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# STRONG CONVERGENCE THEOREM FOR A COMMON POINT OF SOLUTION OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEM

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**Abstract.** We introduce an iterative process which converges strongly to a common point of solutions of variational inequality problem for $\gamma$-inverse strongly monotone mapping and fixed points of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping in Banach spaces. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

**Keywords**: Monotone mappings; relatively asymptotically nonexpansive mappings; relatively nonexpansive, strong convergence; variational inequality problems.

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## 1. Introduction

Let $E$ be a real Banach space with dual $E^*$. A mapping $A : D(A) \subset E \to E^*$ is said to be *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay\rangle \geq 0. \quad (1)$$

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$A$ is said to be $\gamma-$*inverse strongly monotone* if there exists a positive real number $\gamma$ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma||Ax-Ay||^2, \text{ for all } x,y\in D(A). \tag{2}$$

If $A$ is $\gamma-$inverse strongly monotone, then it is *Lipschitz continuous* with constant $\frac{1}{\gamma}$, i.e., $||Ax-Ay|| \leq \frac{1}{\gamma}||x-y||$, for all $x,y\in D(A)$.

Suppose that $A$ is a monotone mapping from $C\subseteq E$ into $E^*$. The variational inequality problem is formulated as finding:

$$\text{a point } u\in C \text{ such that } \langle v-u, Au\rangle \geq 0, \text{ for all } v\in C. \tag{3}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$.

Variational inequalities were initially studied by Stampacchia [7, 9] and ever since have been widely studied. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u\in C$ satisfying $0\in Au$. If $E=H$, a Hilbert space, one method of solving a point $u\in VI(C,A)$ is the projection algorithm which starts with any point $x_1=x\in C$ and updates iteratively as $x_{n+1}$ according to the formula

$$x_{n+1}=P_C(x_n-\alpha_n Ax_n), \text{ for any } n\geq 1, \tag{4}$$

where $P_C$ is the metric projection from $H$ onto $C$ and $\{\alpha_n\}$ is a sequence of positive real numbers. In the case that $A$ is $\gamma-$inverse strongly monotone, Iiduka, Takahashi and Toyoda [4] proved that the sequence $\{x_n\}$ generated by (4) converges *weakly* to some element of $VI(C,A)$.

Our concern now is the following: *Is it possible to construct a sequence $\{x_n\}$ which converges strongly to some point of $VI(C,A)$?*

In this connection, when $E = H$, a Hilbert space and $A$ is $\gamma-$inverse strongly monotone, Iiduka, Takahashi and Toyoda [4] studied the following iterative scheme, the so called *hybrid projection iteration method*:

$$(5) \qquad \begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = P_C(x_n - \alpha_n A x_n), \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \ n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$ and $P_C$ is the metric projection of $H$ onto $C$. They proved that the sequence $\{x_n\}$ generated by (5) converges strongly to $P_{VI(C,A)}(x_0)$.

It is well known that if $C$ is a nonempty closed convex subset of a Hilbert space $H$ the metric projection $P_C : H \to C$ is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue of the metric projection in Hilbert spaces. Next, we assume that $E$ is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2, \forall x, y \in E,$$

where $J$ is the normalized duality mapping from $E$ into $2^{E^*}$ defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\}.$$

It is well known that $E$ is smooth if and only if $J$ is single-valued and if $E$ is uniformly smooth then $J$ is uniformly continuous on bounded subsets of $E$. Moreover, if $E$ is a reflexive and strictly convex Banach space with a strictly convex dual, then $J^{-1}$ is single valued, one-to-one, surjective, and it is the duality mapping from $E^*$ into $E$ and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, [16]).

Following Alber [1], the generalized projection $\Pi_C : E \to C$, is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x).$$

If $E$ is a Hilbert space, then $\phi(y, x) = ||y - x||^2$ and $\Pi_C = P_C$ is the metric projection of $H$ onto $C$.

In the case that $E$ is 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [3] studied the following iterative scheme for a variational inequality problem for $\gamma-$inverse strongly monotone mapping $A$:

$$(6) \qquad \begin{cases} x_0 \in K, \text{ chosen arbitrary,} \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \\ C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \ n \geq 1, \end{cases}$$

where $\Pi_{C_n \cap Q_n}$ is the generalized projection from $E$ onto $C_n \cap Q_n$, $J$ is the normalized duality mapping from $E$ into $E^*$ and $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of $VI(C, A)$ provided that $VI(C, A) \neq \emptyset$ and $A$ satisfies $||Ax|| \leq ||Ax - Ap||$, for all $x \in C$ and $p \in VI(C, A)$.

