Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 4, 387-395
ISSN: 1927-6303

A CONVERGENCE THEOREM OF THE PICARD ITERATION WHOSE MAPPING HAS MULTIPLE FIXED POINTS

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Abstract. In this paper, we give a sufficient condition for a self-mapping T on X which has multiple fixed points satisfying the Picard iteration $\{T^n x\}$ converges to a fixed point of T for every starting point in a subset of X.

Keywords: Fixed point theorem; Banach contraction principle; Picard and weakly Picard operator.

2010 AMS Subject Classification: 47H10, 37C25.

1. Introduction

In a complete metric space (X,d), fixed point theorems were categorized by Suzuki [8] as follows: let T be a self-mapping on X,

(T1) Leader-type : T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$.

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Received January 27, 2015

(T2) Unnamed-type : T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point.

(T3) Subrahmanyam-type : T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

(T4) Caristi-type : T may have more one than fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point of T.

It is clear that (T1) implies (T3) and (T2) implies (T4). In particular, (T1) and (T3) are based on the following Picard iteration: for a fixed $x \in X$,

$$x_0 = x,\ x_n = Tx_{n-1}\ (n = 1, 2, \dots).$$

In this paper, we study convergence theorems concerned with the Picard iteration.

Many fixed point theorems of (T1) were studied, for example, see [1,2,3,4,6,9]. The most famous fixed point theorem of (T1) is the following Banach contraction principle:

Theorem 1.1 (Banach, [1]). *Let (X,d) be a complete metric space, and let T be a self-mapping on X. If T is a contraction, that is,*

$$\textit{there exists } r \in [0,1) \textit{ such that for all } x,y \in X,\ d(Tx,Ty) \le rd(x,y).$$

Then T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$.

Also the following theorem of (T1) is a well-known generalization of Theorem 1.1:

Theorem 1.2 (Meir and Keeler, [2]). *Let (X,d) be a complete metric space, and let T be a self-mapping on X. If T is a weakly uniformly strict contraction, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in X$,*

$$\varepsilon \le d(x,y) < \varepsilon + \delta \textit{ implies } d(Tx,Ty) < \varepsilon.$$

Then T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$.

Finally, a necessary and sufficient condition for (T1) were given in 2008 as follows:

Theorem 1.3 (Suzuki, [8]). *Let T be a mapping on a complete metric space (X,d). Then (T1) holds if and only if T satisfies the following two conditions:*

(1) For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbb{N}$ such that

$$d(T^i x, T^j y) < \varepsilon + \delta \textit{ implies } d(T^{i+v}x, T^{j+v}y) < \varepsilon$$

for all $i, j \in \mathbb{N} \cup \{0\}$.

(2) For $x, y \in X$, there exist $v \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j y) < \alpha_n \textit{ implies } d(T^{i+v}x, T^{j+v}y) < \frac{1}{n}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

On the other hand, there are a few fixed point theorems of (T3), see [5, 7, 8]. The following is the most famous fixed point theorem in (T3) :

Theorem 1.4 (Subrahmanyam, [5]). *Let (X, d) be a complete metric space, and let T be a self-mapping on X. Assume that there exists $r \in [0, 1)$ such that for all $x \in X$,*

$$d(T^2 x, Tx) \leq r d(Tx, x).$$

Then T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

Similar to (T1), a necessary and sufficient condition for (T3) were given as follows:

Theorem 1.5 (Suzuki, [9]). *Let T be a mapping on a complete metric space (X, d). Then (T3) holds if and only if T satisfies the following two conditions:*

(1) For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbb{N}$ such that

$$d(T^i x, T^j x) < \varepsilon + \delta \textit{ implies } d(T^{i+v}x, T^{j+v}x) < \varepsilon$$

for all $i, j \in \mathbb{N} \cup \{0\}$.

(2) For $x, y \in X$, there exist $v \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that

$$d(T^i x, T^j y) < \alpha_n \textit{ implies } d(T^{i+v}x, T^{j+v}y) < \frac{1}{n}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

In the above theorems, sufficient conditions for a self-mapping T on X satisfying the Picard iteration $\{T^n x\}$ converges to a fixed point of T for every starting point x in X, are given. In this paper, we give a sufficient condition for a self-mapping T on X which has multiple fixed points satisfying the Picard iteration $\{T^n x\}$ converges to a fixed point of T for every starting point x in a given subset of X.

2. Main results

Now we give a main result with respect to a sufficient condition for a self-mapping T on X which has multiple fixed points satisfying the Picard iteration $\{T^n x\}$ converges to a fixed point of T for every starting point x in a given subset of X.

