AN ALGORITHM FOR APPROXIMATING A COMMON FIXED POINT OF A FINITE FAMILY OF LIPSCHITZ PSEUDOCONTRACTIVE MULTI-VALUED MAPPINGS

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Abstract. The purpose of this paper is twofold. We first give erratum to a proof given by Woldeamanuel et al. [Strong convergence theorems for a common fixed point of a finite family of Lipschitz hemicontractive-type multivaled mappings, Adv. Fixed Point Theory, 5 (2015), No. 2, 228-253]. In addition, we study an algorithm which approximates a common fixed point of a finite family of Lipschitz pseudocontractive multi-valued mappings under appropriate conditions.

Keywords: Demiclosed, Hausdorff metric; k-strictly pseudocontractive multi-valued mapping; Lipschitz pseudocontractive multi-valued mapping; Monotone multi-valued mapping; Strong convergence.

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1. Introduction

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Let $E$ be a nonempty real normed linear space. A subset $K$ of $E$ is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$
\|x - k\| = \inf\{\|x - y\| : y \in K\} = d(x, K).
$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. In fact, if $K$ is a closed and convex subset of a uniformly convex Banach space $E$, then for any $x \in E$ there exists a unique point $u_x \in K$ such that (see, e.g., [26], [25], [15] and [8])

$$
\|x - u_x\| = \inf\{\|x - y\| : y \in K\} = d(x, K).
$$

Let $E$ be a nonempty real normed space. We denote the family of all nonempty proximinal subsets of $E$ by $P(E)$, the family of all nonempty closed, convex and bounded subsets of $E$ by $CBC(E)$, the family of all nonempty closed and bounded subsets of $E$ by $CB(E)$ and the family of all nonempty subsets of $E$ by $2^E$ for a nonempty normed space $E$.

Let $D$ be the Hausdorff metric induced by the metric $d$ on $E$, that is, for every $A, B \in CB(E)$,

$$
D(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.
$$

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called $L$-Lipschitzian if there exists $L \geq 0$ such that,

$$
\forall x, y \in D(T), D(Tx, Ty) \leq L\|x - y\|.
$$

In (1), if $L \in [0, 1)$, $T$ is said to be a contraction, while $T$ is nonexpansive if $L = 1$. A point $x \in C$ is a fixed point of $T$ if $x \inTx$ and we denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : x \in Tx\}$.

A mapping $T : D(T) \subseteq E \to CB(E)$ is said to be hemicontractive-type in the terminology of Hicks and Cubilek [17], if $F(T) \neq \emptyset$ and for all $p \in F(T), x \in D(T)$

$$
D^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2, \forall u \in Tx,
$$

while, a mapping $T : D(T) \subseteq E \to CB(E)$ is said to be demicontractive-type, if $F(T) \neq \emptyset$ and for all $p \in F(T), x \in D(T)$ there exists $k \in [0, 1)$ such that

$$
D^2(Tx, Tp) \leq \|x - p\|^2 + k\|x - u\|^2, \forall u \in Tx.
$$
For the definitions of $k$-strictly pseudocontractive-type, quasi-nonexpansive-type, pseudocontractive-type and nonexpansive-type multivalued mappings we refer the reader to the paper [31].

Recently, Woldeamanuel et. al. [31] introduced an iteration scheme $x_1 = w \in K$ by

\[
\begin{align*}
    y_n &= (1 - \beta_n)x_n + \beta_n u_n, \quad u_n \in T_n x_n, \\
    z_n &= \gamma_n w_n + (1 - \gamma_n) x_n, \quad w_n \in T_n y_n, \\
    x_{n+1} &= \alpha_n w + (1 - \alpha_n) z_n, \quad n \geq 1,
\end{align*}
\]

where $T_n := T_n (mod \ N)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy some conditions.

They stated a theorem (Theorem 3.1 [31]) and proved strong convergence of the scheme to a common fixed point $p$, which is nearest to $w$, of $T_i, i = 1, \ldots, N$. The proof depends on the argument that $T : K \to CB(K)$ satisfies $\|u - v\| \leq 2D(Tx, Ty), \forall x, y \in K, u \in Tx, v \in Ty$.