Let $T$ be a mapping from $C$ into itself. We denote by $F(T)$ the fixed points set of $T$. A point $p$ in $C$ is said to be an *asymptotic fixed point of* $T$ (see [14]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is said to be *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$ for each $x, y \in C$ and is called *relatively nonexpansive* if (R1) $F(T) \neq \emptyset$; (R2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and (R3) $F(T) = \hat{F}(T)$. $T$ is called *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$, and $p \in F(T)$.

A mapping $T$ from $C$ into itself is said to be *asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ and $||T^n x - T^n y|| \leq k_n ||x - y||$ for each $x, y \in C$ and is called *relatively asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that (N1) $F(T) \neq \emptyset$; (N2) $\phi(p, T^n x) \leq k_n \phi(p, x)$ for $x \in C$ and $p \in F(T)$, and (N3) $F(T) = \hat{F}(T)$, where $k_n \to 1$, as $n \to \infty$. A self mapping on $C$ is called *asymptotically regular* on $C$, if for any bounded subset $\overline{C}$ of $C$, there holds the following equality:

$$\lim_{n\to\infty} \sup\{||T^{n+1}x - T^n x\} : x \in \overline{C}\} = 0.$$

$T$ is called *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Clearly, the class of relatively asymptotically nonexpansive mappings contains the class of relatively nonexpansive mappings.

In 2003, Nakajo and Takahashi [12] proposed the following modification of the Mann iteration method for a nonexpansive mapping $T$ in a Hilbert space $H$:

$$(7) \qquad \begin{cases} x_0 \in C, \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C; \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 1, \end{cases}$$

where $C$ is a closed convex subset of $H$, $P_C$ denotes the metric projection from $H$ onto a closed convex subset $C$ of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (7) converges strongly to $P_{F(T)}(x_0)$.

In spaces more general than Hilbert spaces, Matsushita and Takahashi [11] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping $T$ in a Banach space $E$:

$$(8)\quad \begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C; \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \geq 1, \end{cases}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (8) converges strongly to $\Pi_{F(T)}x_0$.

Recently, many authors have considered the problem of finding a common element of the fixed points set of relatively nonexpansive mapping and the solution set of variational inequality problem for $\gamma-$inverse monotone mapping (see, e.g., [8, 13, 15, 17, 20, 21]).

In [20], Zegeye *et al.* studied the following iterative scheme for a common point of solutions of a variational inequality problem for $\gamma-$inverse strongly monotone mapping $A$ and fixed points of a closed relatively quasi-nonexpansive mapping $T$ in a 2-uniformly convex and uniformly smooth Banach space $E$:

$$(9)\quad \begin{cases} C_1 = C, \text{ chosen arbitrary,} \\ z_n = \Pi_C(x_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\beta Jx_n + (1-\beta)JTz_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ is a sequence satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A) \neq \emptyset$ provided that $A$ satisfies $||Ax|| \leq ||Ax - Ap||$, for all $x \in C$, and $p \in F$.

Recently, Zegeye and Shahzad [24] studied the following iterative scheme for a common point of solutions of a variational inequality problem for $\gamma-$inverse strongly monotone mapping $A$ and fixed points of an asymptotically nonexpansive mapping on a closed convex and bounded set $C$ which is a subset of a real Hilbert space $H$:

$$(10)\qquad \begin{cases} C_1 = C, \text{ chosen arbitrary,} \\ z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) S^n z_n, \\ C_{n+1} = \{z \in C_n : ||z-u_n||^2 \le ||z-x_n||^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), n \ge 1, \end{cases}$$

where $P_{C_n}$ is the metric projection from $H$ into $C_n$ and $\theta_n = (1-\alpha_n)(k_n^2-1)(diam(C))^2$ and $\{\alpha_n\}, \{\lambda_n\}$ are sequences satisfying certain condition. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A) \neq \emptyset$ provided that $A$ satisfies $||Ax|| \le ||Ax - Ap||$ for all $x \in C$ and $p \in F$.

We note that the computation of $x_{n+1}$ in Algorithms (5),(6) and (7)-(10) is not simple because of the involvement of computation of $C_{n+1}$ from $C_n$, for each $n \ge 1$.

