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VISCOSITY APPROXIMATION PROCESS FOR FIXED POINTS OF ASYMPTOTICALLY PSEUDOCONTRACTIVE SEMIGROUPS

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Abstract. Moudafi's viscosity approximation with a strongly continuous uniformly asymptotically regular and uniformly L-Lipschitz semigroup of asymptotically pseudocontractive mappings are considered. A strong convergence theorem of fixed points is established.

Keywords: Strongly pseudocontractive mapping; Asymptotically pseudocontractive semigroup; Common fixed point; Variational inequality.

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1. Introduction

Let E be a real Banach space with norm $\|.\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \forall x \in C,$$

where E^* denotes the dual space of E and $\langle ., . \rangle$ denotes the generalized duality pairing between E and E^*. In the sequel, we denote a single valued normalized duality mapping by j.

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Let C be a nonempty closed convex subset of a Banach space E and T a nonlinear mapping.

Definition 1.1. $T : C \to C$ is said to be pseudocontractive if there exists some $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \forall x,y \in C$. T is said to be strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ such that $\langle Tx - Ty, j(x-y)\rangle \le \alpha\|x-y\|^2, \forall x,y \in C$, for some $j(x-y) \in J(x-y)$. T is said to be Lipschitz if there exists a constant $L > 0$ such that $\|Tx - Ty\| \le L\|x-y\|, \forall x,y \in C$. The class of asymptotically pseudocontractive mapping was introduced by Schu [1]. T is said to be asymptotically pseudocontractive if there exists some $j(x-y) \in J(x-y)$ and $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n\|x-y\|^2, \forall x,y \in C.$$

T is called uniformly asymptotically regular if $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$ for all $x \in C$. T is called uniformly L-Lipschitz if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x-y\|, \forall x,y \in C$. From now on, we use $F(T)$ to denote the fixed point set of T.

Definition 1.2. $\Gamma = \{T(t) : t \ge 0\}$ is said to be a strongly continuous semigroup of asymptotically pseudocontractive mappings from C to C if the following conditions are satisfied:

(a) $T(0)x = x$ for all $x \in C$;

(b) $T(s+t) = T(s)T(t)$ for all $s,t \ge 0$;

(c) there exists some $j(x-y) \in J(x-y)$ and $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle (T(t_n))^n x - (T(t_n))^n y, j(x-y)\rangle \le k_n\|x-y\|^2, \forall t_n > 0, x,y \in C.$$

(d) for each $x \in K$, the mapping $T(\cdot)x$ is continuous. Γ is said to be uniformly asymptotically regular if $\|(T(t))^{n+1}x - (T(t))^n x\| \to 0$ as $n \to \infty$. $\forall t > 0, x \in C$ and it is called uniformly L-Lipschitz if there exists a constant $L > 0$ such that $\|(T(t_n))^n x - (T(t_n))^n y\| \le L\|x-y\|, \forall t_n > 0, x,y \in C$. In this paper, we use F to denote the set of common fixed points of Γ; that is,

$$F = \{x \in C : T(t)x = x, t > 0\} = \bigcap_{t>0} F(T(t)).$$

Moudafi's viscosity approximation methods have been recently studied; see [2] and [3] and the references therein. However, the involved mapping f is usually considered as a contraction. Cho and Kang [4] consider a pseudocontraction semigroup based on Moudafi's viscosity approximation with continuous strong pseudocontractions in the framework of Banach spaces.

The purpose of the article is to consider asymptotically pseudocontraction semigroup based on Moudafi's viscosity approximation in the framework of Banach spaces. The results presented in this paper mainly improved and extended the corresponding results.

Let l^∞ be the Banach space of all bounded real-valued sequences. A Banach limit LIM is a linear continuous functional on l^∞ such that

$$\|\mathrm{LIM}\| = \mathrm{LIM}(1) = 1, \mathrm{LIM}(t_1, t_2, \cdots) = \mathrm{LIM}(t_2, t_3, \cdots)$$

for each $t = (t_1, t_2, \cdots) \in l^\infty$. If LIM is a Banach limit, then

$$\liminf_{n\to\infty} t_n \leq \mathrm{LIM}(t) \leq \limsup_{n\to\infty} t_n, \forall t = (t_1, t_2, \cdots) \in l^\infty.$$

Lemma 1.1. [5] *Let E be a Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous and strong pseudocontraction. Then T has a unique fixed point in C.*

Lemma 1.2. *In a Banach space, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

where $j(x-y) \in J(x-y)$.

Lemma 1.3. [6] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a strictly increasing, continuous, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r(0)$ and $\lambda \in [0,1]$.

