A COMMON FIXED POINT THEOREM OF PRESIC TYPE FOR FOUR MAPS IN G-METRIC SPACES

U. C. GAIROLA¹, N. DHASMANA²,*

Department of Mathematics, H. N. B. Garhwal University, Campus Pauri,
Pauri Garhwal- 246001, Uttarakhand India

Copyright © 2015 Gairola and Dhasmana. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we extended the idea of Presic type contraction for G-metric space to obtain a unique common fixed point result for four maps. The result generalizes several well known comparable results in the literature.

Keywords: Presic type fixed point theorem; G-metric space; G-Coincidence point; Weakly commuting, Coincidently commuting.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

In 1922, Banach proved a fixed point theorem for contraction mapping in metric space. This result has been extended and generalized for various settings (see, for instance [10], [12] and the references therein). The study of fixed points of mappings satisfying certain contractive condition has been at the centre of vigorous research activity. Mustafa and Sims [19] introduced the new concept of G-metric space. Since then many authors have been studying fixed point

*Corresponding author

E-mail addresses: ucgairola@rediffmail.com (U. Gairola), nirmaladhasmana@rediffmail.com (N. Dhasmana)

Received May 16, 2015
results in G-metric spaces and subsequently many fixed point results on such spaces appeared
(see, for instance [1-6], [8], [18], [20-21], [27] and the references therein).

On the other hand, amongst the various generalization of Banach contraction principle, Presic
[22] in 1965 gave a contractive condition on finite product of metric spaces and proved a fixed
al. [23-24] extended and generalized these results. Also with a view to generalize the fixed
point theorem for commuting maps, Sessa [25] introduced the concepts of weakly commuting
maps. Later on, Singh and Gairola [26] extended the notion of weakly commuting maps to co-
ordinatewise commuting and weakly commuting maps for two system of maps on finite product
of metric spaces and proved some fixed point theorems. Gairola and Jangwan [15], Singh
Gairola and [16] and Baillon and Singh [7] conceptualize co-ordinatewise R-weakly commuting
mappings and compatible maps. George and Khan [17] used the concept of weakly commuting
and coincidently commuting maps for k-tuples and generalized Presic type fixed point theorem
for two maps and then later on Rao et al. [23] extended this work for three maps using the
concept of 2k-weakly compatible pair.

The aim of this paper is to prove a Presic type common fixed point theorem for four mappings
in complete G-metric space which extend and unify the results of Ciric-Presic [9], Dhasmana
[11], Gairola-Dhasmana [13] and Rao et al. [23].

2. Definitions and propositions

We begin by briefly recalling some basic definitions and results will be needed in the sequel.
Let \((X,d)\) be a metric space, \(k\) a positive integer, \(T : X^k \to X\) and \(f : X \to X\) be mappings. An
element \(x \in X\) is said be a coincidence point of \(f\) and \(T\) if \(fx = T(x,x,\ldots,x)\), \(x\) is a common
fixed point of \(f\) and \(T\) if \(x = fx = T(x,x,\ldots,x)\). The set of coincidence point of \(f\) and \(T\) is
denoted by \(C(f,T)\).

Definition 2.1. [17] (see also [26]) Mappings \(f\) and \(T\) are said to be commuting if \(f(T(x,x,\ldots,x)) =
T(fx,fx,\ldots,fx)\) for all \(x \in X\).

Definition 2.2. [17] (see also [26]) Mappings \(f\) and \(T\) are said to be weakly commuting if
\(d(f(T(x,x,\ldots,x)),T(fx,fx,\ldots,fx)) \leq d(f(x),T(x,x,\ldots,x))\) for all \(x \in X\).
Definition 2.3. [17] Mappings $f$ and $T$ are said to be coincidentally commuting if they commute at their coincidence points.

Remark 2.4. [17] (see also [26]) For $k = 1$, above definitions reduce to the usual definition of commuting and weakly commuting mappings in a metric space.

Remark 2.5. It is notable that the above Definitions 2.1, 2.2 and 2.3 are special cases of definition 1 and 2 of Singh-Gairola [26]. See also the remarks of Gairola et al. [15-16].