More recently, Zegeye and Shahzad [25] studied the following iterative scheme for a common point of solutions of finite family of $\gamma$-inverse strongly monotone mappings and fixed points of two $\phi$-uniformly $L$-Lipschitzian and quasi-$\phi$-asymptotically nonexpansive mappings in a 2-uniformly convex and uniformly smooth Banach space $E$:

$$(11)\qquad \begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ u_n = \Pi_C J^{-1}(Jx_n - \lambda_n A_n x_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + \beta_n J S_1^n u_n + \theta_n J S_2^n u_n), \end{cases}$$

where $A_n =: A_n(mod\ N)$ and $\alpha_n, \beta_n, \theta_n \subset [c_1, 1]$, for some $c_1 > 0$, satisfying some mild conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := \Big[\bigcap_{i=1}^{N} VI(C, A_i)\Big] \cap \Big[\bigcap_{l=1}^{2} F(S_l)\Big]$ provided that *interior of $F$ is nonempty*. We recall that $T : C \to C$ is called *$\phi$-uniformly $L$-Lipschitzian* if there exists $L > 0$ such that $\phi(T^n x, T^n y) \le L\phi(x, y), \forall x, y \in C$ and it called *quasi-$\phi$-asymptotically nonexpansive* if there exists $k_n \subseteq [1, \infty)$ such that $\phi(p, T^n x) \le k_n \phi(p, x), \forall x \in C, p \in F(T)$. But it is worth mentioning, the assumption, that *the interior of* $F$ *is nonempty* is severe restriction.

It is our purpose in this paper to introduce an iterative scheme $\{x_n\}$ which converges strongly to a common point of solutions of variational inequality problem for $\gamma$-inverse monotone mapping and fixed points of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping in Banach spaces. Our scheme does not involve computation of $C_{n+1}$ from $C_n$ for each $n \geq 1$ and the requirement that interior of $F$ is nonempty is dispensed with. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

## 2. Preliminaries

Let $E$ be a normed linear space with dim $E \geq 2$. The *modulus of smoothness* of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| \ + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}.$$

The space $E$ is said to be *smooth* if $\rho_E(\tau) > 0$, $\forall\, \tau > 0$ and $E$ is called *uniformly smooth* if and only if $\lim\limits_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$.

The *modulus of convexity* of $E$ is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1;\ \epsilon = \|x - y\|\right\}.$$

$E$ is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0, 2]$. Let $p > 1$. Then $E$ is said to be $p-$*uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$, for all $\epsilon \in [0, 2]$. Observe that every $p$-uniformly convex space is uniformly convex.

It is well known (see for example [19]) that

$$L_p\ (\ l_p)\ or\ W_m^p \text{ is } \begin{cases} p - uniformly\ convex, & if\ p \geq 2, \\ 2 - uniformly\ convex, & if\ 1 < p \leq 2. \end{cases}$$

In the sequel, we shall need the following lemmas:

**Lemma 2.1.** [19] *Let $E$ be a $2-$uniformly convex Banach space. Then, for all $x, y \in E$, we have*

$$||x-y|| \leq \frac{2}{c^2}||Jx - Jy||, \tag{12}$$

*where $J$ is the normalized duality mapping of $E$ and $0 < c \leq 1$.*

**Lemma 2.2.** [22] *Let $C$ be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space $E$. If $A : C \to E^*$ is continuous monotone mapping, then $VI(C, A)$ is closed and convex.*

**Proposition 2.3.** *Let $C$ be a closed convex subset of a uniformly convex and uniformly smooth Banach space $E$, and let $S$ be closed relatively asymptotically nonexpansive mapping from $C$ into itself. Then $F(S)$ is closed and convex.*

**Proof.** The method of proof of Proposition 2.11 of [23] provides the required conclusion.

**Lemma 2.4.** [1] *Let $K$ be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space $E$ and let $x \in E$. Then $\forall y \in K$,*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x).$$

**Lemma 2.5.** [5] *Let $E$ be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x||^2, \text{ for all } x \in E \text{ and } x^* \in E,$$

studied by Alber [1]. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

**Lemma 2.6.** [1] *Let $E$ be a reflexive strictly convex and smooth Banach space with $E^*$ as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*),$$

*for all $x \in E$ and $x^*, y^* \in E^*$.*

**Lemma 2.7.** [1] *Let $C$ be a convex subset of a real smooth Banach space $E$. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \forall z \in C.$$

**Lemma 2.8.** [20] *Let $E$ be a uniformly convex Banach space and $B_R(0)$ be a closed ball of $E$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\alpha x + (1-\alpha)y||^2 \leq \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)g(||x-y||),$$

*for $\alpha \in (0,1)$ and for $x, y \in B_R(0) := \{x \in E : ||x|| \leq R\}$.*

**Lemma 2.9.** [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\beta_n)a_n + \beta_n\delta_n, n \geq n_0,$$

*where $\{\beta_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2.10.** [10] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

*In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

## 3. Main results

We remark that, as it is mentioned in [24], if $C$ is a subset of a real Banach space $E$ and $A : C \to E^*$ is a mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$, then $VI(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}$. We shall make use of this remark to prove the next theorem.

