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SOME NEW NOTIONS OF CONVERGENCE AND STABILITY OF COMMON FIXED POINTS IN 2-METRIC SPACES

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Abstract. Stability results for a pair of sequences of mappings and their common fixed points in a 2-metric space using certain new notions of convergence are proved. The results obtained here in extend and unify several known results.

Keywords: Stability, common fixed points, 2-metric spaces, J-Lipschitz.

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1. Introduction

The relationship between the convergence of a sequence of self mappings $\{T_n\}$ of a metric (resp. topological space) X and their fixed points, known as the stability (or continuity) of fixed points, has been widely studied in fixed point theory in various settings (cf. [1-5], [13-20], [22-28])). The origin of this problem seems into a classical result of Bonsall [5] (see also Sonnenschein [28]) for contraction mappings. Subsequent results by Nadler Jr. [20] and others address mainly the problem of replacing the completeness of the

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space X by the existence of fixed points (which was ensured otherwise by the completeness of X) and various relaxations on the contraction constant. Recently, in attempt to provide the localized versions of certain stability results of Bonsall [5] and Nadler [20], significant weakenings were made to the well-known notions of pointwise and uniform convergence by Barbet and Nachi [4]. In addition, these notions have been successfully utilized by them to obtain certain stability results in a metric space. These results have been further generalized by Mishra and Kalinde [17], Mishra, Singh and Pant [18] and Mishra, Singh and Stofile [19]. The purpose of this paper is to extend the results of Barbet and Nachi [4] to 2-metric spaces due to Gähler [6]. We note that the results obtained here in may be considered as significant in the sense that these spaces differ substantially in terms of their topological properties from those of metric spaces (see Remark 1.4).

We first recall some basic concepts on 2-metric spaces. For details we refer to Gähler [6] and Iséki [7-9].

Definition 1.1. Let X be a nonempty set, consisting of at least three points. A 2-metric on X is a real-valued function ρ on $X \times X \times X$ which satisfies the following conditions:

- **(a):** To each pair of distinct points $x, y \in X$ there exists a point $a \in X$ such that $\rho(x, y, a) \neq 0$.
- **(b):** If at least two of x, y, a are equal then $\rho(x, y, a) = 0$.
- **(c):** $\rho(x, y, a) = \rho(y, a, x) = \rho(x, a, y)$ for all $x, y, a \in X$.
- **(d):** $\rho(x, y, a) \leq \rho(x, y, z) + \rho(x, z, a) + \rho(z, y, a)$ for all $x, y, z, a \in X$.

It is easily seen that ρ is non-negative. The pair (X, ρ) is called a 2-metric space.

Definition 1.2. A sequence $\{ x_n\}$ in a 2-metric space (X, ρ) is said to be convergent with limit $z \in X$ if

$$\lim_{n\to\infty} \rho(x_n, z, a) = 0 \text{ for all } a \in X.$$

Notice that if the sequence $\{ x_n\}$ converges to z, then

$$\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b) \text{ for all } a, b \in X.$$

A sequence $\{ x_n \}$ in a 2-metric space (X, ρ) is said to be a Cauchy sequence if

$$\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0 \text{ for all } a \in X.$$

A 2-metric space (X, ρ) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.3. A 2-metric space (X, ρ) is said to be bounded if there is a constant K such that $\rho(a, b, c) \leq K$ for all $a, b, c \in X$.

Remark 1.4. The following remarks capture some distinct features of topological properties of 2-metric spaces which differ from those of metric spaces.

- **(i):** Given any metric space which consists of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not always true as one can find a 2-metric space which does not have a countable basis associated with one of its arguments (see Gähler [6, page 123]).
- **(ii):** It is known that a 2-metric ρ is continuous in any one of its arguments . Generally, we cannot however assert the continuity of ρ in all the three arguments. But if it is continuous in any two arguments, then it is continuous in all the three arguments (see Gäher [6, Theorem 20 and example on page 145]).
- **(iii):** In a complete 2-metric space a convergent sequence need not be Cauchy (see Naidu and Prasad [21, Example 0.1]).
- **(iv):** In a 2-mertic space (X, ρ) every convergent sequence is Cauchy whenever ρ is continuous. However, the converse need not be true (see Naidu and Prasad [21, Example 0.2]).