Theorem 2.1. *Let (X,d) be a complete metric space, let T be a self-mapping on X, and let B be a subset of X satisfies $T(B) \subseteq B$. Assume that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\text{for all } x,y \in B,\ \varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \varepsilon. \tag{2.1}$$

Then $\{T^n x\}$ converges to a fixed point of T for all $x \in B$.

Proof At first, we prove that

$$\text{for all } x,y \in B \text{ with } x \neq y,\ d(Tx,Ty) < d(x,y).$$

If not, there exist $x_0, y_0 \in B$ with $x_0 \neq y_0$ such that $d(Tx_0,Ty_0) \geq d(x_0,y_0)$. Put $\varepsilon_0 = d(x_0,y_0) > 0$, then there exists $\delta_0 > 0$ such that (2.1) holds by the assumption. From $\varepsilon_0 = d(x_0,y_0) < \varepsilon_0 + \delta_0$, we have $d(Tx_0,Ty_0) < \varepsilon_0 = d(x_0,y_0)$. This is a contradiction. Next, for any given $x \in B$, define a sequence $\{x_n\}$ as

$$x_0 = x,\ x_n = Tx_{n-1}\ (n = 1,\ 2, \ldots).$$

If $x_n = x_{n-1}$ holds, x_{n-1} is the fixed point. Then we may assume that $x_n \neq x_{n-1}$ for all n. Put $c_n = d(x_n,x_{n-1})$ for all $n \in \mathbb{N}$. Since $c_n \geq 0$ and

$$c_{n+1} = d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) < d(x_n,x_{n-1}) = c_n,$$

$\{c_n\}$ is a lower bounded and decreasing sequence. Then there exists $c \in [0,\infty)$ such that $c_n \to c$. We show $c = 0$. If $c > 0$, by putting $\varepsilon_1 = c$, there exists $\delta_1 > 0$ such that (2.1) holds. From $c \leq c_n$ for all $n \in \mathbb{N}$ and $c_n \to c$, we have $c \leq c_n < c + \delta_1$ for sufficiently large n. Since $\varepsilon_1 \leq d(x_n,x_{n-1}) < \varepsilon_1 + \delta_1$, then $c_{n+1} = d(Tx_n,Tx_{n-1}) < \varepsilon_1 = c$ and this is a contradiction. Therefore

$c = 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. If not, there exists $\varepsilon_2 > 0$ such that for all $N \in \mathbb{N}$, there exist $l, m \geq N$ such that $d(x_l, x_m) > 2\varepsilon_2$ and $l < m$. Also there exists $\delta_2 > 0$ such that (2.1) holds. Put $\delta' = \min\{\varepsilon_2, \delta_2\}$. We have $\varepsilon_2 \leq d(x,y) < \varepsilon_2 + \delta'$ implies $d(Tx, Ty) < \varepsilon_2$. Form $c_n \to 0$, there exists $M \in \mathbb{N}$ such that $c_n < \delta'/3$, for all $n \geq M$. Put $N = M$, then there exist $l, m \geq M$ such that $l < m$ and $d(x_l, x_m) > 2\varepsilon_2$. Also we have, for all $j \in \{l, l+1, \ldots, m\}$,

$$|d(x_l, x_j) - d(x_l, x_{j+1})| \leq d(x_j, x_{j+1}) = c_j < \frac{\delta'}{3}.$$

From this and

$$c_l = d(x_l, x_{l+1}) < \frac{\delta'}{3} < \varepsilon_2 + \frac{2}{3}\delta' < \varepsilon_2 + \delta' \leq 2\varepsilon_2 < d(x_l, x_m),$$

there exists $k \in \mathbb{N}$ such that $\varepsilon_2 + 2\delta'/3 < d(x_l, x_k) < \varepsilon_2 + \delta'$. Then we have $d(x_{l+1}, x_{k+1}) < \varepsilon_2$, and then

$$\begin{aligned} d(x_l, x_k) &\leq d(x_l, x_{l+1}) + d(x_{l+1}, x_{k+1}) + d(x_{k+1}, x_k) \\ &< c_l + \varepsilon_2 + c_k \\ &< \varepsilon_2 + \frac{2}{3}\delta'. \end{aligned}$$