**Remark 1.1.** A close look at the property of $T$ shows that the argument considered may not be in general true. To see this, one may consider the following example.

**Example 1.1.** Let $T : \mathbb{R} \to 2^\mathbb{R}$ be given by

\[
    Tx = \begin{cases} 
        [-\sqrt{2}x, 0], & x \in [0, \infty], \\
        [0, -\sqrt{2}x], & x \in [-\infty, 0]. 
    \end{cases}
\]

It can be shown that $T$ is hemicontractive-type. Now, for $x = 3$ and $y = 2$, we have $Tx = [-3\sqrt{2}, 0]$ and $Ty = [-5\sqrt{2}, 0]$, so that

\[
D(tx, Ty) = D([-3\sqrt{2}, 0], [-5\sqrt{2}, 0]) = 2\sqrt{2}.
\]

Now for $u = o \in Tx$ and $v = -5\sqrt{2} \in Ty$, we have

\[
    \|u - v\| = 5\sqrt{2} > 4\sqrt{2} = 2D(Tx, Ty).
\]

A mapping $T : K \to CB(H)$ is said to be pseudocontractive (see [19, 20, 24]), if the inequality

\[
    \langle u - v, x - y \rangle \leq \|x - y\|^2,
\]
holds for each \( x, y \in K, u \in Tx, v \in Ty \). In this case,
\[
\|x - y - (u - v)\|^2 + 2 \langle u - v, x - y \rangle \leq 2\|x - y\|^2 + \|x - y - (u - v)\|^2,
\]
which implies that
\[
\|u - v\|^2 \leq \|x - y\|^2 + \|x - y - (u - v)\|^2.
\]

Hence, \( T : K \rightarrow CB(H) \) is said to be pseudocontractive multi-valued mapping, if \( \forall x, y \in K \)
\[
\|u - v\|^2 \leq \|x - y\|^2 + \|x - y - (u - v)\|^2, \quad \forall u \in Tx, v \in Ty.
\]

We observe that (6) implies that \( \forall x, y \in K \),
\[
D^2(Tx, Ty) \leq \|x - y\|^2 + \|x - y - (u - v)\|^2, \quad \forall u \in Tx, v \in Ty,
\]
known as pseudocontractive-type multi-valued mapping (see, [31]).

For an example of pseudocontractive multi-valued mapping, see [32].

A mapping \( T : K \rightarrow CB(H) \) is said to be \( k \)-strongly pseudocontractive (see [19, 20]), if there exists \( k \in (0, 1) \) such that the inequality
\[
\langle u - v, x - y \rangle \leq k\|x - y\|^2,
\]
holds for each \( x, y \in K, u \in Tx, v \in Ty \).

Again we refer the reader to [32] for an example of \( k \)-strongly pseudocontractive multi-valued mapping.

**Remark 1.2.** Note that the class of pseudocontractive multi-valued mappings properly includes the class of \( k \)-strongly pseudocontractive multi-valued mappings.

Multi-valued pseudocontractive mappings are also related with the important class of non-linear monotone mappings, where \( A : K \rightarrow CB(H) \) is called monotone, if for any \( x, y \in K \),
\[
\langle u - v, x - y \rangle \geq 0, \quad \forall u \in Ax, v \in Ay.
\]

A mapping \( A : K \rightarrow CB(H) \) is said to be \( k \)-strongly monotone mapping if for all \( x, y \in K \), there exists \( k \in [0, 1) \), such that
\[
\langle u - v, x - y \rangle \geq k\|x - y\|^2, \quad \forall u \in Ax, v \in Ay.
\]
We note that $T$ is pseudocontractive if and only if $A := I - T$ is monotone and hence $x \in F(T)$ if and only if $x \in N(A) := \{x \in K : 0 \in Ax\}$.

Recently, Woldeamanuel et al. [32] introduced an iteration scheme $x_1 = w \in K$ by

$$
\begin{align*}
  y_n &= (1 - \beta_n)x_n + \beta_n u_n, \\
  z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \\
  x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
$$

(11)

where $u_n \in Tx_n, w_n \in Ty_n$ such that $||u_n - w_n|| \leq 2D(Tx_n, Ty_n)$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy certain mild conditions.