Definition 1.5. Let (X, ρ) be a 2-metric space. A mapping $T : X \to X$ is called *k-Lipschitz* (or simply *Lipschitz*) if there exists a real $k > 0$ such that

$$\rho(Tx, Ty, a) \leq k\rho(x, y, a) \text{ for all } x, y, a \in X.$$

In case the above condition is satisfied for $k \in (0, 1)$, T is called k-contraction (or simply contraction).(cf. [10], [12]).

Definition 1.6. Let (X, ρ) be a 2-metric space, $S, T : Y \subseteq X \to X$. Then the pair (S, T) will be called *J-Lipschitz* (Jungck Lipschitz) if there exists a constant $\mu > 0$ such that

$$\rho(Sx, Sy, a) \leq \mu\rho(Tx, Ty, a) \text{ for all } \; x, y, a \in Y.$$

The pair (S, T) is generally called *Jungck contraction* (or simply $J-$*contraction*) when $\mu \in (0, 1)$ and the constant μ in this case is a called *Jungck constant* (see, for instance, [24]). Indeed, *J-contractions* and their generalized versions became popular because of the constructive approach of proof adopted by Jungck [11]. Now onwards, a *J-Lpschitz* (resp. *J-contraction*) with Jungck constant μ will be called *J-Lipschitz* (resp. *J-contraction*) with constant μ (cf. [19] for details).

Throughout, (X, ρ) will denote a 2-metric space with ρ continuous, $\mathbb{N}$, the set of naturals and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

2. (G)-convergence and stability

Definition 2.1. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then $\{S_n\}_{n\in\mathbb{N}}$ is said to converge *(G)-pointwise* to a map S_∞: $X_\infty \to X$, or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), if the following condition holds:

(G): $Gr(S_\infty) \subset \liminf Gr(S_n)$: for every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that for any $a \in X$,

$$\lim_n \rho(x_n, x, a) = 0 \; \text{ and } \; \lim_n \rho(S_n x_n, S_\infty x, a) = 0,$$

where $Gr(T)$ stands for the graph of T.

In view of Barbet and Nachi [4], we note that:

(i): A (G)-limit need not be unique.

(ii): The property (G) is more general than pointwise convergence. However, the two notions are equivalent provided the sequence $\{S_n\}_{n\in\mathbb{N}}$ is equicontinuous when the domains of definitions are identical.

The following theorem gives a sufficient condition for the existence of a unique (G)-limit.

Theorem 2.2. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\ \mathbb{N}}$ a sequence of J-contraction mappings relative to a continuous mapping $T : X \to X$ with constant μ. If $S_\infty : X_\infty \to X$ is a (G)-limit of the sequence $\{S_n\}$, then S_∞ is unique.

Proof. Suppose that $S_\infty, S_\infty^* : X_\infty \to X$ are (G)-limit maps of the sequence $\{S_n\}$. Then for every $x \in X_\infty$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\underset{n\in\mathbb{N}}{\Pi} X_n$ such that for any $a \in X$,

$$\lim_n \rho(x_n, x, a) = 0 \text{ and } \lim_n \rho(S_n x_n, S_\infty x, a) = 0,$$

$$\lim_n \rho(y_n, x, a) = 0 \text{ and } \lim_n \rho(S_n y_n, S_\infty^* x, a) = 0.$$

Further, since S_n is a J-contraction for each $n \in \mathbb{N}$, there exists a constant $\mu \in (0, 1)$ such that for any $a \in X$,

$$\rho(S_n x_n, S_n y_n, a) \le \mu\rho(Tx_n, Ty_n, a).$$

Therefore for any $n \in \mathbb{N}$ and $a \in X$,

$$\begin{aligned}
\rho(S_\infty x, S_\infty^* x, a) &\le \rho(S_\infty x, S_\infty^* x, S_n x_n) + \rho(S_\infty x, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a)\\
&\le \rho(S_\infty x, S_\infty^* x, S_n x_n) + \rho(S_\infty x, S_n x_n, S_n y_n) + \rho(S_\infty x, S_n y_n, a)\\
&+ \rho(S_n y_n, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a)\\
&\le \rho(S_\infty x, S_\infty^* x, S_n x_n) + \mu\rho(S_\infty x, Tx_n, Ty_n) + \rho(S_\infty x, S_n y_n, a)\\
&+ \rho(S_n y_n, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a)\\
&\le \rho(S_\infty x, S_\infty^* x, S_n x_n) + \mu[\rho(S_\infty x, Tx_n, Tx) + \rho(S_\infty x, Tx, Ty_n)\\
&+ \rho(Tx, Tx_n, Ty_n)] + \rho(S_\infty x, S_n y_n, a) + \rho(S_n y_n, S_n x_n, a)\\
&+ \rho(S_n x_n, S_\infty^* x, a).
\end{aligned}$$