**Theorem 3.1.** *Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space $E$. Let $A : C \to E^*$ be a $\gamma$-inverse strongly monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T : C \to C$ be an asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequences $\{k_n\}$. Assume that $F := VI(C, A) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \ \text{chosen arbitrarily}, \\ w_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)JT^n y_n), \end{cases} \tag{13}$$

*where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$ for some real numbers $a, b$ such that $0 < a \leq \lambda_n \leq b < \frac{c^2\gamma}{2}$, for $\frac{1}{c}$ a 2-uniformly convex constant of $E$. Then $\{x_n\}$ converges strongly to an element of $F$.*

**Proof.** Let $p := \Pi_F w$. Then by Lemma 2.4 and Lemma 2.6 we get that

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, J^{-1}(Jx_n - \lambda_n Ax_n)) = V(p, Jx_n - \lambda_n Ax_n) \\
&\leq V(p, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n\rangle \\
&\quad -2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\
&= V(p, Jx_n) - 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, Ax_n\rangle \\
&= \phi(p, x_n) - 2\lambda_n \langle x_n - p, Ax_n\rangle - 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, Ax_n\rangle \qquad (14)
\end{aligned}$$

Thus, since $p \in F$ and $A$ is $\gamma-$inverse strongly monotone, Lemma 2.1 and the fact that $\lambda_n < \frac{c^2}{2}\gamma$, we have from (14) that

$$\begin{aligned}
\phi(p, w_n) &\leq \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n - Ap\rangle \\
&\quad -2\lambda_n\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, Ax_n\rangle \\
&\leq \phi(p, x_n) - 2\lambda_n\gamma||Ax_n||^2 \\
&\quad +2\lambda_n||J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)||||Ax_n|| \\
&\leq \phi(p, x_n) - 2\lambda_n\gamma||Ax_n||^2 + \frac{4}{c^2}\lambda_n^2||Ax_n||^2 \\
&= \phi(p, x_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \gamma)||Ax_n||^2 \qquad (15) \\
&\leq \phi(p, x_n). \qquad (16)
\end{aligned}$$

Now from (13), Lemma 2.4, property of $\phi$ and (16) we get that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n) \\
&\leq \phi(p, J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n) \\
&= ||p||^2 - 2\langle p, \alpha_n Jw + (1-\alpha_n)Jw_n\rangle + ||\alpha_n Jw + (1-\alpha_n)Jw_n||^2 \\
&\leq ||p||^2 - 2\alpha_n\langle p, Jw\rangle - 2(1-\alpha_n)\langle p, Jw_n\rangle \\
&\quad +\alpha_n||Jw||^2 + (1-\alpha_n)||Jw_n||^2 \\
&= \alpha_n\phi(p, w) + (1-\alpha_n)\phi(p, w_n) \\
&\leq \alpha_n\phi(p, w) + (1-\alpha_n)\phi(p, x_n). \qquad (17)
\end{aligned}$$

Then, from (13) and property of $\phi$ we get that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)JT^n y_n) \\
&\leq \phi(p, J^{-1}(\beta_n Jw_n + (1-\beta_n)JT^n y_n) \\
&\leq \beta_n\phi(p, w_n) + (1-\beta_n)\phi(p, JT^n y_n)),
\end{aligned}$$

which implies using relatively asymptotic nonexpansiveness of $T$, (16) and (17) that

$$\begin{aligned} \phi(p, x_{n+1}) &\leq \beta_n \phi(p, w_n) + (1-\beta_n) k_n \phi(p, y_n) \\ &\leq \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p, y_n)) + (1-\beta_n)(k_n - 1)\phi(p, y_n) \\ &\leq \beta_n \phi(p, x_n) + (1-\beta_n)\big[\alpha_n \phi(p, w) + (1-\alpha_n)\phi(p, x_n)\big] \\ &\quad +(1-\beta_n)(k_n - 1)\big[\alpha_n \phi(p, w) + (1-\alpha_n)\phi(p, x_n)\big], \\ &\leq \big[\alpha_n(1-\beta_n) + (1-\beta_n)(k_n-1)\alpha_n\big]\phi(p, w) \\ &\quad +\big[(1-\alpha_n(1-\beta_n)) + (1-\beta_n)(k_n-1)(1-\alpha_n)\big]\phi(p, x_n) \\ &\leq \delta_n \phi(p, w) + \big[1 - (1-\epsilon)\delta_n\big]\phi(p, x_n), \end{aligned} \tag{18}$$