They proved the strong convergence of the Scheme (11) to the fixed point of Lipschitz pseudocontractive multi-valued mapping $T$. This brings us to the next question.

**Question:** Can we extend the results of [32] to a common fixed point of a finite family of Lipschitz pseudocontractive multi-valued mappings?

The purpose of this paper is twofold. In section three, motivated by the result of Woldeamanuel et al. [31] and Remark 1.1, we consider the scheme studied in [31] with appropriate assumptions on $T$ and give a modified proof which will enable us to correct the anomalies pointed out in Remark 1.1. In section four, we extend the work of Woldeamanuel et al. [32] to a finite family of Lipschitz pseudocontractive multi-valued mappings under appropriate conditions.

### 2. Preliminaries

**Definition 2.1** Let $E$ be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multi-valued mapping. $(I - T)$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to $p$ and $D(\{x_n\}, Tx_n) \to 0$, then $p \in Tp$.

**Lemma 2.1.** [30] Let $H$ be a real Hilbert space. Then, the following equations hold:

1. $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $\forall t \in [0, 1]$,
2. Given any $x, y \in H$, $\|x - y\|^2 = \|x - z\|^2 + \|z - y\|^2 + 2\langle x - z, z - y \rangle$. 

Lemma 2.2. [11] Let \( H \) be a real Hilbert space. Then, the following equation holds: If \( \{x_n\} \) is a sequence in \( H \) such that \( x_n \rightharpoonup z \in H \), then

\[
\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.
\]

Lemma 2.3. [1] Let \( K \) be a nonempty, closed and convex subset of a real Hilbert space \( H \). Let \( x \in H \). Then, \( x_0 = P_K(x) \) if and only if \( \langle z - x_0, x - x_0 \rangle \leq 0, \forall z \in K \).

Lemma 2.4. [4] Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T : K \to CBC(K) \) be a multivalued mapping and \( P_T(x) = \{y \in Tx : \|x - y\| = d(x,Tx)\} \). Then, for any \( x \in K, x_0 \in P_T(x) \) if and only if \( \langle z - x_0, x - x_0 \rangle \leq 0, \forall z \in Tx \).

Lemma 2.5. [17] Let \( \{a_n\} \) be a sequence of real numbers such that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that \( a_{n_i} < a_{n_i+1} \), for all \( i \in \mathbb{N} \). Then, there exists a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty \) and the following properties are satisfied by all (sufficiently large) numbers \( k \in \mathbb{N} \): \( a_{m_k} \leq a_{m_k+1} \) and \( a_{m_k} \leq a_{k+1} \). In fact, \( m_k := \max \{j \leq k : a_j < a_{j+1}\} \).

Lemma 2.6. [28] Let \( K \) be a metric space. Let \( T : K \to P(K) \) be a multivalued mapping. Then, the following are equivalent:

1. \( x \in Tx \),
2. \( P_Tx = \{x\} \),
3. \( x \in F(P_T) \).

Moreover, \( F(T) = F(P_T) \).

Lemma 2.7. [33] Let \( H \) be a real Hilbert space, \( C \) a closed convex subset of \( H \) and \( T : C \to C \) be a continuous pseudo-contractive mapping, then \((I - T)\) is demiclosed at zero, i.e., if \( \{x_n\} \) is a sequence in \( C \) such that \( x_n \rightharpoonup x \) and \( Tx_n - x_n \to 0 \), as \( n \to \infty \), then \( x = Tx \).