Since T is continuous and $x_n \to x$ and $y_n \to x$ as $n \to \infty$, it follows that $Tx_n \to Tx$, $Ty_n \to Tx$. Hence the R.H.S. of the above expression tends to 0 as $n \to \infty$. Therefore $S_\infty x = S_\infty^* x$.□

Corollary 2.3. Theorem 2.2 with J-contraction replaced by J-Lipschitz.

The following result is an extension of Barbet and Nachi [4, Proposition 1] to 2-metric spaces (see [18, Proposition 2.2]).

Corollary 2.4. Let (X,ρ) be a 2-metric space,$\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $S_n : X_n \to X$ a $k-$ contraction (resp. k-Lipschitz) mapping for each $n \in \mathbb{N}$. If $S_\infty : X_\infty \to X$ is a $(G)-$limit of $\{S_n\}_{n\in\mathbb{N}}$, then S_∞ is unique.

Proof. It comes from Theorem 2.2 when T is the identity mapping and $\mu \in (0,1)$ (resp. $\mu > 0$).□

Now we present our first stability result.

Theorem 2.5. Let (X,ρ) be a 2-metric space, $\{X_n\}_{n\in\ \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n,\ T_n : X_n \to X\}_{n\in\ \overline{\mathbb{N}}}$ two families of mappings, each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous. If for all $n \in \overline{\mathbb{N}}$, z_n is a common fixed point of S_n and T_n, then the sequence $\{z_n\}$ converges to z_∞.

Proof. Since z_n is a common fixed point of S_n and T_n for each $n \in \overline{\mathbb{N}}$, the property (G) holds and $z_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ with $y_n \in X_n$ (for all $n \in \overline{\mathbb{N}}$) such that for any $a \in X$,

$$\lim_n \rho(y_n, z_\infty, a) = 0,\ \ \lim_n \rho(S_n y_n, S_\infty z_\infty, a) = 0 \text{ and } \ \lim_n \rho(T_n y_n, T_\infty z_\infty, a) = 0.$$

Using the fact that the pair (S_n, T_n) is a J-contraction for all $n \in \mathbb{N}$, we have for any $a \in X$,

$$\begin{aligned}
\rho(z_n, z_\infty, a) &= \rho(S_n z_n,\ S_\infty z_\infty, a) \\
&\le \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \rho(S_n z_n, S_n y_n, a) + \rho(S_n y_n, S_\infty z_\infty, a) \\
&\le \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \mu\rho(T_n z_n, T_n y_n, a) + \rho(S_n y_n, S_\infty z_\infty, a) \\
&\le \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \rho(S_n y_n, S_\infty z_\infty, a) \\
&+ \mu[\rho(T_n z_n, T_n y_n, T_\infty z_\infty) + \rho(T_n z_n, T_\infty z_\infty, a) + \rho(T_\infty z_\infty, T_n y_n, a)].
\end{aligned}$$

The R.H.S. of the above expression tends to 0 as $n \to \infty$ and the conclusion follows.□

When for each $n \in \overline{\mathbb{N}}$, T_n is an identity mapping on X_n in Theorem 2.5, we have the following result as an extension of Barbet and Nachi [4, Theorem 2] to 2-metric spaces (see [18, Theorem 2.3]).

Corollary 2.6. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and S_n is a $k-$ contraction for each $n \in \overline{\mathbb{N}}$. If x_n is a fixed point of S_n for each $n \in \overline{\mathbb{N}}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.

The following result gives a comparison with Rhoades [22, Theorem 2] and presents a 2-metric space version of Bonsall [5, Theorem 1.2, page 6].

Corollary 2.7. Let X be a complete 2-metric space and $\{ S_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a family of contraction mappings with the same Lipschitz constant $k < 1$ and such that the sequence $\{S_n\}_{n\in\mathbb{N}}$ converges pointwise to S_∞. Then, for all $n \in \overline{\mathbb{N}}$, T_n has a unique fixed point x_n and the sequence $\{ x_n\}_{n\in\mathbb{N}}$ converges to x_∞.

