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FIXED POINT THEOREMS FOR SET-VALUED QUASI-CONTRACTION MAPS IN A MENGER SPACE

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Abstract. In this paper, the concept of set-valued quasi-contraction mappings in Menger spaces is introduced and the proof of a set-valued fixed point theorem in this space is presented.

Keywords: Fixed point theorem; Quasi-contraction maps; Set-valued maps; Menger Space.

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1. Introduction

Probabilistic metric spaces originated in 1942 by Karl Menger [1]. The notion of Menger was to replace non-negative real numbers with distribution functions as the values of the metric. An analogy between probabilistic metric spaces and metric spaces can be drawn in a situation when we only know probabilities of possible values of distances between two points and we are not sure of the exact distance. Sehgal and Bharucha-Reid [2] introduced preliminary concepts and definitions on the theory of probabilistic metric spaces and proved several fixed point theorems in that space.

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2. Preliminaries

Definition 1.1. [3,4] A Triangular norm * also known as t-norm is a binary operation on the unit interval [0,1] such that for all $a,b,c,d \in [0,1]$, the following are satisfied

(i) $a*1 = a$
(ii) $a*b = b*a$
(iii) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$
(iv) $a*(b*c) = (a*b)*c$.

Three popular examples of t-norms are
$a*b = ab$, $a*b = min\{a,b\}$ and $a*b = max\{a+b-1,0\}$.

Definition 1.2.[3, 4] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing, left continuous, $inf?\{F(x) : x \in \mathbb{R}\} = 0$ and $sup?\{F(x) : x \in \mathbb{R}\} = 1$.
Informally the distribution function or cumulative frequency function, describes the probability that a variate X takes on a value less than or equal to $x \in X$.

Definition 1.3. A statistical measure that defines a probability distribution for a random variable, denoted by $f(x)$ is defined as the probability density function. The probability density function when graphically portrayed, indicates by the area under the graph the interval under which the variable falls.

Example 1.1 Flip an unbiased coin two times. Let H denote the outcome that a head is obtained, T denote the outcome that a tail is obtained and the variate X be number of heads.
Clearly $X = \{0,1,2\}$.
Now $F(0) = P(X \leq 0) = P(X = 0) = P(T,T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
$F(1) = P(X \leq 1) = P(H,T) + P(T,H) + P(T,T) = (\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = \frac{3}{4}$
$F(2) = P(X \leq 2) = P(T,T) + P(T,H) + P(H,T) + P(H,H) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$

Therefore,

$$F(x) = \begin{cases} \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \\ 0, & x < 0 \end{cases}$$

Example 1.2 Given

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

we can find the distribution function by considering the three regions of $f(x)$, namely $x < 0$, $0 \leq x \leq 1$ and $x > 1$ made obvious by a sketch of the graph of f(x) in Figure 1.1 below

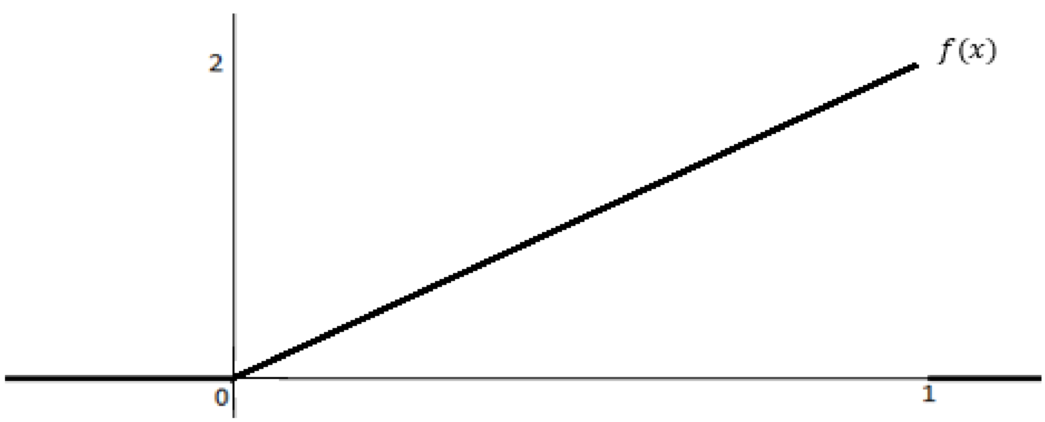

Figure 1.1: Diagram showing sketch of the function $f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$

