Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 1, 88-100
ISSN: 1927-6303

POSITIVE SOLUTIONS FOR FOURTH-ORDER SECOND-POINT NONHOMOGENEOUS SINGULAR BOUNDARY VALUE PROBLEMS

WENGUO SHEN

Department of Basic Courses, Lanzhou Institute of Technology, Lanzhou 730050, China

Abstract. In this paper, we investigate the fourth-order second-point nonhomogeneous singular boundary value problem $x'''' + a(t)f(x) = 0$, $0 < t < 1$, $x(0) = \alpha, x(1) = \beta, x'(0) = \lambda, x'(1) = -\mu$, where, $a(t)$ may be singular at $t = 0$, $t = 1$, $a \in C((0,1),[0,\infty))$ satisfying $0 < \int_0^1 K(\tau(s),s)a(s)ds < \infty$. $f(x) \in C([0,\infty),[0,\infty))$. We study the existence and nonexistence of positive solutions and the dependence of the solutions on the parameters α, β, λ, μ for the above boundary value problems. The proof of our main results is based upon the Guo-Krasnoselskii fixed point theorem and Schauders fixed point theorem.

Keywords: singular fourth-order differential equation; positive solutions; nonhomogeneous boundary-value problem; fixed point theorem in cones.

2010 AMS Subject Classification: 34B10, 34B18.

1. Introduction

The deformations of an elastic beam in equilibrium state with fixed both endpoints can be described by the fourth-order boundary value problem

*Corresponding author.

E-mail address: shenwg1963@126.com

Received September 25, 2014

$$\begin{cases} x''''(t)+a(t)f(x)=0,\ 0<t<1, \\ x(0)=\alpha, x(1)=\beta, x'(0)=\lambda, x'(1)=-\mu. \end{cases} \tag{1.1}$$

Since the problem (1.1) cannot transform into a system of second-order equation, the treatment method of second-order system does not apply to the problem (1.1). Thus, existing literature on the problem (1.1) is limited. Recently, when $\alpha=\beta=\lambda=\mu=0$, the existence of positive solutions of the problem (1.1) has been studied by several authors, see [1-6].

Also the existence of positive solutions for the nonhomogeneous boundary value problems have been studied by some authors, see [7-14]. Among them, it is worth mentioning that Chen [10] and Ma [12] studied the existence of positive solutions of three-point nonhomogeneous boundary value problems of second-order ordinary differential equations. Kong and Kong [11,12] considered multi-point nonhomogeneous boundary value problems of second order ordinary differential equations. By employing the Guo-Krasnoselskii fixed point theorem and Schauder's fixed point theorem, Sun [9] studied the existence and nonexistence of positive solutions to the third order three-point nonhomogeneous BVP. Zhang [10] studied the existence of positive solutions of three-point nonhomogeneous boundary value problems of second-order ordinary differential equations. Hao, Liu and Wu [14] has used the Guo-Krasnoselskii fixed point theorem, the upper-lower solutions method and topological degree theory to study the existence, nonexistence and multiplicity of positive solutions for the second order m-point nonhomogeneous singular boundary value problem. However, to the author's knowledge, fewer results on the fourth-order second-point nonhomogeneous singular boundary value problem (1.1) can be found in the literature.

Inspired and motivated by the works mentioned above, in this paper, we will consider the existence or nonexistence of positive solutions to BVP (1.1).

2. Preliminaries

For convenience, we list the following conditions:

$(H1)$ $\alpha>0, \beta>0, \lambda>0, \mu>0$.

$(H2)$ $f(x)\in C([0,\infty),[0,\infty))$.

$(H3)$ $a \in C((0,1),[0,\infty))$, and $0 < \int_0^1 K(\tau(s),s)a(s)ds < \infty$.

$(H4)$ $f_0 = 0$, $f_\infty = \infty$.

$(H5)$ $f_0 = \infty$, $f_\infty = 0$.

Where $f_0 = \lim_{x\to 0^+} \frac{f(x)}{x}, f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$.