Proof. This comes from Corollary 2.6 and the fact that X is complete.□

Again, when $X_n = X$, for all $n \in \overline{\mathbb{N}}$, we obtain, as a consequence of Theorem 2.5 the following result.

Corollary 2.8. Let (X, ρ) be a 2-metric space and $S_n,\ T_n : X \to X$ be such that the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous, and with at least one common fixed point z_n for all $n \in \overline{\mathbb{N}}$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge pointwise respectively to $S_\infty,\ T_\infty : X \to X$, then the sequence $\{z_n\}$ converges to z_∞.

Notice that Corollary 2.8 presents an extension of a result of Singh [24, Theorem 1] to 2-metric spaces.

We remark that under the conditions of Theorem 2.5 the pair (S_∞, T_∞) of (G)-limit maps is also a J-contraction. Indeed, we have the following stability result.

Theorem 2.9. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two families of mappings, each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant $\{\mu_n\}_{n\in\mathbb{N}}$, a bounded (resp. convergent) sequence and T_n continuous. Then the pair (S_∞, T_∞) is a J-contraction with constant $\mu = \sup_{n\in\mathbb{N}} \mu_n$ (resp. $\mu = \lim_n \mu_n$).

Proof. Let $x,\ y \in X_\infty$. Then by the property (G), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that:

$$\begin{aligned}\lim_n \rho\left(x_n, x, a\right) &= 0, \lim_n \rho\left(S_n x_n, S_\infty x, a\right) = 0, \lim_n \rho\left(T_n x_n, T_\infty x, a\right) = 0,\\ \lim_n \rho\left(y_n, y, a\right) &= 0, \lim_n \rho\left(S_n y_n, S_\infty y, a\right) = 0, \lim_n \rho\left(T_n y_n, T_\infty y, a\right) = 0\end{aligned}$$

for all $a \in X$.

Since for any $n \in \mathbb{N}$ and each $a \in X$,

$$\limsup_n \mu_n \rho(T_n x_n,\ T_n y_n, a) \leq \mu\rho(T_\infty x,\ T_\infty y, a),$$

the above inequality yields

$$\rho(S_\infty x,\ S_\infty y, a) \leq \mu\rho(T_\infty x,\ T_\infty y, a),$$

and the conclusion follows.□

Corollary 2.10. Theorem 2.9 with J-contraction replaced by J- Lipscitz.

When for each $n \in \overline{\mathbb{N}}$, T_n is an identity mapping in Theorem 2.9, we have the following extension of Barbet and Nachi [4, Proposition 4] (see [18, Proposition 2.7]).

Corollary 2.11. Let X be a 2-metric space, $\{\ X_n\}_{\ n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{\ n\in\overline{\mathbb{N}}}$ a family of mappings, satisfying the property (G) and such that, for any $n \in \overline{\mathbb{N}}$, S_n is J-Lipschitz (resp. J-contraction) with constant $\{\ k_n\}_{\ n\in\mathbb{N}}$ a bounded (resp. convergent) sequence. Then S_∞ is J-Lipschitz (resp. J-contraction) with constant $k := \sup_{n\in\mathbb{N}} k_n$ (resp. $k := \lim k_n$).

The existence of a fixed point for a (G)-limit mapping is characterized by the following result when it is a contraction.

Proposition 2.12. Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two families of mappings, each satisfying the property (G) and such that, for all $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and

T_n continuous. Assume that, for any $n \in \mathbb{N}$, x_n is a common fixed point of S_n and T_n. Then:

$$\begin{aligned} S_\infty \text{ and } T_\infty \text{ admit a common fixed point} \quad &\Leftrightarrow \quad \{x_n\} \text{ converges and } \lim_n x_n \in X_\infty \\ &\Leftrightarrow \quad \{x_n\} \text{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Proof. In view of Theorem 2.9, we only have to prove the sufficient condition. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim_j x_{n_j} = x_\infty \in X_\infty$. By (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_n \rho(y_n, z_\infty, a) = 0, \ \ \lim_n \rho(S_n y_n, S_\infty x_\infty, a) = 0 \text{ and } \ \lim_n \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