where $\delta_n = (1-\beta_n)k_n\alpha_n$, since there exists $N_0 > 0$ such that $\frac{(k_n-1)}{\alpha_n} \leq \epsilon k_n$ for all $n \geq N_0$ and for some $\epsilon > 0$ satisfying $(1-\epsilon)\delta_n \leq 1$. Thus, by induction,

$$\phi(p, x_{n+1}) \leq \max\{\phi(p, x_0), (1-\epsilon)^{-1}\phi(p, w)\}, \forall n \geq N_0.$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemma 2.4, Lemma 2.6 and property of $\phi$ we obtain that

$$\begin{aligned} \phi(p, y_n) &\leq \phi(p, z_n) = V(p, Jz_n) \\ &\leq V(p, Jz_n - \alpha_n(Jw - Jp)) - 2\langle z_n - p, -\alpha_n(Jw - Jp)\rangle \\ &= \phi(p, J^{-1}(\alpha_n Jp + (1-\alpha_n)Jw_n) + 2\alpha_n\langle z_n - p, Jw - Jp\rangle \\ &\leq \alpha_n \phi(p, p) + (1-\alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, Jw - Jp\rangle \\ &= (1-\alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, Jw - Jp\rangle \\ &\leq (1-\alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, Jw - Jp\rangle. \end{aligned} \tag{19}$$

Furthermore, from (13), Lemma 2.8 and relatively asymptotic nonexpansiveness of $T$ we have that

$$\begin{aligned}\phi(p,x_{n+1}) &= \phi(p,\Pi_C J^{-1}(\beta_n Jw_n+(1-\beta_n)JT^n y_n))\\ &\leq \beta_n\phi(p,w_n)+(1-\beta_n)\phi(p,JT^n y_n)\\ &\quad -(1-\beta_n)\beta_n g(||Jw_n-JT^n y_n||)\\ &\leq \beta_n\phi(p,w_n)+(1-\beta_n)\phi(p,y_n)\\ &\quad +(1-\beta_n)(k_n-1)\phi(p,y_n)-(1-\beta_n)\beta_n g(||Jw_n-JT^n y_n||),\end{aligned}$$

which implies from (15) and (19) that

$$\begin{aligned}\phi(p,x_{n+1}) &\leq \beta_n\Big[\phi(p,x_n)+2\lambda_n(\frac{2}{c^2}\lambda_n-\gamma)||Ax_n||^2\Big]\\ &\quad +(1-\beta_n)\Big[(1-\alpha_n)\phi(p,x_n)+2\alpha_n\langle z_n-p,Jw-Jp\rangle\Big]\\ &\quad +(1-\beta_n)(k_n-1)\phi(p,y_n)-(1-\beta_n)\beta_n g(||Jw_n-JT^n y_n||)\\ &\leq (1-\theta_n)\phi(p,x_n)+2\theta_n\langle z_n-p,Jw-Jp\rangle+(k_n-1)M\\ &\quad -(1-\beta_n)\beta_n g(||Jw_n-JT^n y_n||)-2\lambda_n\beta_n(\gamma-\frac{2}{c^2}\lambda_n)||Ax_n||^2 \qquad (20)\\ &\leq (1-\theta_n)\phi(p,x_n)+2\theta_n\langle z_n-p,Jw-Jp\rangle+(k_n-1)M, \qquad (21)\end{aligned}$$

for some $M>0$, where $\theta_n:=\alpha_n(1-\beta_n)$ for all $n\in N$. Note that $\theta_n$ satisfies $\lim_n\theta_n=0$ and $\sum_{n=1}^{\infty}\theta_n=\infty$.

Now, the rest of the proof is divided into two parts:

**Case 1.** Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\phi(p,x_n)\}$ is non-increasing. In this situation, $\{\phi(p,x_n)\}$ is convergent. Then from (20) we have that

$$2\lambda_n\beta_n(\gamma-\frac{2}{c^2}\lambda_n)||Ax_n||^2+(1-\beta_n)\beta_n g(||Jw_n-JT^n y_n||\to 0, \qquad (22)$$

which implies, by the property of $g$ and the fact that $\lambda_n<\frac{c^2}{2}\gamma$, that

$$||Ax_n||\to 0 \text{ and } Jw_n-JT^n y_n\to 0,\ \text{ as } n\to\infty, \qquad (23)$$