For $x < 0$, $F(x) = P(X \leq x) = \int_{-\infty}^{x} 0dt = 0$.
For $0 \leq x \leq 1$, $F(x) = P(X \leq x) = \int_{-\infty}^{0} 0dt + \int_{0}^{x} 2tdt = x^2$.
For $x > 1$, $F(x) = P(X \leq x) = \int_{-\infty}^{0} 0dt + \int_{0}^{1} 2tdt + \int_{1}^{x} 0dt = 1$.
Therefore,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Definition 1.3 Let S be the set of all distribution functions on $[-\infty,\infty]$. The specific distribution function $L:\mathbb{R}\to\mathbb{R}^{+}$ is defined by

$$L(t)=\begin{cases}0, & t\leq 0\\ 1, & t>0\end{cases}$$

Definition 1.4 Let X be a non-empty set and S the set of all distribution functions on $[-\infty,\infty]$. Then $F:X\times X\to S$ is called a probabilistic distance on X and for all $x,y\in X$, we write F_{xy} to denote $F(x,y)$.

Definition 1.5 Let X be a non-empty set and F be a probabilistic distance satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$:

(i) $F_{xy}(t)=1$ if and only if $x=y$
(ii) $F_{xy}(0)=0$
(iii) $F_{xy}(t)=F_{yx}(t)$
(iv) If $F_{xy}(t)=1$, $F_{yz}(s)=1$ then $F_{xz}(t+s)=1$.

Then the pair (X,F) is called a probabilistic metric space or a PM-space.
Informally based on Definition 1.5 we can infer that $F_{xy}(t)$ is the probability that the distance between x and y is less than t.

Definition 1.6 The triple $(X,F,*)$ is called a Menger space if (X,F) is a probabilistic metric space, $*$ is a t-norm and for all $x,y,z\in X$ with $t,s>0$, $F_{xy}(t+s)\geq F_{xz}(t)*F_{zy}(s)$.

Definition 1.7 Let $(X,F,*)$ be a Menger space and * be a continuous t-norm.

(i) A sequence x_n in X is said to be convergent to $x\in X$ if and only if for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exists an integer $n_0=n_0(\varepsilon,\lambda)$ such that $F_{x_nx}(\varepsilon)>1-\lambda$ for $n\geq n_0$ and we write $x_n\to x$ as $n\to\infty$ or $\lim_{n\to\infty}?x_n=x$.
(ii) A sequence x_n in X is said to be a Cauchy sequence if for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exist an integer $n_0=n_0(\varepsilon,\lambda)$ such that $F_{x_nx_{n+p}}(\varepsilon)>1-\lambda$ for $n\geq n_0$ and $p>0$.
(iii) A Menger space is said to be complete if every Cauchy sequence is convergent.

The following theorem was proven by Singh and Jain [5].

Theorem 1.1 *Let $(X,F,*)$ be a Menger Space. If there exist $k \in (0,1)$ such that for all $x,y \in X$ and $t > 0$, $F_{xy}(kt) \geq F_{xy}(t)$, then $x = y$.*

3. Main results

Before getting to the main result we must first define the following. Definitions 2.1 and 2.2 given below are analogous to the definitions defined by Sookoo and Gunakala [6, 7].

Definition 2.1 Let $(X,F,*)$ be a Menger space. Then $CB_F(X)$ denotes the set of all non-empty closed and bounded subsets of X in the Menger space $(X,F,*)$.

Definition 2.2 Let $A,B \in CB_F(X)$. Then the Hausdorff Menger distance between the two sets A and B is denoted by $M_{AB}(\alpha)$, where $\alpha \in \mathbb{R}$, and is defined as

$$M_{AB}(\alpha) = max\{\delta(A,B), \delta(B,A)\}$$

where $\delta(A,B) = sup\{F_{aB}(\alpha) : a \in A\}$, $\delta(B,A) = sup\{F_{Ab}(\alpha) : b \in B\}$,
with

$F_{aB}(\alpha) = inf\{F_{ab}(\alpha) : b \in B\}$ and $F_{Ab}(\alpha) = inf\{F_{ab}(\alpha) : a \in A\}$.