Definition 2.6. [19] Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying:

$$(G_1) G(x, y, z) = 0; \text{ if } x = y = z,$$

$$(G_2) 0 < G(x, x, y); \text{ for all } x, y \in X \text{ with } x \neq y$$

$$(G_3) G(x, x, y) \leq G(x, y, z); \text{ for all } x, y, z \in X \text{ with } z \neq y.$$

$$(G_4) G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots; \text{ (symmetry in all three variables) and}$$

$$(G_5) G(x, y, z) \leq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X \text{ (rectangle inequality).}$$

Then the function $G$ is called a generalized metric or more specifically a G-metric on $X$, and the pair $(X, G)$ is called a G-metric space.

Definition 2.7. [19] Let $(X, G)$ be a G-metric space and let $\{x_n\}$ be a sequence of points of $X$. We say that $\{x_n\}$ is G-convergent to $x$ if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$; that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We refer to $x$ as the limit of the sequence $\{x_n\}$ and write $x_n \xrightarrow{G} x$.

Proposition 2.8. [19] Let $(X, G)$ be a G-metric space. The following statements are equivalent.

1. $\{x_n\}$ is G-convergent to $x$.
2. $G(x_n, x) \to 0$, as $n \to \infty$.
3. $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

Definition 2.9. [19] Let $(X, G)$ be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$. 
Proposition 2.10. [19] In a G-metric space \((X, G)\), the following two statements are equivalent.

1) The sequence \(\{x_n\}\) is G-Cauchy.

2) For every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(G(x_n, x_m, x_m) < \varepsilon\) for all \(n, m \geq N\).

Definition 2.11. [19] A G-metric space \((X, G)\) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in \((X, G)\) is G-convergent in \((X, G)\).

Definition 2.12. [19] A G-metric space \((X, G)\) is called symmetric if \(G(x, y, y) = G(y, x, x)\) for all \(x, y \in X\).

Proposition 2.13. [19] Let \((X, G)\) be a G-metric space. Then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

Proposition 2.14. [19] Every G-metric space \((X, G)\) defines a metric space \((X, d_G)\) by

\[d_G(x, y) = G(x, y, y) + G(y, x, x)\] for all \(x, y \in X\).

Note that if \((X, G)\) is a symmetric G-metric space, then

\[d_G(x, y) = 2G(x, y, y)\] \(\forall x, y \in X\).

3. Main results

Now we state our main result.

Theorem 3.1. Let \((X, G)\) be a G-metric space, \(k\) a positive integer and \(S, T, R : X^k \rightarrow X, f : X \rightarrow X\) be mappings satisfying the following conditions

1) \(S(X^k) \cup T(X^k) \cup R(X^k) \subseteq f(X)\)

2) \(G(S(x_1, x_2, \ldots, x_{k-1}, x_k), T(x_2, x_3, \ldots, x_k, x_{k+1}), R(x_3, x_4, \ldots, x_{k+1}, x_{k+2})) \leq \lambda \max\{G(fx_i, fx_{i+1}, fx_{i+2}), 1 \leq i \leq k\}\)

for all \(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2} \in X\)

3) \(G(T(y_1, y_2, \ldots, y_{k-1}, y_k), R(y_2, y_3, \ldots, y_k, y_{k+1}), S(y_3, y_4, \ldots, y_{k+1}, y_{k+2})) \leq \lambda \max\{G(fy_i, fy_{i+1}, fy_{i+2}), 1 \leq i \leq k\}\)
for all $y_1, y_2, \ldots, y_k, y_{k+1}, y_{k+2}$ in $X$.

\[(4) \quad G(R(z_1, z_2, \ldots, z_{k-1}, z_k), S(z_2, z_3, \ldots, z_k, z_{k+1}), T(z_3, z_4, \ldots, z_{k+1}, z_{k+2})) \leq \lambda \max\{G(fz_i, fz_{i+1}, fz_{i+2}) : 1 \leq i \leq k\}\]

for all $z_1, z_2, \ldots, z_k, z_{k+1}, z_{k+2}$ in $X$, where $0 \leq \lambda < 1$.

\[(5) \quad d\left(S(u, u, \ldots u), T(v, v, \ldots v), R(w, w, \ldots w)\right) < G(fu, fv, fw),\]

for all $u, v, w \in X$ with $u \neq v \neq w$. Suppose that $f(X)$ is complete and one of $(f, S), (f, T) or (f, R)$ is coincidently commuting pair. Then there exist a unique point $p \in X$ such that $fp = p = S(p, p, \ldots, p) = T(p, p, \ldots, p) = R(p, p, \ldots, p)$.