and hence, since $J^{-1}$ is uniformly continuous on bounded sets we obtain that

$$w_n - T^n y_n \to 0, \text{ as } n \to \infty. \tag{24}$$

Furthermore, Lemma 2.4, property of $\phi$ and the fact that $\alpha_n \to 0$, as $n \to \infty$, imply that

$$\begin{aligned}
\phi(w_n, y_n) &= \phi(w_n, \Pi_C z_n) \leq \phi(w_n, z_n) \\
&= \phi(w_n, J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n) \\
&\leq \alpha_n \phi(w_n, w) + (1-\alpha_n)\phi(w_n, w_n) \\
&\leq \alpha_n \phi(w_n, w) + (1-\alpha_n)\phi(w_n, w_n) \to 0 \text{ as } n \to \infty, \qquad (25)
\end{aligned}$$

and hence

$$w_n - y_n \to 0 \text{ and } w_n - z_n \to 0, \text{ as } n \to \infty. \tag{26}$$

Therefore, from (24) and (26) we obtain that

$$y_n - z_n \to 0 \text{ and } y_n - T^n y_n \to 0, \text{ as } n \to \infty. \tag{27}$$

Therefore, since

$$\begin{aligned}
||y_n - Ty_n|| &\leq ||y_n - T^n y_n|| + ||T^n y_n - T^{n+1} y_n|| + ||T^{n+1} y_n - Ty_n||, \\
&= ||y_n - T^n y_n|| + ||T^n y_n - T^{n+1} y_n|| + ||T(T^n y_n) - Ty_n||,
\end{aligned}$$

(28)

we have from (27), asymptotic regularity and uniform continuity of $T$ that

$$||y_n - Ty_n|| \to 0, \text{ as } n \to \infty. \tag{29}$$

Since $\{z_n\}$ is bounded and $E$ is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty} \langle z_n - p, Jw - Jp\rangle = \lim_{i\to\infty} \langle z_{n_i} - p, Jw - Jp\rangle$. Then, from (27) we get that

$$y_{n_i} \rightharpoonup z, w_{n_i} \rightharpoonup z, \text{ as } i \to \infty. \tag{30}$$

Thus, since $T$ satisfies condition (N3) we obtain from (29) that $z \in F(T)$.

Next, we show that $z \in A^{-1}(0)$. Now, from Lemma 2.4 and Lemma 2.6 we have that

$$\begin{aligned}\phi(x_n, w_n) &= \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \le V(x_n, Jx_n - \lambda_n Ax_n)) \\ &\le V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\ &= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -A_n x_n\rangle \\ &= 2\lambda_n\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -A_n x_n\rangle \\ &\le 2\lambda_n||J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n||.||Ax_n|| \le \frac{4}{c^2}\lambda_n^2||Ax_n||^2,\end{aligned}$$

then, using (23) we obtain that

$$\phi(x_n, w_n) \to 0, \text{ as } n \to \infty, \tag{31}$$

which implies by Lemma 2.5 that

$$x_n - w_n \to 0, \text{ as } n \to \infty, \tag{32}$$

and hence from (30) we have that $x_{n_i} \rightharpoonup z$. Now, since $A$ is $\gamma$-inverse strongly monotone, we have

$$\gamma||Ax_{n_i} - Az||^2 \le \langle x_{n_i} - z, Ax_{n_i} - Az\rangle \to 0, \text{ as } i \to \infty. \tag{33}$$

In particular, $Ax_{n_i} \to Az$. Because, $Ax_n \to 0$, so $Az = 0$. Hence, $z \in A^{-1}(0)$.

Thus, from the above discussions we obtain that $z \in F := F(T) \cap VI(C, A)$. Therefore, by Lemma 2.7, we immediately obtain that $\limsup_{n\to\infty}\langle z_n - p, Jw - Jp\rangle = \lim_{i\to\infty}\langle z_{n_i} - p, Jw - Jp\rangle = \langle z - p, Jw - Jp\rangle \le 0$. It follows from Lemma 2.9 and (21) that $\phi(p, x_n) \to 0$, as $n \to \infty$. Consequently, $x_n \to p$.