Lemma 2.8. Let \( H \) be a real Hilbert space. Then,

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.
\]

Proposition 2.1. [3] Let \( H \) be a Hilbert space. Let \( K \) be a nonempty closed and convex subset of \( H \). Let \( T : K \to CB(K) \) be \( k \)-strictly pseudocontractive-type multivalued mapping. Then \( T \) is \( L \)-Lipschitz mapping.
Lemma 2.9. [34] Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following relation: 
\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, n \geq n_0,
\]
where \( \{\alpha_n\} \subset (0, 1) \) and \( \{\delta_n\} \subset \mathbb{R} \) satisfying the following conditions:
\[
\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \limsup_{n \to \infty} \delta_n \leq 0. \text{ Then, } \lim_{n \to \infty} a_n = 0.\]

Lemma 2.10. [21] Let \( E \) be a complete metric space. Let \( A, B \in CB(E) \) and \( a \in A \).

1. If \( \gamma > 0 \), then there exists \( b \in B \) such that \( d(a, b) \leq H(A, B) + \gamma \).
2. If \( x \in E \), then \( d(x, A) \leq d(x, B) + D(A, B) \).

The proof of the following are given in [32].

Lemma 2.11. Let \( H \) be a real Hilbert space. Suppose \( K \) is a closed, convex, nonempty subset of \( H \). Assume that \( T : K \to CB(K) \) is pseudocontractive multi-valued mapping with \( F(T) \neq \emptyset \). Then, \( F(T) \) is closed and convex.

Lemma 2.12. Let \( H \) be a real Hilbert space. Suppose \( K \) is a closed, convex, nonempty subset of \( H \). Assume that \( T : K \to CB(K) \) is Lipschitz pseudocontractive multi-valued mapping. Then, there is a single-valued nonexpansive mapping \( S : K \to K \), such that for some \( \lambda > 0 \) and for any \( y \in K \), \( S(y) \) is a fixed point of \( T_y(x) := (1 - \lambda)y + \lambda Tx \).

Lemma 2.13. Let \( H \) be a real Hilbert space. Suppose \( K \) is a closed, convex, nonempty subset of \( H \). Assume that \( T : K \to CB(K) \) is Lipschitz pseudocontractive multi-valued mapping. Then \( (I - T) \) is demiclosed at zero.

3. Convergence results for a finite family of lipschitz hemicontractive-type mappings

Now, we give the modification of the statement and proof of Theorem 3.1 of [31].

Theorem 3.1. Let \( K \) be a non-empty, closed and convex subset of a real Hilbert space \( H \). Let \( T_i : K \to CB(K), i = 1, 2, \ldots, N \), be a finite family of Lipschitz hemicontractive-type mappings with Lipschitz constants \( L_i, i = 1, 2, \ldots, N \), respectively. Assume that \( (I - T_i), i = 1, \ldots, N \) are demiclosed at zero and \( \mathcal{F} = \cap_{i=1}^{N} F(T_i) \) is non-empty, closed and convex with \( T_i(p) = \{p\}, \forall p \in F(T) \) and for each \( i = 1, 2, \ldots, N \). Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 = \).
which implies that

\[
\begin{align*}
    y_n &= (1 - \beta_n)x_n + \beta_n u_n, \\
    z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \\
    x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
\]

(12)

where, \( u_n \in T_n x_n, w_n \in T_n y_n \) such that \( ||u_n - w_n|| \leq 2D(T_n x_n, T_n y_n) \) and \( T_n := T_{n(\text{mod } N) + 1} \). \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1) \) satisfy the following conditions:

i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \)

ii. \( 0 < \alpha \leq \gamma_n \leq \beta \leq 1 - 1 / \sqrt{4L^2 + 1} \), \( \forall n \geq 1 \) for \( L := \max\{L_i : 1, 2, \ldots, N\} \).

Then, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w \).

**Proof.** Let \( p = P_\mathcal{F}(w) \). Now, using (1) of Lemma 2.1,

\[
||x_{n+1} - p||^2 = ||\alpha_n (w - p) + (1 - \alpha_n)(z_n - p)||^2 \\
\leq \alpha_n ||w - p||^2 + (1 - \alpha_n) ||z_n - p||^2 \\
= \alpha_n ||w - p||^2 + (1 - \alpha_n) ||\gamma_n (w_n - p) + (1 - \gamma_n)(x_n - p)||^2 \\
= \alpha_n ||w - p||^2 + (1 - \alpha_n) \gamma_n ||w_n - p||^2 + (1 - \alpha_n)(1 - \gamma_n)||x_n - p||^2 \\
-(1 - \alpha_n) \gamma_n (1 - \gamma_n)||w_n - x_n||^2, \\
= \alpha_n ||w - p||^2 + (1 - \alpha_n)(1 - \gamma_n)||x_n - p||^2 + (1 - \alpha_n) \gamma_n ||w_n - p||^2 \\
-(1 - \alpha_n) \gamma_n (1 - \gamma_n)||w_n - x_n||^2,
\]