First we show that $S_\infty x_\infty = x_\infty$. For any $a \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \rho(x_\infty, S_\infty x_\infty, a) &\leq \rho(x_\infty, x_{n_j}, a) + \rho(S_{n_j} x_{n_j}, S_\infty x_\infty, a) + \rho(x_\infty, S_\infty x_\infty, S_{n_j} x_{n_j}) \\ &\leq \rho(x_\infty, x_{n_j}, a) + \rho(S_{n_j} x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) + \\ &\quad\ \rho(S_{n_j} x_{n_j}, S_{n_j} y_{n_j}, a) + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) \\ &\quad\ +\rho(x_\infty, S_\infty x_\infty, S_{n_j} x_{n_j}) \\ &\leq \rho(x_\infty, x_{n_j}, a) + \rho(S_{n_j} x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) + \mu\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) \\ &+ \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_\infty, S_\infty x_\infty, S_{n_j} x_{n_j}). \end{aligned}$$

The R.H.S. of the above expression tends to zero as $n \to \infty$ and hence $S_\infty x_\infty = x_\infty$.

Next, by the triangle inequality we have

$$\rho(x_\infty, T_\infty x_\infty, a) \leq \rho(x_\infty, x_{n_j}, a) + \rho(T_{n_j} x_{n_j}, T_\infty x_\infty, a) + \rho(x_\infty, T_\infty x_\infty, T_{n_j} x_{n_j}).$$

The R.H.S. of the above expression tends to zero as $n \to \infty$ and hence $T_\infty x_\infty = x_\infty$. Therefore $S_\infty x_\infty = T_\infty x_\infty = x_\infty$ and x_∞ is a common fixed point of S_∞ and T_∞.□

Remark 2.13. Under the assumptions of Proposition 2.12 and if

(i): $\liminf X_n \subset X_\infty$ (i.e., the limit of any convergent sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then:

$$S_\infty \text{ and } T_\infty \text{ admit a common fixed point} \Leftrightarrow \{x_n\} \text{ converges.}$$

(ii): $\limsup X_n \subset X_\infty$ (i.e., the cluster point of any sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞) then:

$$S_\infty \text{ and } T_\infty \text{ admit a common fixed point} \Leftrightarrow \{x_n\} \text{ admits a convergent subsequence.}$$

Under a compactness assumptions, we have the following.

Theorem 2.14. Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space, (X,ρ) $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two families of mappings, each satisfying the property (G) and such that, for any $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous. Assume that, $\limsup X_n \subset X_\infty$ and $\underset{n\in\mathbb{N}}{\cup} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, x_n is a common fixed point of S_n and T_n, then the pair (S_∞, T_∞) of (G)-limit mappings of S_n and T_n admits a common fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be the common fixed point of S_n and T_n. Then by the compactness assumption, $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_j}\}$. Now, by Remark 2.13, S_∞ and T_∞ admit a common fixed point x_∞ and by Theorem 2.5 $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.□

Remark 2.15. By choosing X_n and T_n suitably in Proposition 2.12 and Theorem 2.14, we obtain the extensions of the corresponding results of Barbet and Nachi [4, Corollary 6 and Theorem 7] (see [18, Corollary 2.5 and Theorem 2.10]).

We now introduce another notion of convergence which is weaker than (G)-convergence.

Definition 2.16. Let (X,ρ) be a 2-metrc space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then $\{S_n\}_{n\in\mathbb{N}}$ is said to converge (G^-) to a mapping S_∞: $X_\infty \to X$, or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G$^-$), if the following condition holds:

(G$^-$): $Gr(T_\infty) \subset \limsup Gr(T_n)$: for all $z \in X_\infty$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X_n$, and which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_j \rho(x_{n_j}, z, a) = 0 \text{ and } \lim_j \rho(T_{n_j} x_{n_j}, T_\infty z, a) = 0 \text{ for all } a \in X.$$

We shall establish in the next result that a fixed point of a (G$^-$)-limit map is a cluster point of the sequence of fixed points associated with $\{T_n\}$.

Theorem 2.17. Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space (X, ρ) and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two families of J-contraction mappings with constant μ and T_n continuous, each satisfying the property (G$^-$) . If, for any $n \in \overline{\mathbb{N}}$, x_n is a common fixed point of S_n and T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