**Proof.** Suppose $x_1, x_2, \ldots, x_k$ are arbitrary points in $X$ and for $n \in N$ and define

\[
\begin{align*}
fx_{k+3n-2} &= S(x_{3n-2}, x_{3n-1}, \ldots, x_{3n+k-3}), \\
fx_{k+3n-1} &= T(x_{3n-1}, x_{3n}, \ldots, x_{3n+k-2}), \\
fx_{k+3n} &= R(x_{3n}, x_{3n+1}, \ldots, x_{3n+k-1}).
\end{align*}
\]

Let

\[(6) \quad \alpha_n = G(fx_n, fx_{n+1}, fx_{n+2}).\]

Let $\theta = \lambda^{\frac{1}{k}}$ and $K = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Claim $\alpha_n \leq K\theta^n$ for all $n \in N$. By selection of $K$ we have $\alpha_n \leq K\theta^n$ for $n = 1, 2, \ldots, k$. Now,

\[
\alpha_{k+1} = G(fx_{k+1}, fx_{k+2}, fx_{k+3})
\]

\[
= G(S(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1}), R(x_3, x_4, \ldots, x_{k+2}))
\]

\[
\leq \lambda \max\{G(fx_i, fx_{i+1}, fx_{i+2}) : i = 1, 2, \ldots, k\} \text{ by } (2)
\]

\[
= \lambda \max\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k\}
\]

\[
\leq \lambda \max\{K\theta, K\theta^2, \ldots, K\theta^{k-1}, K\theta^k\}
\]

\[
= \lambda K\theta = \theta^k K\theta \text{ as } \theta = \lambda^{\frac{1}{k}}.
\]
Thus $\alpha_{k+1} \leq K\theta^{k+1}$. Similarly, we have

$$
\alpha_{k+2} = G(fx_{k+2}, fx_{k+3}, fx_{k+4})
$$

$$
= G(T(x_2, x_3, \ldots, x_{k+1}), R(x_3, x_4, \ldots, x_{k+2}), S(x_4, x_5, \ldots, x_{k+3}))
$$

$$
\leq \lambda \max \{G(fx_i, fx_{i+1}, fx_{i+2}) : i = 2, 3, \ldots, k+1\} \text{ by (3)}
$$

$$
= \lambda \max \{\alpha_2, \alpha_3, \ldots, \alpha_k, \alpha_{k+1}\}
$$

$$
\leq \lambda \max \{K\theta^2, K\theta^3, \ldots, K\theta^k, K\theta^{k+1}\}
$$

$$
= \lambda K\theta^2 = \theta^k K\theta^2 \text{ as } \theta = \lambda^{\frac{1}{k}} = K\theta^{k+2}.
$$

Thus $\alpha_{k+2} \leq K\theta^{k+2}$. Also,

$$
\alpha_{k+3} = G(fx_{k+3}, fx_{k+4}, fx_{k+5})
$$

$$
= G(R(x_3, x_4, \ldots, x_{k+2}), S(x_4, x_5, \ldots, x_{k+3}), T(x_5, x_6, \ldots, x_{k+4}))
$$

$$
\leq \lambda \max \{G(fx_i, fx_{i+1}, fx_{i+2}) : i = 3, 4, \ldots, k+2\} \text{ by (4)}
$$

$$
= \lambda \max \{\alpha_3, \alpha_4, \ldots, \alpha_{k+1}, \alpha_{k+2}\}
$$

$$
\leq \lambda \max \{K\theta^3, K\theta^4, \ldots, K\theta^{k+1}, K\theta^{k+2}\}
$$

$$
= \lambda K\theta^3 = \theta^k K\theta^3 \text{ as } \theta = \lambda^{\frac{1}{k}} = K\theta^{k+3}.
$$

Thus $\alpha_{k+3} \leq K\theta^{k+3}$. Hence the claim is true.