(13)

which implies that

\[
||x_{n+1} - p||^2 \leq \alpha_n ||w - p||^2 + (1 - \alpha_n)(1 - \gamma_n)||x_n - p||^2 + (1 - \alpha_n) \gamma_n D(T_n y_n, T_n p)^2 \\
-(1 - \alpha_n) \gamma_n (1 - \gamma_n)||w_n - x_n||^2 \\
\leq \alpha_n ||w - p||^2 + (1 - \alpha_n)(1 - \gamma_n)||x_n - p||^2 + (1 - \alpha_n) \gamma_n \\
[||y_n - p||^2 + ||y_n - w_n||^2] - (1 - \alpha_n) \gamma_n (1 - \gamma_n)||w_n - x_n||^2.
\]
Thus,

\[(14) \quad \|x_{n+1} - p\|^2 \leq \alpha_n \|w - p\|^2 + (1 - \alpha_n)(1 - \gamma_n) \|x_n - p\|^2 + (1 - \alpha_n) \gamma_n \|y_n - p\|^2 + (1 - \alpha_n) \gamma_n (1 - \gamma_n) \|w_n - x_n\|^2.\]

On the other hand, using (12) and using the assumption that \(|u_n - w_n| \leq 2D(T_n x_n, T_n y_n)\) we have

\[
\|y_n - w_n\|^2 = \|(1 - \beta_n)(x_n - w_n) + \beta_n(u_n - w_n)\|^2 \\
= (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
\leq (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n 4D^2(T_n x_n, T_n y_n) - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
\leq (1 - \beta_n)\|x_n - w_n\|^2 + 4\beta_n^2\|x_n - y_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
\leq (1 - \beta_n)\|x_n - w_n\|^2 + 4\beta_n^2\|x_n - u_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2.
\]

Hence,

\[(15) \quad \|y_n - w_n\|^2 \leq (1 - \beta_n)\|x_n - w_n\|^2 - \beta_n(1 - \beta_n - 4\beta_n^2)\|x_n - u_n\|^2.\]

Again,

\[
\|y_n - p\|^2 = \|(1 - \beta_n)x_n + \beta_n u_n - p\|^2 \\
= \|(1 - \beta_n)(x_n - p) + \beta_n(u_n - p)\|^2 \\
= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2,
\]

which gives that

\[
\|y_n - p\|^2 \leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n D^2(T_n x_n, T_n p) - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n [\|x_n - p\|^2 + \|x_n - u_n\|^2] \\
- \beta_n(1 - \beta_n)\|x_n - u_n\|^2.
\]

Thus,

\[(16) \quad \|y_n - p\|^2 \leq \|x_n - p\|^2 + \beta^2\|x_n - u_n\|^2.\]
Now substituting (16), (15) into (14), we have

\[ \| x_{n+1} - p \|^2 \leq \alpha_n \| w - p \|^2 + (1 - \alpha_n)(1 - \gamma_n)\| x_n - p \|^2 + (1 - \alpha_n)\gamma_n \| x_n - p \|^2 \]

\[ + (1 - \alpha_n)\gamma_n\beta_n^2\| x_n - u_n \|^2 + (1 - \alpha_n)\gamma_n(1 - \beta_n)\| x_n - w_n \|^2 \]

\[ - \beta_n(1 - \alpha_n)\gamma_n(1 - \beta_n - 4L^2\beta_n^2)\| u_n - x_n \|^2 \]

\[ - (1 - \alpha_n)\gamma_n(1 - \gamma_n)\| w_n - x_n \|^2, \]

which reduces to

\[ \text{(17)} \quad \| x_{n+1} - p \|^2 \leq \alpha_n \| w - p \|^2 + (1 - \alpha_n)\| x_n - p \|^2 - \beta_n(1 - \alpha_n) \]

\[ \times \gamma_n(1 - 2\beta_n - 4L^2\beta_n^2)\| u_n - x_n \|^2 + (1 - \alpha_n)\gamma_n\gamma_n - \beta_n)\| x_n - w_n \|^2. \]