Lemma 2.1. [2, 3] *The Green's function $K(t,s)$ for the homogeneous BVP*

$$\begin{cases} x''''(t) = 0,\ 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0 \end{cases}$$

is given by

$$K(t,s) = \frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s], & 0 \le t \le s \le 1, \\ s^2(1-t)^2[(t-s)+2(1-s)t], & 0 \le s \le t \le 1. \end{cases} \tag{2.1}$$

Lemma 2.2. [2, 3] *Let $(H1)$ hold, then the unique solution of the following BVP*

$$\begin{cases} x''''(t) = 0,\ 0 < t < 1, \\ x(0) = \alpha, x(1) = \beta, x'(0) = \lambda, x'(1) = -\mu \end{cases}$$

is given by

$$x(t) = \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t), \tag{2.2}$$

where

$$\phi_1(t) = (1-t)^2(2t+1), \phi_2(t) = t(1-t)^2, \phi_3(t) = t^2(3-2t), \phi_4(t) = t^2(1-t). \tag{2.3}$$

Proof. The proof is obvious, so we omit it.

Lemma 2.3. [2, 3] *$K(t,s)$ defined by (2.1) satisfies*

$$c(t)K(\tau(s),s) \le K(t,s) \le K(\tau(s),s),\ \forall t,\ s \in [0,1], \tag{2.4}$$

where

$$\tau(s) = \begin{cases} \frac{1}{3-2s}, & 0 \le s \le \frac{1}{2}, \\ \frac{2s}{1+2s}, & \frac{1}{2} \le s \le t \le 1, \end{cases} \quad K(\tau(s),s) = \begin{cases} \frac{2s^2(1-s)^3}{3(3-2s)^2}, & 0 \le s \le \frac{1}{2}, \\ \frac{2s^3(1-s)^2}{3(1+2s)^2}, & \frac{1}{2} \le s \le t \le 1, \end{cases} \tag{2.5}$$

$$c(t) = \frac{2}{3}\min\{t^2, (1-t)^2\}. \tag{2.6}$$

Remark 2.4. (i) For any $\delta \in (0, \frac{1}{2})$, there exists a constant $\gamma_\delta > 0$, for any $t \in [\delta, 1-\delta]$, such that

$$K(t,s) \geq \frac{2}{3}\delta^2 K(\tau(s),s),\ \forall s \in [0,1], \tag{2.7}$$

where $K(t,s)$ defined by (2.1).

(ii) Let $(H1)$ hold, then the unique solution x defined by (2.2) satisfies

$$x(t) > 0,\ \forall t \in [0,1]. \tag{2.8}$$

Lemma 2.5. [2, 3] *Let $(H1)$ hold, for $y \in C(0,1)$ and $\int_0^1 K(\tau(s),s)y(s)ds < \infty$, then the unique solution of the following BVP*

$$\begin{cases} x''''(t) + y(t) = 0,\ 0 < t < 1, \\ x(0) = \alpha, x(1) = \beta, x'(0) = \lambda, x'(1) = -\mu. \end{cases} \tag{2.9}$$

is given by

$$x(t) = \int_0^1 K(t,s)y(s)ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t). \tag{2.10}$$

Proof. By Lemma 2.1, Lemma 2.2 and the condition $\int_0^1 K(\tau(s),s)y(s)ds < \infty$, we easily get Lemma 2.5.

Lemma 2.6. *Let $(H1)$ hold. For $y \in C(0,1)$ and $\int_0^1 K(\tau(s),s)y(s)ds < \infty$, If $x \in C^+[0,1]$, then the unique solution $x(t)$ of the BVP (2.9) is nonnegative and satisfies*

$$\min_{t\in[\delta,1-\delta]} x(t) \geq \gamma_\delta \|x\|. \tag{2.11}$$

Proof. Let $x \in C^+[0,1]$, it is obvious that x is nonnegative. For any $t \in [0,1]$, by (2.4) and Lemma 2.5, it follows that

$$\begin{aligned} x(t) &= \int_0^1 K(t,s)y(s)ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\ &\leq \int_0^1 K(\tau(s),s)y(s)ds + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu, \end{aligned} \tag{2.12}$$

and thus,

$$\|x(t)\| \leq \int_0^1 K(\tau(s),s)y(s)ds + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu. \tag{2.13}$$