**Case 2.** Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(p, x_{n_i}) < \phi(p, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\phi(p, x_{m_k}) \le \phi(p, x_{m_k+1})$ and $\phi(p, x_k) \le \phi(p, x_{m_k+1})$, for all $k \in \mathbb{N}$. Then from (20) and the fact that $\theta_n \to 0$ we have

$$||Ax_{m_k}|| \to 0 \text{ and } g(||Jw_{m_k} - JT^{m_k}y_{m_k}||) \to 0, \text{ as } k \to \infty.$$

Thus, using the same proof as in Case 1, we obtain that $w_{m_k} - Ty_{m_k} \to 0$, $w_{m_k} - y_{m_k} \to 0$, $w_{m_k} - z_{m_k} \to 0$, $w_{m_k} - x_{m_k} \to 0$, as $k \to \infty$ and hence we obtain that

$$\limsup_{k\to\infty} \langle z_{m_k} - p, Jw - Jp \rangle \leq 0. \tag{34}$$

Then from (21) we have that

$$\phi(p, x_{m_k+1}) \leq (1 - \theta_{m_k})\phi(p, x_{m_k}) + 2\theta_{m_k}\langle z_{m_k} - p, Jw - Jp\rangle + (k_{m_k} - 1)M. \tag{35}$$

Since $\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1})$, (35) implies that

$$\begin{aligned} \theta_{m_k}\phi(p, x_{m_k}) &\leq \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) + 2\theta_{m_k}\langle z_{m_k} - p, Jw - Jp\rangle \\ &\quad + (k_{m_k} - 1)M \\ &\leq 2\theta_{m_k}\langle z_{m_k} - p, Jw - Jp\rangle + (k_{m_k} - 1)M. \end{aligned}$$

In particular, since $\theta_{m_k} > 0$, we get

$$\phi(p, x_{m_k}) \leq 2\langle z_{m_k} - p, Jw - Jp\rangle + \frac{(k_{m_k} - 1)}{\theta_{m_k}}M.$$

Then, from (34) and the fact that $\frac{(k_{m_k}-1)}{\theta_{m_k}} \to 0$ we obtain $\phi(p, x_{m_k}) \to 0$, as $k \to \infty$. This together with (35) gives $\phi(p, x_{m_k+1}) \to 0$, as $k \to \infty$. But $\phi(p, x_k) \leq \phi(p, x_{m_k+1})$, for all $k \in \mathbb{N}$, thus we obtain that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p$ and the proof is complete.

It is worth to mention that the method of proof of Theorem 3.1 provides the following theorem.

**Theorem 3.2.** *Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space $E$. Let $A : C \to E^*$ be a $\gamma$-inverse strongly* *monotone mapping. Let $T : C \to C$ be an asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequences $\{k_n\}$. Assume that $F := A^{-1}(0) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(36) \qquad \begin{cases} x_0 = w \in C, \ \text{chosen arbitrarily}, \\ w_n = J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)JT^n y_n), \end{cases}$$

*where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_n\}$ is a sequence in $[a,b]$ for some real numbers $a, b$ such that $0 < a \leq \lambda_n \leq b < \frac{c^2\gamma}{2}$, for $\frac{1}{c}$ a 2-uniformly convex constant of $E$. Then $\{x_n\}$ converges strongly to an element of $F$.*

The following is an example of an asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping.

**Example 3.3.** Let $C := [\frac{-1}{\pi}, \frac{1}{\pi}]$ and define $T : C \to C$ by

$$T(x) = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}), & x \neq 0, \\ x, & x = 0. \end{cases}$$

Then following an argument used in [6], it can be seen that $T$ is relatively asymptotically nonexpansive, asymptotically regular and uniformly continuous mapping. For detail, see [26].

If in Theorem 3.1, we assume that $A \equiv 0$, then the assumption that $E$ be 2-uniformly convex may not be needed. In fact, we have the following corollary.

**Corollary 3.4.** *Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $T : C \to C$ be an asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequences $\{k_n\}$.*

*Assume that $F := F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$
(37)\qquad \begin{cases} x_0 = w \in C, \ \text{chosen arbitrarily}, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jx_n), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n y_n), \end{cases}
$$

*where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$, $\{\beta_n\} \subset [c,d] \subset (0,1)$. Then $\{x_n\}$ converges strongly to an element of $F$.*

**Proof.** If we put $A \equiv 0$ in (13) then we get that $w_n = x_n$ and (13) reduces to (37). Therefore, the conclusion follows from Theorem 3.1 without the requirement that $E$ be 2-uniformly convex.

If in Theorem 3.1, we assume that $T \equiv I$, identity map on $C$ then we get the following corollary.

**Corollary 3.5.** *Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space $E$. Let $A : C \to E^*$, be a $\gamma$-inverse strongly monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Assume that $F := VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$
(38)\qquad \begin{cases} x_0 = w \in C, \ \text{chosen arbitrarily}, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)(Jx_n - \lambda_n Ax_n)), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)Jy_n), \end{cases}
$$

*where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_n\}$ is a sequence in $[a,b]$ for some real numbers $a, b$ such that $0 < a \leq \lambda_n \leq b < \frac{c^2\gamma}{2}$, for $\frac{1}{c}$ a 2-uniformly convex constant of $E$. Then $\{x_n\}$ converges strongly to an element of $F$.*

**Proof.** If we put $T \equiv I$, identity map on $C$, then (13) reduces to (38). Therefore, the conclusion follows from Theorem 3.1.