2. Main results

Theorem 2.1. *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E. Let $\Gamma = \{T(t) : t \geq 0\}$ be a strongly continuous uniformly asymptotically regular and uniformly L-Lipschitz semigroup of asymptotically pseudocontractive mappings from C to C with $\{k_n\} \subset [1,\infty)$ and $F \neq \emptyset$. Let $f : C \to C$ be a fixed bounded, continuous and strongly pseudocontraction with the coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be the sequences of real number satisfying $\alpha_n \in (0,1)$, $t_n > 0$ and*

$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = \lim_{n\to\infty}(k_n-1)/\alpha_n = 0$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)(T(t_n))^n x_n,\ n \geq 1. \tag{2.1}$$

Assume that $\mathrm{LIM}\|T(t)x_n - T(t)x^*\| \leq \mathrm{LIM}\|x_n - x^*\|,\ \forall x^* \in K, t > 0$ *where* $K = \{x^* \in C : \Phi(x^*) = \min_{x\in C}\Phi(x)\}$ *with* $\Phi(x) = \mathrm{LIM}\|x_n - x\|^2, \forall x \in C$. *Then* $\{x_n\}$ *converges strongly to* $x^* \in F$ *which solves the following variational inequality:*

$$\langle (I-f)x^*, j(x^*-x)\rangle \leq 0, \forall x \in F. \tag{2.2}$$

Proof. Clearly, for each $n \in N$, define a mapping T_n as follows

$$T_n x = \alpha_n f(x) + (1-\alpha_n)(T(t_n))^n x, \forall x \in C.$$

We observe that

$$\begin{aligned}
&\langle T_n x - T_n y, j(x-y)\rangle \\
&= \alpha_n \langle f(x) - f(y), j(x-y)\rangle + (1-\alpha_n)\langle (T(t_n))^n x - (T(t_n))^n y, j(x-y)\rangle \\
&\leq \alpha_n \alpha \|x-y\|^2 + (1-\alpha_n)k_n\|x-y\|^2 \\
&= (k_n - \alpha_n k_n + \alpha\alpha_n)\|x-y\|^2, \forall x, y \in C.
\end{aligned}$$

The condition $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$ imply $\frac{k_n-1}{\alpha_n} < 1-\alpha \leq k_n - \alpha$, for sufficient large $n \geq 0$, *i.e.*, $0 < k_n - \alpha_n k_n + \alpha\alpha_n < 1$. This shows that T_n is a continuous and strong pseudocontractive. By Lemma 1.1, we have that T_n has a unique fixed point, denoted as x_n. That is (2.1) is well defined.

Next, we show that $\{x_n\}$ is bounded. Indeed, for any $p \in F$, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \alpha_n\langle f(x_n) - f(p), j(x_n - p)\rangle + \alpha_n\langle f(p) - p, j(x_n - p)\rangle \\
&\quad + (1-\alpha_n)\langle (T(t_n))^n x_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n\alpha\|x_n - p\|^2 + \alpha_n\langle f(p) - p, j(x_n - p)\rangle + (1-\alpha_n)k_n\|x_n - p\|^2 \\
&= (k_n - \alpha_n k_n + \alpha\alpha_n)\|x_n - p\|^2 + \alpha_n\langle f(p) - p, j(x_n - p)\rangle,
\end{aligned}$$

we have

$$\|x_n - p\|^2 \le \frac{\alpha_n\langle f(p)-p, j(x_n-p)\rangle}{(k_n-\alpha)\alpha_n-(k_n-1)} \le \frac{1}{1-\alpha}\langle f(p)-p, j(x_n-p)\rangle. \tag{2.3}$$

That is, $\|x_n - p\| \le \frac{1}{1-\alpha}\|f(p)-p\|$. This implies that $\{x_n\}$ is bounded. From the boundedness of f, we have that $\{f(x_n)\}$ is also bounded. If we fix $t>0$, then

$$\begin{aligned}
&\|x_n-(T(t))^n x_n\| \\
&\le \sum_{k=0}^{[t/t_n]-1}\|(T((k+1)t_n))^n x_n-(T(kt_n))^n x_n\|+\|(T([t/t_n]t_n)^n x_n-(T(t))^n x_n\| \\
&\le [t/t_n]L\|(T(t_n))^n x_n-x_n\|+L\|(T(t-[t/t_n]t_n))^n x_n-x_n\| \\
&= [t/t_n]\alpha_n L\|(T(t_n))^n x_n-f(x_n)\|+L\|(T(t-[t/t_n]t_n))^n x_n-x_n\| \\
&\le Lt\alpha_n/t_n\|(T(t_n))^n x_n-f(x_n)\|+L\max\{\|(T(s))^n x_n-x_n\|: 0\le s\le t_n\}.
\end{aligned}$$

From $\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{\alpha_n}{t_n}=0$, we have

$$\lim_{n\to\infty}\|x_n-(T(t))^n x_n\|=0. \tag{2.4}$$

Thus

$$\begin{aligned}
&\|x_n-T(t)x_n\| \\
&\le \|x_n-(T(t))^n x_n\|+\|(T(t))^n x_n-(T(t))^{n+1}x_n\|+\|(T(t))^{n+1}x_n-(T(t))x_n\| \\
&\le (1+L)\|x_n-(T(t))^n x_n\|+\|(T(t))^n x_n-(T(t))^{n+1}x_n\|.
\end{aligned}$$