From hypothesis (ii) in (12) we have that

\[ \text{(18)} \quad 1 - 2\beta_n - 4L^2\beta_n^2 \geq 1 - 2\beta - 4L^2\beta^2 \quad \text{and} \quad \gamma_n \leq \beta_n. \]

Using (18) in (17), we get that

\[ \text{(19)} \quad \| x_{n+1} - p \|^2 \leq (1 - \alpha_n)\| x_n - p \|^2 + \alpha_n\| w - p \|^2. \]

Thus, by induction

\[ \| x_{n+1} - p \|^2 \leq \max\{\| x_1 - p \|^2, \| w - p \|^2\}, \forall n \geq 1. \]
This implies that \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are all bounded. Furthermore, from (12), Lemma 2.8 and (17), we get that

\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n) (y_n w_n + (1 - \gamma_n) x_n) + \alpha_n w - p\|^2 \\
= \|(1 - \alpha_n) ((y_n w_n + (1 - \gamma_n) x_n) - p) + \alpha_n(w - p)\|^2 \\
\leq (1 - \alpha_n) \|y_n w_n + (1 - \gamma_n) x_n - p\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle \\
= (1 - \alpha_n) \gamma_n \|w_n - p\|^2 + (1 - \gamma_n) \|x_n - p\|^2 \\
- \gamma_n (1 - \gamma_n) \|x_n - w_n\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle \\
\leq (1 - \alpha_n) \gamma_n \|x_n - p\|^2 + (1 - \alpha_n) \gamma_n \|x_n - w_n\|^2 + (1 - \alpha_n) \gamma_n \\
\times [(1 - \beta_n) \|x_n - w_n\|^2 - \beta_n (1 - \beta_n - 4L^2 \beta_n^2) \|x_n - u_n\|^2] \\
- (1 - \alpha_n) \gamma_n (1 - \gamma_n) \|w_n - x_n\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle \\
= (1 - \alpha_n) \|x_n - p\|^2 - (1 - \alpha_n) \gamma_n \beta_n (1 - 2\beta_n - 4L^2 \beta_n^2) \|x_n - u_n\|^2 \\
+ 2\alpha_n \langle w - p, x_{n+1} - p \rangle + (1 - \alpha_n) \gamma_n (\gamma_n - \beta_n) \|x_n - w_n\|^2.
\]

That is, we get that

\[
\|x_{n+1} - p\|^2 = (1 - \alpha_n) \|x_n - p\|^2 - (1 - \alpha_n) \gamma_n \beta_n (1 - 2\beta_n - 4L^2 \beta_n^2) \|x_n - u_n\|^2 \\
+ 2\alpha_n \langle w - p, x_{n+1} - p \rangle + (1 - \alpha_n) \gamma_n (\gamma_n - \beta_n) \|x_n - w_n\|^2,
\]

(20)
which implies

\[ \|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\beta_n\|1 - 2\beta_n - 4L^2\beta_n^2\] \times \|x_n - u_n\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle, \]

and

\[ \|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\alpha^2(1 - 2\beta - 4L^2\beta^2) \] \times \|x_n - u_n\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p\rangle. \]

Now we consider the following two cases:

**Case 1.** Suppose that there exists \( n_0 \in \mathbb{N} \) such that \( \{\|x_n - p\|\} \) is non-increasing, \( \forall n \geq n_0 \). Then, we get that \( \{\|x_n - p\|\} \) is convergent. So, from (22) and the fact that \( \alpha_n \to 0 \), we have that

\[ (1 - c)\alpha^2(1 - 2\beta - 4L^2\beta^2)\|x_n - u_n\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2, \]

which gives that

\[ \|x_n - u_n\| \to 0. \] \( (23) \)