On the other hand, (2.4) and Lemma 2.5 imply that, for any $t \in [\delta, 1-\delta]$,

$$\begin{aligned} x(t) &= \int_0^1 K(t,s)y(s)ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\ &\geq \frac{2}{3}\delta^2 \int_0^1 K(\tau(s),s)y(s)ds + \alpha\delta^2(3-2\delta) + \beta\delta^2(1-\delta) + \lambda\delta^2(3-2\delta) + \mu\delta^2(1-\delta), \\ &\geq \gamma_\delta[\int_0^1 K(\tau(s),s)y(s)ds + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu] = \gamma_\delta\|x\|, \end{aligned} \tag{2.14}$$

where

$$\gamma_\delta = \min\{\frac{2}{3}\delta^2, \delta^2(3-2\delta), \frac{27}{4}\delta^2(1-\delta)\}.$$

Define an operator A by

$$x(t) = \int_0^1 K(t,s)a(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) := (Ax)(t), \tag{2.15}$$

where $K(t,s)$is given in (2.1). Let

$$P = \left\{ x \mid x \in C[0,1],\, x(t) \geq 0,\, \min_{t\in[\delta,1-\delta]} x(t) \geq \gamma_\delta\|x\| \right\},$$

where, γ_δ is given by Lemma 2.6.

Lemma 2.7. *If conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied, then BVP (1.1) has a positive solution $x = x(t)$ if and only if x is a fixed point of A.*

Proof. By Lemma 2.5 and the conditions $(H1)$, $(H2)$ and $(H3)$, we easily get Lemma 2.7.

Lemma 2.8. *If conditions $(H1)$, $(H2)$ and $(H3)$ are satisfied, then $A : P \to P$ is completely continuous.*

Proof. For any $y \in P$, let $x(t) = Ty(t), t \in [0,1]$. Obviously, $x(t)$ is a solution of the following BVP

$$\begin{cases} x''''(t) + a(t)f(x) = 0,\ 0 < t < 1, \\ x(0) = \alpha, x(1) = \beta, x'(0) = \lambda, x'(1) = -\mu. \end{cases}$$

From Lemma 2.6, we have $x(t) \geq 0, t \in [0,1]$, $\min_{t\in[\delta,1-\delta]} x(t) \geq \gamma_\delta \|x\|$, and $A(P) \subset P$.

Next, we show that A is completely continuous. Define an $a_n(t) : (0,1) \to [0,+\infty)$ by

$$a_n(t) = \begin{cases} \inf\{a(t), a(\frac{1}{n})\}, & 0 < t \leq \frac{1}{n}, \\ a(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \inf\{a(t), a(\frac{n-1}{n})\}, & \frac{n-1}{n} \leq t < 1. \end{cases}$$

It is easy to see that $a_n \in C(0,1)$ and $0 < a(t) \leq a_n(t), t \in (0,1)$. Furthermore, we define an operator $A_n : P \to P$ as follows:

$$(A_n x)(t) = \int_0^1 K(t,s)a_n(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t), n \geq 2.$$

Obviously, $A_n : P \to P$ is a completely continuous operator on P for each $n \geq 2$. For $r > 0$, set $B_r = \{x \in P : \|x\| \leq r\}$, then A_n converges uniformly to A on B_r.

In fact, for $r > 0$ and $x \in B_r$, by $(H1)$, $(H2)$, $(H3)$ and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} |A_n x(t) - Ax(t)| &= |\int_0^1 K(t,s)[a_n(s) - a(s)]f(x(s))ds| \\ &\leq \int_0^{\frac{1}{n}} K(t,s)|a_n(s) - a(s)|f(x(s))ds \\ &+ \int_{\frac{n-1}{n}}^1 K(t,s)|a_n(s) - a(s)|f(x(s))ds \\ &\leq M\left[\int_0^{\frac{1}{n}} K(t,s)|a_n(s) - a(s)|ds + \int_{\frac{n-1}{n}}^1 K(t,s)|a_n(s) - a(s)ds\right] \\ &\to 0\ (n \to \infty), \end{aligned}$$

where $M = \max_{x\in[0,r]} f(x)$. So we conclude that A_n converges uniformly to A on B_r as $n \to \infty$, and therefore A is completely continuous.

Theorem 2.1. [20] *Let E be a Banach space and $P \subset E$ be a cone . Assume Ω_1 and Ω_2 are open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator. In addition suppose either*

$$\begin{aligned}
&(i)\|Au\| \le \|u\|, u \in P \cap \partial\Omega_1, \text{ and } \|Au\| \ge \|u\|, u \in P \cap \partial\Omega_2, or \\
&(ii)\|Au\| \ge \|u\|, u \in P \cap \partial\Omega_1, \text{ and } \|Au\| \le \|u\|, u \in P \cap \partial\Omega_2
\end{aligned} \tag{2.16}$$

holds. Then A has a fixed point $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Throughout this section, we shall use the following notation:

$$M_1 = \frac{1}{5\int_0^1 K(\tau(s),s)a(s)ds}, M_2 = \frac{3}{4\delta^2\gamma_\delta \int_\delta^{1-\delta} K(\tau(s),s)a(s)ds}, \ \mathbb{R}_+ = [0,\infty).$$

It is obvious that $M_2 > M_1 > 0$.