If in Theorem 3.1, we assume that $T$ is relatively nonexpansive we get the following corollary.

**Corollary 3.6.** *Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space $E$. Let $A : C \to E^*$, be a $\gamma$-inverse strongly monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $F := VI(C, A) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(39) \qquad \begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ w_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)JTy_n), \end{cases}$$

*where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$ for some real numbers $a, b$ such that $0 < a \leq \lambda_n \leq b < \frac{c^2\gamma}{2}$, for $\frac{1}{c}$ a 2-uniformly convex constant of $E$. Then $\{x_n\}$ converges strongly to an element of $F$.*

**Proof.** We note that the method of proof of Theorem 3.1 provides the required assertion.

If $E = H$, a real Hilbert space, then $E$ is 2-uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on $H$ and $\Pi_C = P_C$, projection mapping from $H$ onto $C$. Thus, the following corollary holds.

**Corollary 3.7.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $A : C \to H$ be a $\gamma$-inverse strongly monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T : C \to C$ be an asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequences $\{k_n\}$. Assume that $F := VI(C, A) \cap F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(40) \qquad \begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ w_n = x_n - \lambda_n Ax_n, \\ y_n = P_C(\alpha_n w + (1-\alpha_n)w_n), \\ x_{n+1} = P_C(\beta_n w_n + (1-\beta_n)T^n y_n), \end{cases}$$

*where* $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $[a,b]$ *for some real numbers* $a, b$ *such that* $0 < a \leq \lambda_n \leq b < \gamma$. *Then* $\{x_n\}$ *converges strongly to an element of* $F$.

## 4. Applications

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in Banach spaces. We shall make use of the following lemma by Baillon and Haddad [2].

**Lemma 4.1.** *Let* $E$ *be a Banach space, Let* $f$ *be a continuous Fréchet differentiable convex functional on* $E$ *and let* $\bigtriangledown f$ *be the gradient of* $f$. *If* $\bigtriangledown f$ *is* $\frac{1}{\alpha}$*-Lipschitzian continuous, then* $\bigtriangledown f$ *is* $\alpha$*-inverse-strongly monotone.*

**Theorem 4.2.** *Let* $E$ *be a 2-uniformly convex and uniformly smooth real Banach space. Let* $f$ *be a continuously Fréchet differentiable convex functional on* $E$ *and* $\bigtriangledown f$ *is* $\frac{1}{\alpha}$*-Lipschitzian continuous and* $F := (\bigtriangledown f)^{-1}(0) = \{z \in E : f(z) = \min_{y\in E} f(y)\} \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)(Jx_n - \lambda_n \bigtriangledown f(x_n)), \\ x_{n+1} = \Pi_C J^{-1}(\beta_n Jw_n + (1-\beta_n)Jy_n), \end{cases} \tag{41}$$

*where* $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $[a,b]$ *for some real numbers* $a, b$ *such that* $0 < a \leq \lambda_n \leq b < \frac{c^2\alpha}{2}$, *for* $\frac{1}{c}$ *a 2-uniformly convex constant of* $E$. *Then* $\{x_n\}$ *converges strongly to an element of* $F$.

**Proof.** We note from Lemma 4.1 that $\bigtriangledown f$ is $\alpha$-inverse strongly monotone operator from $E$ into $E^*$. Thus, using Theorem 3.2 with $T \equiv I$, $\{x_n\}$ converges strongly to $F$.

**Remark 4.3.**

(1) Theorem 3.1 improves and extends the corresponding results of Zegeye *et al.* [20], Zegeye and Shahzad [24] and [25] in the sense that either our scheme does not require computation of $C_{n+1}$ for each $n \geq 1$ or the assumption that the interior of $F$ is nonempty is not required.

(2) Corollary 3.4 improves the corresponding results of Nakajo and Takahashi [12] and Matsushita and Takahashi [11] in the sense that either our scheme does not require computation of $C_{n+1}$ for each $n \geq 1$ or the class of mappings considered in our corollary is more general.

(3) Corollary 3.5 improves the corresponding results of Iiduka and Takahashi [3] and Iiduka, Takahashi and Toyoda [4] in the sense that our scheme does not require computation of $C_{n+1}$ for each $n \geq 1$.

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