Now, by claim, for $l, n, p$ with $l > n > p$ and the rectangular inequality of G-metric space, we have

$$
G(fx_n, fx_p, fx_l) \leq G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_{n+2}, fx_{n+2}) + \ldots + G(fx_{l-1}, fx_l, fx_l)
$$

$$
\leq G(fx_n, fx_{n+1}, fx_{n+2}) + G(fx_{n+1}, fx_{n+2}, fx_{n+3}) + \ldots + G(fx_{l-2}, fx_{l-1}, fx_l)
$$

$$
= \alpha_n + \alpha_{n+1} + \ldots + \alpha_{l-2}
$$

$$
\leq K\theta^n + K\theta^{n+1} + \ldots + K\theta^{l-2}
$$

$$
\leq K[\theta^n + \theta^{n+1} + \ldots + \theta^{l-2} + \ldots]
$$

$$
= K\frac{\theta^n}{1-\theta} \to 0 \text{ as } n \to \infty.
$$
Hence \( \{f x_n\} \) is a G-Cauchy sequence. Since \( f(X) \) is a G-complete and there exists \( z \) in \( f(X) \) such that \( z = \lim f x_n \). There exist \( p \in X \) such that \( z = f p \). Then for any integer \( n \), using (2), (3) and (4) we have

\[
G(S(p, p, \ldots, p), f x_n, f x_{n+1})
\]

\[
= G(S(p, p, \ldots, p), S(x_n, x_{n-1}, \ldots, x_{n+3}), S(x_n, x_{n-1}, \ldots, x_{n+3}))
\]

\[
\leq G(S(p, p, \ldots, p), T(p, p, \ldots, x_{n-2}), T(p, p, \ldots, x_{n-2}))
\]

and

\[
+ G(T(p, p, \ldots, x_{n-2}), R(p, p, \ldots, x_{n-1}), R(p, p, \ldots, x_{n-1}))
\]

\[
+ G(R(p, p, \ldots, x_{n-1}), S(p, p, \ldots, x_n), S(p, p, \ldots, x_n))
\]

\[
+ G(S(p, p, \ldots, x_n), T(p, p, \ldots, x_{n+1}), T(p, p, \ldots, x_{n+1}))
\]

\[
+ \ldots
\]

\[
+ G(T(p, p, x_{n-2}, \ldots, x_{n+3}), R(p, x_{n-2}, \ldots, x_{n+3}), R(p, x_{n-2}, \ldots, x_{n+3}))
\]

\[
+ G(R(p, x_{n-2}, \ldots, x_{n+3}), S(x_{n-2}, \ldots, x_{n+3}), S(x_{n-2}, \ldots, x_{n+3}))
\]

\[
\leq G(S(p, p, \ldots, p), T(p, p, \ldots, x_{n-2}), R(p, p, \ldots, x_{n-1}))
\]

\[
+ G(T(p, p, \ldots, x_{n-2}), R(p, p, \ldots, x_{n-1}), S(p, p, \ldots, x_n))
\]

\[
+ G(R(p, p, \ldots, x_{n-1}), S(p, p, \ldots, x_n), T(p, p, \ldots, x_{n+1}))
\]

\[
+ G(S(p, p, \ldots, x_n), T(p, p, \ldots, x_{n+1}), R(p, p, \ldots, x_{n+2}))
\]

\[
+ \ldots
\]

\[
+ G(S(p, p, \ldots, x_{n+3}), T(p, p, \ldots, x_{n+3}), R(p, p, \ldots, x_{n+3}))
\]

\[
+ G(T(p, p, x_{n+2}, \ldots, x_{n+5}), R(p, x_{n+2}, \ldots, x_{n+5}), S(x_{n+2}, \ldots, x_{n+5}))
\]

\[
\leq \lambda G(f p, f x_{n-2}, f x_{n-1})
\]

\[
+ \lambda \max\{G(f p, f x_{n-2}, f x_{n-1}), G(f x_{n-2}, f x_{n-1}, f x_{n+1})\}
\]

\[
+ \lambda \max\{G(f p, f x_{n-2}, f x_{n-1}), G(f x_{n-2}, f x_{n-1}, f x_{n+1}), G(f x_{n-1}, f x_{n+1}), f x_{n+1})\}
\]
\[ + \lambda \max \{ G fp, f x_{3n-2}, f x_{3n-1}, f x_{3n}, f x_{3n+1}, f x_{3n+2} \} \]
\[ + \ldots \]
\[ + \lambda \max \{ G fp, f x_{3n-2}, f x_{3n-1}, \ldots, G f x_{k+3n-6}, f x_{k+3n-5}, f x_{k+3n-4} \} \]
\[ + \lambda \max \{ G fp, f x_{3n-2}, f x_{3n-1}, \ldots, G f x_{k+3n-6}, f x_{k+3n-5}, f x_{k+3n-4} \} \].