Lemma 3.1. [20] *If conditions $(H1)$, $(H2)$, $(H3)$ and $(H4)$ are satisfied, then BVP (1.1) has at least one positive solution for all $(\alpha,\beta,\lambda,\mu) \in \mathbb{R}_+^4 \setminus \{0,0,0,0\}$ with $\alpha+\beta+\lambda+\mu$ small enough.*

Proof. Since $f_0 = 0$, there exists $R_1 > 0$ such that $\frac{f(x)}{x} \le M_1$, $x \in (0,R_1]$. Therefore,

$$f(x) \le M_1 x, \ x \in (0,R_1]. \tag{3.1}$$

Set $\Omega_1 = \{x \in C[0,1] : \|x\| < R_1\}$, and let α,β,λ,μ satisfy

$$0 < \alpha \le \frac{1}{5}R_1, 0 < \beta \le \frac{27}{20}R_1, 0 < \lambda \le \frac{1}{5}R_1, 0 < \mu \le \frac{27}{20}R_1. \tag{3.2}$$

Then, for any $x \in P$ and $\|x\| = R_1$, it follows from Lemma 2.3, Lemma 2.5, (3.1) and (3.2) that

$$\begin{aligned}
Ax(t) &\le \int_0^1 K(\tau(s),s)a(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\
&\le M_1 \int_0^1 K(\tau(s),s)a(s)ds \cdot \|x\| + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu \\
&\le \frac{1}{5}R_1 + \frac{1}{5}R_1 + \frac{1}{5}R_1 + \frac{1}{5}R_1 + \frac{1}{5}R_1 = \|x\|,
\end{aligned}$$

and thus

$$\|Ax\| \le \|x\|,\ \forall x \in P \cap \partial\Omega_1. \tag{3.3}$$

Since $f_\infty = \infty$, for $M_2 > 0$, there exists $R_2 > R_1$ such that $\frac{f(x)}{x} \ge 2M_2$, for $x \in [\gamma_\delta R_2, \infty)$. Thus we have

$$f(x) \ge 2M_2 x,\ x \in [\gamma_\delta R_2, \infty). \tag{3.4}$$

Set $\Omega_2 = \{x \in C[0,1] \,|\, \|x\| < R_2\}$, For any $x \in P \cap \partial\Omega_2$, by Lemma 2.4, one has $\min_{t\in[\delta,1-\delta]} x(t) \ge \gamma_\delta \|x\| \ge \overline{R}_2$. Thus, from (2.5) and (3.4), we can conclude that

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &= \int_0^1 K\left(\frac{1}{2}, s\right) a(s) f(x(s))ds + \frac{1}{2}\alpha + \frac{1}{8}\beta + \frac{1}{4}\lambda + \frac{1}{16}\mu \\ &\ge \int_0^1 \min_{t\in[\delta,1-\delta]} K(t,s)a(s)f(x(s))ds \\ &\ge \frac{2}{3}\delta^2 \cdot 2M_2 \int_\delta^{1-\delta} K(\tau(s),s)a(s)x(s)ds \\ &\ge \frac{4}{3}\delta^2 M_2 \gamma_\delta \int_\delta^{1-\delta} K(\tau(s),s)a(s)ds \cdot \|x\| = \|x\|, \end{aligned}$$

which implies that

$$\|Ax\| \ge \|x\|,\ \forall x \in P \cap \partial\Omega_2. \tag{3.5}$$

Therefore, by (3.3), (3.5) and the first part of Theorem 2.1, we know that the operator A has at least one fixed point $x^* \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which is a positive solution of BVP (1.1).