Informally $M_{AB}(\alpha)$ is the greatest probability that the distance one must traverse to get from a point on one set to the closest point in the other set is less than α.

Definition 2.3 Let $(X,F,*)$ be a Menger space. The set valued mapping $T : X \to CB_F(X)$ is said to be a Menger q-set-valued quasi-contraction if for any $x,y \in X$, $\alpha > 0$ and $1 < q < 2$,

$$M_{TxTy}(\alpha) \leq (q-1)max\{F_{xy}(\alpha), F_{xTx}(\alpha), F_{yTy}(\alpha), F_{xTy}(\alpha), F_{yTx}(\alpha)\}$$

We will now proceed to give our main result.

Theorem 3.1 *Let $(X,F,*)$ be a complete Menger space. If the set-valued mapping $T : X \to CB_F(X)$ is a Menger q-set-valued quasi-contraction mapping where $1 < q < 2$, then T has a fixed point in X. That is, there exist $u \in X$ such that $u \in Tu$.*

Proof: T is given to be a Menger q-set-valued quasi-contraction mapping.

Therefore, for all $x, y \in X$ we have,

$$M_{TxTy}(\alpha) \leq (q-1)max\{F_{xy}(\alpha), F_{xTx}(\alpha), F_{yTy}(\alpha), F_{xTy}(\alpha), F_{yTx}(\alpha)\}.$$

Now let $c > 0$ be such that $q < c < 2$. This implies,

$$M_{TxTy}(\alpha) < (c-1)max\{F_{xy}(\alpha), F_{xTx}(\alpha), F_{yTy}(\alpha), F_{xTy}(\alpha), F_{yTx}(\alpha)\}.$$

Now we note that for each $A, B \in CB_F(X)$, with $a \in A$ there exist $b \in B$ such that $F_{ab}(\alpha) \leq M_{AB}(\alpha)$.

Let $x_1 \in Tx_0$ and $x_2 \in Tx_1$. Therefore we have,

$$F_{x_1x_2}(\alpha) \leq M_{Tx_0Tx_1}(\alpha) < (c-1)max\{F_{x_0x_1}(\alpha), F_{x_0Tx_0}(\alpha), F_{x_1Tx_1}(\alpha), F_{x_0Tx_1}(\alpha), F_{x_1Tx_0}(\alpha)\}$$

Similarly, it follows by induction that we have $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$. This implies,

$$F_{x_{n-1}x_n}(\alpha) \leq M_{Tx_{n-2}Tx_{n-1}}(\alpha) < (c-1)max\left\{\begin{array}{c}(F_{x_{n-2}x_{n-1}}(\alpha), F_{x_{n-2}Tx_{n-2}}(\alpha), F_{x_{n-1}Tx_{n-1}}(\alpha) \\ F_{x_{n-2}Tx_{n-1}}(\alpha), F_{x_{n-1}Tx_{n-2}}(\alpha)\end{array}\right\}$$

Using the first condition of Definition 1.5 and taking $\alpha = (\frac{t}{p} + \frac{s}{p})$ where $t, s, p > 0$, we get,

$$F_{x_{n-1}x_n}\left(\frac{t}{p} + \frac{s}{p}\right) < (c-1)max\left\{\begin{array}{c}(F_{x_{n-2}x_{n-1}}(\frac{t}{p} + \frac{s}{p}), F_{x_{n-2}Tx_{n-2}}(\frac{t}{p} + \frac{s}{p}), F_{x_{n-1}Tx_{n-1}}(\frac{t}{p} + \frac{s}{p}) \\ F_{x_{n-2}Tx_{n-1}}(\frac{t}{p} + \frac{s}{p}), F_{x_{n-1}Tx_{n-2}}(\frac{t}{p} + \frac{s}{p})\end{array}\right\}$$

$$= (c-1)F_{x_{n-1}x_{n-1}}\left(\frac{t}{p} + \frac{s}{p}\right) = (c-1)$$

For $n = 1$, and using the fact that $c \leq \frac{c}{c-1}$ for $1 < q < c < 2$, we have,

$$F_{x_0x_1}\left(\frac{t}{p} + \frac{s}{p}\right) < (c-1)F_{x_0x_0}\left(\frac{t}{p} + \frac{s}{p}\right) = (c-1) \leq \frac{c}{c-1} - 1$$

