \varphi_N(\Delta_k)$ and $\hat{x} \in F(\hat{y})$ such that $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$.*

(B) *the following minimax inequality holds:*

$$\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{N\in\langle Y\rangle} \sup_{x\in F(y)} \sup_{y\in\varphi_N(\Delta_k)} f_i(x, \pi_i(y)).$$

Proof. By using similar argument as in the proof of Theorem 3.5 in [12], we can show that (Y, φ_N) is also a CFC-space.

(A) For any $i \in I$ and $\lambda \in R$, define a mapping $A_i : X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\}, \ \forall\, x \in X.$$

Then by condition (i), for each $x \in X$, $A_i(x)$ is an FC-subspace of Y_i and for each $y_i \in Y_i$, $A_i^{-1}(y_i) = \{x \in X : f_i(x, y_i) > \lambda\}$ is transfer compactly open-valued on Y_i by condition (ii). Now we assume that the statement (A) (2) is not true. We claim that A_i is a generalized $G_{\mathbf{B}}$-mapping. Indeed, if A_i is not a generalized $G_{\mathbf{B}}$-mapping, then there exists $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$ such that

$$F(\varphi_N(\Delta_k)) \bigcap (\bigcap_{j=0}^{k} cintA_i^{-1}(\pi_i(y_{r_j}))) \neq \emptyset.$$

It follows that there exist $\hat{y} \in \varphi_N(\Delta_k)$ and $\hat{x} \in F(\hat{y})$ such that $\pi_i(N_1) \subset cintA_i(\hat{x}) \subset A_i(\hat{x})$. Since $A_i(x)$ is an FC-subspace of Y_i, we have

$$\hat{y}_{i_0} = \pi_{i_0}(\hat{y}) \in \pi_{i_0}(\varphi_N(\Delta_k)) = \varphi_{N_{i_0}}(\Delta_k) \subset A_{i_0}(\hat{x}),$$

i.e., $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$. This contradicts the assumption that (A) (2) is not true. Therefore A_i is a generalized $G_{\mathbf{B}}$-mapping. Noting that each $A_i^{-1}(y_i)$ is transfer compactly open-valued on Y_i, hence all conditions of Lemma 2.3 are satisfied and it follows that there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for all $i \in I$. Hence $f_i(\hat{x}, y_i) \leq \lambda$ for all $i \in I$ and $y_i \in Y_i$, so that the statement (A) (1) holds.

(B) Let $\lambda_0 = \sup_{i\in I} \sup_{N\in\langle Y\rangle} \sup_{x\in F(y)} \sup_{y\in\varphi_N(\Delta_k)} f_i(x, \pi_i(y))$, then the the statement (A) (2) is false, so that the statement (A) (1) must hold, i.e., there exists $\hat{x} \in X$ such that $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leq \lambda_0$. Hence, we have

$$\inf_{x\in X} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{N\in\langle Y\rangle} \sup_{x\in F(y)} \sup_{y\in\varphi_N(\Delta_k)} f_i(x, \pi_i(y)).$$

This completes the proof.

Corollary 4.1. *Let X be a topological space, K be a nonempty compact subset of X and I be any index set. For each $i \in I$, suppose (Y_i, φ_{N_i}) is an FC-space and $Y = \prod_{i\in I} Y_i$ such that (Y, φ_N) is an FC-space defined as in Lemma 2.1. Let $F \in \mathbf{B}(Y, X)$ and for each $i \in I$, $f_i : X \times Y_i \to R$ be such that*

- (i) *for each $x \in X$, $f_i(x, y_i)$ is FC-quasiconcave in y_i;*
- (ii) *for each $x \in X$, $f_i(x, y_i)$ be transfer compactly lower semicontinuous in x;*
- (iii) *for each $N_i \in \langle Y_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of Y_i containing N_i and for each $\lambda \in R$ and $x \in X\backslash K$, there exists $i \in I$ and $y_i \in L_{N_i}$ satisfying $y_i \in cint\{u_i \in Y_i : f_i(x, u_i) > \lambda\}$.*

Then we have

- (A) *For any $\lambda \in R$, at least one of the following statements holds:*
- (1) *there exists $\hat{x} \in K$ such that $\sup_{i\in I} \sup_{y_i\in Y_i} f_i(\hat{x}, y_i) \leq \lambda$.*
- (2) *there exist $i \in I$, $N = \{y_0, ..., y_n\} \in \langle Y\rangle$ and $N_1 = \{y_{r_0}, ..., y_{r_k}\} \subset N$ such that $\hat{y} \in \varphi_N(\Delta_k)$ and $\hat{x} \in F(\hat{y})$ such that $f_i(\hat{x}, \pi_i(\hat{y})) > \lambda$.*
- (B) *the following minimax inequality holds:*

$$\inf_{x\in K} \sup_{i\in I} \sup_{y_i\in Y_i} f_i(x, y_i) \leq \sup_{i\in I} \sup_{N\in\langle Y\rangle} \sup_{x\in F(y)} \sup_{y\in\varphi_N(\Delta_k)} f_i(x, \pi_i(y)).$$

Proof. By using the similar argument as in the proof of Theorem 5.2, for any $i \in I$ and $\lambda \in R$, define a mapping $A_i : X \to 2^{Y_i}$ by

$$A_i(x) = \{y_i \in Y_i : f_i(x, y_i) > \lambda\}, \ \forall\, x \in X,$$

we can show that A_i is a generalized $G_{\mathbf{B}}$-mapping and $A_i^{-1}(y_i)$ is transfer compactly open-valued on X, condition (iii) of Corollary 4.1 implies condition (i) of Lemma 2.2 and hence all conditions of Lemma 2.2 are satisfied. By Lemma 2.2 and Theorem 4.2, it is easy to show that the conclusion of Corollary 4.1 holds.

Remark 4.2. Theorem 4.2 generalizes Theorem 7 in [8] in several aspects: (1.1) from CG-convex space to CFC-space without linear structure; (1.2) from lower semicontinuous to transfer compactly lower semicontinuous; (1.3) from FC-quasiconcave to G-quasiconcave. Corollary 4.1 improves and generalizes Theorem 8 in [5] in several aspects: (2.1) from G-convex space to FC-space without linear structure; (2.2) from lower semicontinuous to transfer compactly lower semicontinuous; (2.3) from FC-quasiconcave to G-quasiconcave; (2.4) condition (iii) of Corollary 4.1 is weaker than condition (iii) of Theorem 8 in [8]. Corollary 4.1 improves Theorem 12 in [5].

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11171046), the National Natural Science Foundation of China (11071042) and the Scientific Research Foundation of CUIT under Grant(KYTZ201114).

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