Proof. By the property (G$^-$), there exists a sequence $\{y_n\}$ in $\prod_n X_n$ which has a subsequence $\{y_{n_j}\}$ such that:

$$\lim_j \rho(y_{n_j}, x_\infty, a) = 0,\ \lim_j \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) = 0, \text{ and } \lim_j \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Since the pair (S_{n_j}, T_{n_j}) is a J-contraction for each $j \in \overline{\mathbb{N}}$, for any $a \in X$ we have

$$\begin{aligned}
\rho(x_{n_j}, x_\infty, a) &= \rho(S_{n_j} x_{n_j}, S_\infty x_\infty, a) \\
&\le \rho(S_{n_j} x_{n_j}, S_{n_j} y_{n_j}, a) + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) \\
&\le \mu\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) \\
&\le \mu[\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, x_\infty) + \rho(T_{n_j} x_{n_j}, T_{n_j} x_\infty, a) + \rho(T_{n_j} x_\infty, T_{n_j} y_{n_j}, a)] \\
&+ \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j})
\end{aligned}$$

The R.H.S. of the above expression tends to zero as $j \to \infty$ and hence $\{x_{n_j}\}$ converges to x_∞, the common fixed point of S_∞ and T_∞.$\Box$

When for all $n \in \overline{\mathbb{N}}$, $X_n = X$ and T_n is an identity mapping, we get the following analogue of Barbet and Nachi [4, Theorem 8] to 2-metric spaces (see [18, Theorem 2.12]).

Corollary 2.18. Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space (X, ρ) and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G$^-$) . If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

3. (H)-convergence and Stability

Definition 3.1. Let (X,ρ) be a 2-metric space, $\{X_n\}_{n\in\ \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then S_∞ is called an $(H)-$limit of the sequence $\{S_n\}_{n\in\mathbb{N}}$ or, equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

(H): For all sequences $\{x_n\}$ in $\prod\limits_{n\in\ \mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that for any $a\in X$,

$$\lim_n \rho(x_n, y_n, a) = 0 \text{ and } \lim_n \rho(S_n x_n, S_n y_n, a) = 0.$$

In case X is a metric space, we get the corresponding definitions due to Barbet and Nachi [4]. In view of [4] we note that:

(a): A (G)-limit map is not necessarily an (H)-limit.

(b): If $\{S_n\}_{n\in\mathbb{N}}$ converges uniformly to S_∞ on a common domain Y, then S_∞ is an (H)-limit of $\{S_n\}$.

(c): The converse of (b) holds only when S_∞ is uniformly continuous on Y.

For details and examples we again refer to Barbet and Nachi [4, page 56].

The following theorem presents another stability result.

Theorem 3.2. Let (X,ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and let $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be two families of mappings, each satisfying the property (H). Further, let the pair (S_∞, T_∞) be a J-contraction with constant μ_∞ and T_∞ continuous. If, for any $n\in\overline{\mathbb{N}}$, z_n is a common fixed point of S_n and T_n, then the sequence $\{z_n\}$ converges to z_∞.

Proof. The property (H) implies that, there exists a sequence $\{y_n\}$ in X_∞ such that for any $a\in X$,

$$\lim_n \rho(z_n, y_n) = 0,\ \lim_n \rho(S_n z_n, S_\infty y_n, a) = 0 \text{ and } \lim_n \rho(T_n z_n, T_\infty y_n, a) = 0.$$

Then

$$\begin{aligned}
\rho(z_n,\ z_\infty, a) &= \rho(S_n z_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) + \rho(S_\infty y_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) + \mu_\infty \rho(T_\infty y_n, T_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) \\
&+ \mu_\infty [\rho(T_\infty y_n, T_\infty z_\infty, T_n z_n) + \rho(T_\infty y_n, T_n z_n, a) + \rho(T_n z_n, T_\infty z_\infty, a)].
\end{aligned}$$

Since the right hand side of the above inequality tends to 0 as $n \to \infty$, we deduce that $z_n \to z_\infty$ as $n \to \infty$.□

The following corollary is an extension of Barbet and Nachi [4, Theorem 11] to 2-metric spaces (see [18, Theorem 3.4]).

Corollary 3.3. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n \in \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n \in \overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that S_∞ is a k_∞-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x_∞.

Proof. It comes from Theorem 3.2 when for all $n \in \overline{\mathbb{N}}$, T_n is an identity mapping.□

When $X_n = X$, for all $n \in \overline{\mathbb{N}}$ in Corollary 3.3, we get a special case of Rhoades [22, Theorem 3] which in turn presents a $2-$metric space version of Nadler [20, Theorem 1].

Corollary 3.4. Let (X, ρ) be a 2-metric space, $\{S_n : X \to X\}_{n \in \overline{\mathbb{N}}}$ a sequence of mappings which converges uniformly to a contraction mapping $S_\infty : X \to X$. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x_∞.

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