Taking limit as \( n \to \infty \), we get \( G (S p, p, p, \ldots, p, f p, f p) \leq 0 \) so that
\[ (7) \quad S(p, p, \ldots, p) = f p. \]

Consider,
\[ G (fp, T(p, p, \ldots, p), T(p, p, \ldots, p)) = G (S(p, p, \ldots, p), T(p, p, \ldots, p), T(p, p, \ldots, p)) \leq \lambda(0) = 0. \]

Thus
\[ (8) \quad T(p, p, \ldots, p) = f p. \]

Also
\[ G (fp, R(p, p, \ldots, p), R(p, p, \ldots, p)) = G (T(p, p, \ldots, p), R(p, p, \ldots, p), R(p, p, \ldots, p)) \leq \lambda(0) = 0. \]

Thus
\[ (9) \quad R(p, p, \ldots, p) = f p. \]

Now suppose that \( (f, S) \) is a coincidentally commuting pair. Then we have
\[ f (S(p, p, \ldots, p)) = S(fp, fp, \ldots, fp), \]
\[ f^2 p = f (fp) = f (S(p, p, \ldots, p)) = S(fp, fp, \ldots, fp). \]

Suppose \( fp \neq p \)
\[ G (f^2 p, fp, fp) = G (S(fp, fp, \ldots, fp), T(p, p, \ldots, p), R(p, p, \ldots, p)) < d(f^2 p, fp, fp). \]

It is a contradiction. Therefore \( fp = p \). Now from (7), (8) and (9) we have
\[ fp = p = S(p, p, \ldots, p) = T(p, p, \ldots, p) = R(p, p, \ldots, p). \]
Uniqueness of \( p \): Suppose there exists a point \( q \neq p \) in \( X \) such that

\[
fq = q = S(q, q, ..., q) = T(q, q, ..., q) = R(q, q, ..., q).
\]

Consider from (5)

\[
G(fp, fq, fq) = G(S(p, p, ..., p), T(q, q, ..., q), R(q, q, ..., q)) < G(fp, fq, fq).
\]

It is a contradiction. Therefore \( q = p \).

Now we can get the following result of Gairola-Dhasmana [13] as a corollary.

**Corollary 3.2.** Let \((X, G)\) be a \( G \)-metric space, \( k \) a positive integer and \( T : X^k \to X, f : X \to X \) be mappings satisfying the following conditions

(10) \quad \( T(X^k) \subseteq f(X) \)

(11) \quad \( G(T(x_1, x_2, ..., x_{k-1}, x_k), T(x_2, x_3, ..., x_k, x_{k+1}), T(x_3, x_4, ..., x_{k+1}, x_{k+2})) \leq \lambda \max\{G(fx_i, fx_{i+1}, fx_{i+2}), 1 \leq i \leq k\} \)

for all \( x_1, x_2, ..., x_k, x_{k+1}, x_{k+2} \) in \( X \)

(12) \quad \( G\left( T(u, u, ..., u), T(v, v, ..., v), T(w, w, ..., w) \right) < G(fu, fv, fw), \)

for all \( u, v, w \in X \) with \( u \neq v \neq w \). Suppose that \( f(X) \) is \( G \)-complete and \((f, T)\) is coincidently commuting pair. Then there exist a unique point \( p \in X \) such that \( fp = p = T(p, p, ..., p) \).

**Proof.** Putting \( S = R = I \) (Identity map) in Theorem 3.1 we can get the required proof.

**Remark 3.3.** If \( f = I \) (Identity map) in Corollary 3.2, we get the main Theorem of [11].

**Conflict of Interests**

The authors declare that there is no conflict of interests.
REFERENCES