Therefore, from (2.4) and the uniform asymptotic regularity of $T(t)$, we get

$$\lim_{n\to\infty}\|x_n-T(t)x_n\|=0. \tag{2.5}$$

Since E is uniformly convex and Φ is continuous, convex and $\Phi(z)\to\infty$ as $\|z\|\to\infty$. Hence K is nonempty. It is also closed and convex. For any $t>0$, $x^*\in K$, we obtain from (2.5) and the assumption that

$$\Phi(T(t)x^*)=\mathrm{LIM}\|x_n-T(t)x^*\|^2\le \mathrm{LIM}\|x_n-x^*\|^2=\Phi(x^*).$$

That is, $T(t)K\subseteq K$. For any $x,y\in K$, it follows from Lemma 1.3,

$$\|x_n-\frac{x+y}{2}\|^2\le\frac{\|x_n-x\|^2}{2}+\frac{\|x_n-y\|^2}{2}-\frac{g(\|x-y\|)}{4}.$$

So if $x \neq y$, we have

$$r : \Phi(\frac{x+y}{2}) \leq \frac{\Phi(x)}{2} + \frac{\Phi(y)}{2} - \frac{g(\|x-y\|)}{4} < r.$$

This is a contradiction. Therefore, K has a unique element. So there exists a common fixed point x^* of the semigroup in K.

On the other hand, for any $t \in (0,1)$, by lemma 1.2, we have

$$\begin{aligned}\|x_n - x^* - t(f(x^*) - x^*)\|^2 &\leq \|x_n - x^*\|^2 - 2t\langle f(x^*) - x^*, j(x_n - x^* - t(f(x^*) - x^*))\rangle \\ &= \|x_n - x^*\|^2 - 2t\langle f(x^*) - x^*, j(x_n - x^*)\rangle \\ &\quad - 2t\langle f(x^*) - x^*, j(x_n - x^* - t(f(x^*) - x^*)) - j(x_n - x^*)\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\langle f(x^*) - x^*, j(x_n - x^*)\rangle &\leq \frac{1}{2t}(\|x_n - x^*\|^2 - \|x_n - x^* - t(f(x^*) - x^*)\|^2) \\ &\quad - \langle f(x^*) - x^*, j(x_n - x^* - t(f(x^*) - x^*)) - j(x_n - x^*)\rangle.\end{aligned}$$

Since E has a uniformly Gâteaux differential norm, we know that j is norm to weak* uniformly continuous any bounded subset of E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq \frac{1}{2t}(\|x_n - x^*\|^2 - \|x_n - x^* - t(f(x^*) - x^*)\|^2) + \varepsilon, \forall t \in (0, \delta).$$

Taking Banach limit LIM on the above inequality, we arrive at

$$\mathrm{LIM}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq \varepsilon.$$

Since ε is arbitrary, this implies that

$$\mathrm{LIM}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq 0. \tag{2.6}$$

In the inequality (2.3), replacing x^* with p, we have

$$\mathrm{LIM}\|x_n - x^*\|^2 \leq \mathrm{LIM}\frac{1}{1-\alpha}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \leq 0,$$

which leads to

$$\mathrm{LIM}\|x_n - x^*\|^2 = 0.$$

Hence there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^*$. Form the asymptotic pseudocontractivity of $T(t)$, for any $p \in F$, we have the following estimates:

$$\begin{aligned}
&\langle x_n - (T(t))^n x_n, j(x_n - p\rangle \\
&= \langle x_n - p, j(x_n - p)\rangle + \langle p - (T(t))^n x_n, j(x_n - p)\rangle \\
&\geq \|x_n - p\|^2 - k_n\|x_n - p\|^2 = -(k_n - 1)\|x_n - p\|^2.
\end{aligned}$$

Moreover, from Eq (2.1) we have that

$$\begin{aligned}
\langle x_n - f(x_n), j(x_n - p)\rangle &= \frac{1-\alpha_n}{\alpha_n}\langle (T(t))^n x_n - x_n, j(x_n - p)\rangle \\
&\leq \frac{(1-\alpha_n)(k_n - 1)}{\alpha_n}\|x_n - p\|^2.
\end{aligned}$$

In the inequality replacing n_j with n, taking limit, we have that

$$\langle x^* - f(x^*), j(x^* - p)\rangle \leq 0, \forall p \in F. \tag{2.7}$$

If there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l\to\infty} x_{n_l} = y^*$. From (2.5), then $y^* \in F$. It follows from (2.7) that

$$\langle x^* - f(x^*), j(x^* - y^*)\rangle \leq 0. \tag{2.8}$$

By the same way, we have

$$\langle y* - f(y^*), j(y^* - x^*)\rangle \leq 0. \tag{2.9}$$

Inequalities (2.7) and (2.8) yield that

$$\|y^* - x^*\|^2 \leq \langle f(x^*) - f(y^*), j(x^* - y^*)\rangle \leq \alpha\|y^* - x^*\|^2.$$

Since $\alpha \in (0,1)$, we have $x^* = y^*$ and $\{x_n\}$ converges to $x^* \in F$, which is the unique solution to the variational inequality (2.2). This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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