Lemma 3.2. *If conditions $(H1)$, $(H2)$, $(H3)$ and $(H4)$ are satisfied, then BVP (1.1) has no positive solution for all $(\alpha,\beta,\lambda,\mu) \in \mathbb{R}_+^4 \setminus \{0,0,0,0\}$ with $\alpha+\beta+\lambda+\mu$ large enough.*

Proof. Suppose there exist sequences $\{\alpha_n\}$, $\alpha_n > 0$, $\{\beta_n\}$, $\beta_n > 0$, $\{\lambda_n\}$, $\lambda_n > 0$, $\{\mu_n\}$, $\mu_n > 0$ with $\lim_{n\to\infty}(\alpha_n+\beta_n+\lambda_n+\mu_n) = +\infty$, such that for any positive integer n, the BVP

$$\begin{cases} x''''(t) + a(t)f(x) = 0,\ 0 < t < 1, \\ x(0) = \alpha_n, x(1) = \beta_n, x'(0) = \lambda_n, x'(1) = -\mu_n. \end{cases} \tag{3.6}$$

has a positive solution $x_n(t)$. By (2.10), we have

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &= \int_0^1 K\left(\frac{1}{2}, s\right) a(s) f(x_n(s))ds + \frac{1}{2}\alpha_n + \frac{1}{8}\beta_n + \frac{1}{4}\lambda_n + \frac{1}{16}\mu_n \\ &\ge \frac{1}{16}(\alpha_n+\beta_n+\lambda_n+\mu_n) \to \infty,\ (n\to\infty). \end{aligned}$$

$$\|x_n\| \to \infty,\ (n \to \infty). \tag{3.7}$$

Since $f_\infty = \infty$, for $M_2 > 0$, there exists $\widehat{R} > 0$ such that $\frac{f(x)}{x} \geq 4M_2$ for $x \in [\gamma_\delta \widehat{R}, \infty)$, which implies that

$$f(x) \geq 4M_2 x,\ x \in [\gamma_\delta R, \infty). \tag{3.8}$$

Let n be large enough that $\|x_n\| \geq \widehat{R}$. Then

$$\begin{aligned}
\|x_n\| &\geq x_n(\frac{1}{2}) \\
&= \int_0^1 K\left(\frac{1}{2}, s\right) a(s) f(x_n(s)) ds + \frac{1}{2}\alpha_n + \frac{1}{8}\beta_n + \frac{1}{4}\lambda_n + \frac{1}{16}\mu_n \\
&\geq \int_0^1 \min_{t \in [\delta, 1-\delta]} K(t,s) a(s) f(x_n(s)) ds \\
&\geq \frac{2}{3}\delta^2 \cdot 4M_2 \int_\delta^{1-\delta} K(\tau(s), s) a(s) x_n(s) ds \\
&\geq \frac{8}{3}\delta^2 M_2 \gamma_\delta \int_\delta^{1-\delta} K(\tau(s), s) a(s) ds \cdot \|x_n\| = 2\|x_n\|,
\end{aligned}$$

which is a contradiction. The proof is complete.

Theorem 3.3. (i) *conditions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$ *are satisfied, if f is nondecreasing, then there exist positive constants* $(\alpha^*, \beta^*, \lambda^*, \mu^*) \in \mathbb{R}_+^4 \setminus \{0,0,0,0\}$ *such that BVP (1.1) has at least one positive solution for any* $\alpha \in (0, \alpha^*), \beta \in (0, \beta^*), \lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$ *and has no positive solution as satisfying at least one of* $\alpha \in (\alpha^*, \infty)$, $\beta \in (\beta^*, \infty)$, $\lambda \in (\lambda^*, \infty)$, $\mu \in (\mu^*, \infty)$.

(ii) *conditions* $(H1)$, $(H2)$, $(H3)$ *and* $(H5)$ *are satisfied, then the BVP (1.1) has at least one positive solution for any* $\alpha \in (0, \infty), \beta \in (0, \infty), \lambda \in (0, \infty), \mu \in (0, \infty)$.