This is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. Next, we prove that $T^n x \to \bar{x}$ for all $x \in B$. Assume that there exist $x_0, y_0 \in B$ such that $T^n x_0 \to \bar{x}$ and $T^n y_0 \to \bar{y}$ where $\bar{x} \neq \bar{y}$. Put $\varepsilon_3 = d(\bar{x}, \bar{y}) > 0$, then there exists $\delta_3 > 0$ such that (2.1) holds. From $\{d(T^n x_0, T^n y_0)\}$ is a lower bounded and decreasing sequence, we have $\varepsilon_3 \leq d(T^n x_0, T^n y_0)$ for all $n \in \mathbb{N}$. Since $T^n x \to \bar{x}$ and $T^n y \to \bar{y}$, we have $d(T^n x_0, \bar{x}) < \delta_3/2$ and $d(T^n y_0, \bar{y}) < \delta_3/2$ for sufficiently large n. Using (2.1), we have $d(T^{n+1} x_0, T^{n+1} y_0) < \varepsilon_3$. This is a contradiction. Finally, we prove that $\bar{x} \in F(T)$. Assume that $\bar{x} \notin F(T)$.

$$\begin{aligned} 0 < d(\bar{x}, T\bar{x}) &\leq d(\bar{x}, T^n x) + d(T^n x, T\bar{x}) \\ &< d(\bar{x}, T^n x) + d(T^{n-1} x, \bar{x}) \\ &\to 0. \end{aligned}$$

This is a contradiction.

In the above theorem, if B is closed then Theorem 2.1 is equivalent to Theorem 1.2, however B may not be closed. In the following example, we give a map T to which Theorem 2.1 can be applied but Theorems from 1.1 to 1.3 can not be applied.

Example 2.1. Let $(\mathbb{R}^n, d)$, and let T be defined as follows:

$$Tx = \begin{cases} \frac{1}{2}x & x \in (0,\infty)^n, \\ 2x & x \notin (0,\infty)^n. \end{cases}$$

Then we can apply Theorem 2.1 for open set $B = (0,\infty)^n$. Indeed, for all $\varepsilon > 0$, by putting $\delta = \varepsilon$, for all x, $y \in B$ satisfying $\varepsilon \leq d(x,y) < \varepsilon + \delta$,

$$\begin{aligned} d(Tx,Ty) = \|Tx - Ty\| &= \left\| \frac{1}{2}x - \frac{1}{2}y \right\| \\ &= \frac{1}{2}\|x-y\| \\ &= \frac{1}{2}d(x,y) < \frac{1}{2}(\varepsilon+\delta) = \varepsilon. \end{aligned}$$

So T holds the condition of Theorem 2.1. Therefore T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point of T for all $x \in B$. However, Theorems from 1.1 to 1.3 can not be applied because $\{T^n x\}$ does not converge when $x \in X \setminus (B \cup \{0\})$ and also B is not closed.

In the following example, we can see (T3) holds for a self-mapping T which holds the condition of Theorem 2.1:

Example 2.2. Let $(\mathbb{R}^2, d)$, and let T be defined as follows:

$$Tx = \frac{1}{2}(x + P_A(x)),$$

where $A = [-1,1]^2$, $P_A(x) = \{y \in A \mid d(x,y) \leq d(x,z) \text{ for all } z \in A\}$. Let $F(T)$ be the set of all fixed points of T, then we can see $F(T) = A$, that is, T has multiple fixed points. Since

$$T^n x = \frac{1}{2^n}x + \left(1 - \frac{1}{2^n}\right) P_A(x) \to P_A(x) \in A = F(T)$$

for all $x \in X$, (T3) holds for T. Let $B_{(1,1)} := \{x \in \mathbb{R}^2 \mid T^n x \to (1,1)\}$. Then we can check that the condition of Theorem 2.1 for $B = B_{(1,1)}$ holds. Indeed, for all $\varepsilon > 0$, by putting $\delta = \varepsilon$, for all $x,y \in B_{(1,1)}$, $P_A(x) = P_A(y) = (1,1)$. Therefore

$$\begin{aligned}
\varepsilon \le d(x,y) < \varepsilon+\delta \Rightarrow d(Tx,Ty) &= d\left(\frac{1}{2}(x+P_A(x)), \frac{1}{2}(y+P_A(y))\right) \\
&= \left\| \frac{1}{2}(x+P_A(x)) - \frac{1}{2}(y+P_A(x)) \right\| \\
&= \frac{1}{2}\|x-y\| \\
&= \frac{1}{2}d(x,y) \\
&< \frac{1}{2}(\varepsilon+\delta) = \varepsilon.
\end{aligned}$$

Also we have $T^n x \to (1,1)$ for all $x \in B_{(1,1)}$ and $B_{(1,1)} = [1,\infty)^2$. In a similar way, for each $z \in A$, let $B_z := \{x \in \mathbb{R}^2 \mid T^n x \to z\}$, then we have the condition of Theorem 2.1 for $B = B_z$ holds and $T^n x \to P_A(x) \in A = F(T)$ for all $x \in B_z$.