Now by induction we will show that for $n > 1$,

$$F_{x_{n-1}x_n}\left(\frac{t}{p} + \frac{s}{p}\right) < \left(\frac{c}{c-1}\right)^{n-1} - 1 \tag{1}$$

Now, $F_{x_{n-1}x_n}(\frac{t}{p} + \frac{s}{p}) < (c-1)F_{x_{n-1}x_{n-1}}(\frac{t}{p} + \frac{s}{p})$
$< (c-1)^2 F_{x_{n-2}x_{n-2}}(\frac{t}{p} + \frac{s}{p})$
$< (c-1)^3 F_{x_{n-3}x_{n-3}}(\frac{t}{p} + \frac{s}{p}) < ... < (c-1)^{n-1} F_{x_0x_0}(\frac{t}{p} + \frac{s}{p})$

Since we know that $(c-1)^{n-1} < c^{n-1} - 1$ for $1 < q < c < 2$, we have

$$F_{x_{n-1}x_n}\left(\frac{t}{p}+\frac{s}{p}\right) < c^{n-1} - 1 \leq \left(\frac{c}{c-1}\right)^{n-1} - 1$$

Hence equation (1) holds.

We now proceed to show that $\{x_n\}$ is a Cauchy sequence.

From equation (1) with $p > 0$, it follows that

$-F_{x_{n-1}x_n}\left(\frac{t}{p}+\frac{s}{p}\right) > 1 - \left(\frac{c}{c-1}\right)^{n-1}$

Now by Definition 1.6,

$F_{x_n x_{n+p}}(t+s) > F_{x_n x_{n+1}}\left(\frac{t}{p}+\frac{s}{p}\right) * F_{x_{n+1}x_{n+2}}\left(\frac{t}{p}+\frac{s}{p}\right) * \ldots * F_{x_{n+p-1}x_{n+p}}\left(\frac{t}{p}+\frac{s}{p}\right)$

$\geq -F_{x_n x_{n+1}}\left(\frac{t}{p}+\frac{s}{p}\right) * -F_{x_{n+1}x_{n+2}}\left(\frac{t}{p}+\frac{s}{p}\right) * \ldots * -F_{x_{n+p-1}x_{n+p}}\left(\frac{t}{p}+\frac{s}{p}\right)$

$> \left(1-(\frac{c}{c-1})^n\right) * \left(1-(\frac{c}{c-1})^{n+1}\right) * \ldots * \left(1-(\frac{c}{c-1})^{n+p-1}\right)$

$= (1-\lambda_1) * (1-\lambda_2) * \ldots * (1-\lambda_p)$

$\geq 1-\lambda$

where $\lambda_i = (\frac{c}{c-1})^{n+i-1}$ for $i = 1,2,,p$ and $\lambda = min\{\lambda_i : i = 1,2,,p\}$.

Hence, $F_{x_n x_{n+p}}(t+s) > (1-\lambda)$. This implies $\{x_n\}$ is a Cauchy sequence.

Now $(X,F,*)$ is complete. This implies that there exist $u \in X$ such that $\lim_{n\to\infty} ?x_n = u$.

Now for $t > 0$, we can conclude

$F_{uTu}(t) \leq 1 = F_{uu}(kt)$ with $k \in (0,1)$

$= \lim_{n\to\infty} F_{x_n x_{n+1}}(kt) \leq \lim_{n\to\infty} F_{x_n T x_n}(kt) = F_{uTu}(kt)$

Therefore by Theorem 1.1 we have that $u = Tu$. Hence u is a fixed point in X.

□

Conflict of Interests

The authors declare that there is no conflict of interests.

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