Proof. (i) Let $\Sigma = \{(\alpha, \beta, \lambda, \mu) |$ BVP (1.1) has at least one positive solution$\}$, and $(\alpha^*, \beta^*, \lambda^*, \mu^*) = \sup \Sigma = \{(\sup \alpha, \sup \beta, \sup \lambda, \sup \mu) |$ BVP (1.1) at least one positive solution$\}$; it follows from Lemma 2.1 and Lemma 2.2 that $0 < \alpha^* < \infty$, $0 < \beta^* < \infty$, $0 < \lambda^* < \infty$,$0 < \mu^* < \infty$, From the definition of $(\alpha^*, \beta^*, \lambda^*, \mu^*)$, we know that for any $\alpha \in (0, \alpha^*), \beta \in (0, \beta^*), \lambda \in (0, \lambda^*), \mu \in (0, \mu^*)$ there are $\alpha_* > \alpha_0 > \alpha, \beta_* > \beta_0 > \beta, \lambda_* > \lambda_0 > \lambda, \mu_* > \mu_0 > \mu$ such that BVP

$$\begin{cases} x''''(t) + a(t) f(x) = 0,\ 0 < t < 1, \\ x(0) = \alpha_0, x(1) = \beta_0, x'(0) = \lambda_0, x'(1) = -\mu_0 \end{cases} \tag{3.9}$$

has a positive solution $x_0(t)$. Now we prove that for any $\alpha \in (0,\alpha_0), \beta \in (0,\beta_0), \lambda \in (0,\lambda_0), \mu \in (0,\mu_0)$, BVP (1.1) has a positive solution.

In fact, let

$$P(x_0) = \{x \in P | x(t) \le x_0(t),\ t \in [0,1]\}.$$

For any $\alpha \in (0,\alpha_0), \beta \in (0,\beta_0), \lambda \in (0,\lambda_0), \mu \in (0,\mu_0)$, $x \in P(x_0)$, it follows from (2.10) and the monotonicity of f that we have that

$$\begin{aligned} Ax(t) &\le \int_0^1 K(\tau(s),s)a(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\ &\le \int_0^1 K(\tau(s),s)a(s)f(x(s))ds + \alpha_0\phi_1(t) + \beta_0\phi_2(t) + \lambda_0\phi_3(t) + \mu_0\phi_4(t) \\ &= x_0(t). \end{aligned}$$

Thus $A(P(x_0)) \subseteq P(x_0)$. By Schauders fixed point theorem, we know that there exists a fixed point $x \in P(x_0)$ which is a positive solution of BVP (1.1). Theorem 3.3 (i) of the proof is complete.

(ii) Since $f_0 = \infty$, there exists $R_1 > 0$ such that

$$f(x) \ge 2M_2 x,\ x \in [0,R_1). \tag{3.10}$$

So, for any $x \in P$, and $\|x\| = R_1$, and any $\lambda > 0$, we have

$$\begin{aligned} Ax\left(\frac{1}{2}\right) &= \int_0^1 K\left(\frac{1}{2},s\right)a(s)f(x(s))ds + \frac{1}{2}\alpha + \frac{1}{8}\beta + \frac{1}{4}\lambda + \frac{1}{16}\mu \\ &\ge \int_0^1 \min_{t\in[\delta,1-\delta]} K(t,s)a(s)f(x(s))ds \\ &\ge \frac{2}{3}\delta^2 \cdot 2M_2 \int_\delta^{1-\delta} K(\tau(s),s)a(s)x(s)ds \\ &\ge \frac{4}{3}\delta^2 M_2 \gamma_\delta \int_\delta^{1-\delta} K(\tau(s),s)a(s)ds \cdot \|x\| = \|x\|. \end{aligned}$$

and consequently $\|Ax\| \ge \|x\|$. So, if we set $\Omega_1 = \{x \in C[0,1] | \|x\| < R_1\}$, then

$$\|Ax\| \ge \|x\|,\ \forall x \in P \cap \partial\Omega_1. \tag{3.11}$$

Next we construct the set Ω_2. We consider two cases: f is is bounded or is unbounded.

Case (1). Suppose that f is bounded, say $f(x) \le M$ for all $x \in [0,\infty)$. In this case, we choose

$$R_2 = \max\left\{2R_1,\ \frac{M}{M_1}, 5\alpha, \frac{20}{27}\beta, 5\lambda, \frac{20}{27}\mu\right\},$$

and then for $x \in P$ with $\|x\| = R_2$, we have

$$\begin{aligned} Ax(t) &\le \int_0^1 K(\tau(s),s)a(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\ &\le M\int_0^1 K(\tau(s),s)a(s)ds + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu \\ &\le \frac{M}{5M_1} + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 \\ &\le \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 = \|x\|. \end{aligned}$$