On the other hand, since (T3) holds, the conditions of Theorem 1.3 also hold. However it seems that it is hard to know the limit $\lim_{n\to\infty} T^n x$ by using Theorem 1.3.

Motivated by Example 2.2, we give a result of (T3) from Theorem 2.1 by putting a certain subset B of X. For $A \subset X$ and $n \in \mathbb{N}$, denote that $T^{-n}A := (T^n)^{-1}A$ and $T^0 A := A$.

Corollary 2.1. *Let (X,d) be a complete metric space, and let T be a self-mapping on X. Assume that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in X \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} T^{-n}(F(T))$,*

$$\varepsilon \le d(x,y) < \varepsilon+\delta \text{ implies } d(Tx,Ty) < \varepsilon.$$

Then $\{T^n x\}$ converges to a fixed point of T for all $x \in X$.

Proof Let

$$B = X \setminus \bigcup_{n\in\mathbb{N}\cup\{0\}} T^{-n}(F(T)).$$

We show $T(B) \subset B$. If there exists $y \in T(B)$ such that $y \notin B$, then there exists $x \in B$ such that $y = Tx$. Since $y = Tx \notin B$, $Tx \in \bigcup_{n\in\mathbb{N}\cup\{0\}} T^{-n}(F(T))$, and this shows $Tx \in T^{-n_0}(F(T))$ for some $n_0 \in \mathbb{N}\cup\{0\}$, that is,

$$x \in T^{-n_0-1}(F(T)) \subset \bigcup_{n\in\mathbb{N}\cup\{0\}} T^{-n}(F(T)).$$

This contradicts to $x \in B$. Using Theorem 2.1 $\{T^n x\}$ converges to a fixed point of T for all $x \in B$. On the other hand, when $x \notin B$, since

$$x \in \bigcup_{n \in \mathbb{N} \cup \{0\}} T^{-n}(F(T)),$$

there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $T^{n_0} x \in F(T)$, that is, $T^n x = T^{n_0} x$ hold for all $n \geq n_0$. This means $\{T^n x\}$ converges to a fixed point of T. This completes the proof.

In the end of the paper, we give an observation between our result and the previous ones. Define a binary relation $\sim$ on X by for every $x, y \in X$,

$$x \sim y \text{ if and only if } T^n x \to z \text{ and } T^n y \to z \text{ for some } z \in X \text{ or}$$
$$\text{both } \{T^n x\} \text{ and } \{T^n y\} \text{ do not converge.}$$

Then $\sim$ is an equivalence relation on X, that is, for all x, y and $z \in X$,

(1) $x \sim x$,
(2) if $x \sim y$ then $y \sim x$, and
(3) if $x \sim y$ and $y \sim z$ then $x \sim z$.

Let the equivalence class of x and the quotient set be

$$[x] = \{y \in X \mid x \sim y\} \text{ and } X/\!\sim = \{[x] \mid x \in X\},$$

respectively, and define a function $\varphi : (X/\!\sim) \setminus \{N(T)\} \to X$ by

$$\varphi(x) = \lim_{n \to \infty} T^n x,$$

where $N(T) = \{x \mid \{T^n x\} \text{ does not converge}\}$. By using the notations, fixed point theorems can be categorized as follows:

(1) $N(T) = \emptyset$, $|F(T)| = 1$, and $\varphi(X/\!\sim) \subset F(T)$,
(2) $N(T) = \emptyset$, $F(T) \neq \emptyset$, and $\varphi(X/\!\sim) \subset F(T)$, and
(3) $N(T) \cap B = \emptyset$, $F(T) \neq \emptyset$, and $\varphi(B/\!\sim) \subset F(T)$,

where $B/\!\sim = \{[x] \mid x \in B\}$. We can see that (1) is equivalent to (T1), (2) is equivalent to (T3), and (3) is equivalent to the result of Theorem 2.1. If $B = X$ then (3) coincide with (T3), and if $B = X = B_z$ then (3) coincide with (T1) where $z \in X$. As we have seen in Examples 2.1 and

2.2, including the situation $N(T) \neq \emptyset$, Theorem 2.1 is useful to observe the limits of the Picard iteration.

Acknowledgments

The authors are grateful to an anonymous referee for comments and suggestions improved the quality of the paper. The authors' work was partially supported by JSPS KAKENHI Grant Number 25400205.

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