So,

$$\|Ax\| \le \|x\|. \tag{3.12}$$

Case (2). When f is unbounded. Now, since $f_\infty = 0$, there exists $R_0 > 0$ such that

$$f(x) \le M_1 x,\ x \in [R_0,\infty). \tag{3.13}$$

Let $R_2 : R_2 \ge \max\left\{2R_1,\ R_0,\ 5\alpha,\ \frac{20}{27}\beta,\ 5\lambda,\ \frac{20}{27}\mu\right\}$, and be such that

$$f(x) \le f(R_2),\ for\ 0 < x \le R_2. \tag{3.14}$$

(We are able to do this since f is unbounded.) For $x \in P$ with $\|x\| = R_2$, from (2.5), (3.14) and (3.13), we have

$$\begin{aligned} Ax(t) &= \int_0^1 K(t,s)a(s)f(x(s))ds + \alpha\phi_1(t) + \beta\phi_2(t) + \lambda\phi_3(t) + \mu\phi_4(t) \\ &\le \int_0^1 K(\tau(s),s)a(s)ds \cdot f(R_2) + \alpha + \frac{4}{27}\beta + \lambda + \frac{4}{27}\mu \\ &\le M_1R_2\int_0^1 K(\tau(s),s)a(s)ds + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 \\ &\le \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 + \frac{1}{5}R_2 = R_2 = \|x\|. \end{aligned} \tag{3.15}$$

Thus, $\|Ax\| \ge \|x\|$. Therefore, in either case we may put

$$\Omega_2 = \{x \in C[0,1] \,|\, \|x\| < R_2\}.$$

It follows that

$$\|Ax\| \leq \|x\|, \ \forall x \in P \cap \partial\Omega_2. \tag{3.16}$$

So, it follows from (3.11) and (3.16) and the second part of the Theorem 2.1 that A has a fixed point $x^* \in P \cap \left(\overline{\Omega}_4 \backslash \Omega_3\right)$. Then x is a positive solution of BVP (1.1). The proof is complete.

Conflict of Interests

The author declare that there is no conflict of interests.

Acknowledgements

This work is supported by Gansu Provincial National Science Foundation of China (No. 145R-JZA087) and the Scientific Research Foundation of the Education Department of Gansu Province (No.1114-04).

REFERENCES

[1] R.P. Agarwal, Y.M. Chow, Iterative methods for a fourth-order boundary value problem, J. Comput. Appl. Math. 10 (1984), 203-217.

[2] R. Ma, H. Wu, positive solutions of a fourth-order two-point boundary value problem, Acta Math. Sci. A, 22 (2002), 244-249.

[3] Q. Yao, positive solutions for eigenvalue problems of fourth-order elastic beam equations, Appl. Math. Lett. 17 (2004), 237-243.

[4] Q. Yao, Solvability of an elastic beam equation Twith Caratheodory function, Math. Appl. 17 (2004), 389-392.

[5] P. Korman, Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems, Proc. Royal Soc. Edinburgh A 134 (2004), 179-190.

[6] J. Sun, X. Wang, Uniqueness and parameter dependence of solutions of fourth-order four-point nonhomogeneous BVPs, Electron. J. Different. Equ. 2010 (2010), Article ID 84.

[7] L. Kong, Q. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions I, Math. Nachr, 278 (2005), 173-193.

[8] L. Kong, Q. Kong, Second-order boundary value problems with nonhomogeneous boundary conditions II, J. Math. Anal. Appl, 330 (2007), 1393-1411.

[9] L. Kong, Q. Kong, Uniqueness and parameter dependence of solutions of second-order boundary value problems, Appl. Math. Lett. 22 (2009), 1633-1638.

[10] R. Ma, Positive solutions for secone-order three-point boundary value problems, Appl. Math. Lett. 14 (2001), 1-5.

[11] R. Ma, Positive solutions for nonhomogeneous m-point boundary value problems, Comput. Math. Appl. 47 (2004), 689-698.

[12] H. Chen, Positive solutions for the nonhomogeneous three-point boundary value problem of second-order differential equations, Math. Comput. Modelling 45 (2007), 844-852.

[13] Y. Sun, Positive solutions for third-order three-point nonhomogeneous boundary value problems, Appl. Math. Lett. 22 (2009), 45-51.

[14] X. Hao, L. Liu, Y. Wu, On positive solutions of an m-point nonhomogeneous singular boundary value problem, Nonlinear Anal. 73 (2010), 2532-2540.

[15] D. Guo, V. Lakshmikantham; Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.