Now, from (12) and (23) we get

\[ y_n - x_n = \beta_n(u_n - x_n) \to 0, \]

and hence we get that

\[ \|z_n - x_n\| = \gamma_n\|w_n - x_n\| = \gamma_n\|w_n - u_n + u_n - x_n\| \leq \gamma_n\|w_n - u_n\| + \gamma_n\|u_n - x_n\| \leq \gamma_n2D(T_ny_n, T_nx_n) + \gamma_n\|u_n - x_n\| \leq \gamma_n2L\|y_n - x_n\| + \gamma_n\|u_n - x_n\| \to 0. \] \( (24) \)

By (12), (24), the fact that \( \|w - z_n\| \) is bounded and \( \alpha_n \to 0 \), we have

\[ \|x_{n+1} - x_n\| = \|x_{n+1} - z_n + z_n - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \leq \alpha_n\|w - z_n\| + \|z_n - x_n\| \to 0. \] \( (25) \)
But then, since, \( \|x_{n+i} - x_n\| \leq \|x_{n+i} - x_{n+i-1}\| + \ldots + \|x_{n+1} - x_n\| \), we get that

\[
(26) \quad \|x_{n+i} - x_n\| \to 0, \forall i = 1, 2, \ldots, N.
\]

Thus, from (23) and (26), we obtain that

\[
(27) \quad \|u_{n+i} - x_n\| \leq \|u_{n+i} - x_{n+i}\| + \|x_{n+i} - x_n\| \to 0, \forall i = 1, 2, \ldots, N.
\]

Now we show that for \( i \in \{1, 2, \ldots, N\} \), \( \lim_{n \to \infty} d(x_n, T_{n+i}x_n) = 0 \). But from Lemma 2.10, (23), (26) and Lipschitz property of \( T_i \) for each \( i \in \{1, 2, \ldots, N\} \) we get that

\[
(28) \quad d(x_n, T_{n+i}x_n) = d(x_n, T_{n+i}x_{n+i}) + D(T_{n+i}x_n, T_{n+i}x_{n+i}) \leq \|x_n - u_{n+i}\| + L\|x_n - x_{n+i}\| \to 0,
\]

which is the required result. The rest of the proof is the same as Theorem 3.1 of [31].

If, in Theorem 3.1, we consider a single hemicontractive-type mapping we get the following corollary.

**Corollary 3.1.** Let \( H \) be a real Hilbert space and \( K \) be a non-empty, closed and convex subset of \( H \). Let \( T : K \to CB(K) \), be Lipschitz hemicontractive-type mapping with Lipschitz constant \( L \). Assume that \( I - T \) is demiclosed at zero and \( F(T) \) is non-empty with \( T(p) = \{p\} \), \( \forall p \in F(T) \). Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 = w \in K \) by

\[
\begin{align*}
y_n &= (1 - \beta_n)x_n + \beta_n u_n, \quad u_n \in Tx_n, \\
z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \quad w_n \in Ty_n, \\
x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
\]

where \( u_n \in Tx_n, w_n \in Ty_n \) such that \( \|u_n - w_n\| \leq 2D(Tx_n, Ty_n) \), \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1) \) satisfy the following conditions:

i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

ii. \( 0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1} \).

Then, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w \).
If, in Theorem 3.1 we assume that $P_{T_i}, i = 1, \ldots, N$ are Lipschitz hemicontractive-type mappings, then by Lemma 2.6, the requirement that $T_i(p) = \{p\}$ may not be needed. Thus, we obtain the following corollary.

**Corollary 3.2.** Let $H$ be a real Hilbert space and $K$ be a non-empty, closed and convex subset of $H$. Let $T_i : K \to CB(K), i = 1, 2, \ldots, N$, be a finite family of multivalued mappings. Let $P_{T_i}, i = 1, 2, \ldots, N$, be Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1, 2, \ldots, N$, respectively. Assume that $I - P_{T_i}, i = 1, \ldots, N$ are demiclosed and $F = \cap_{i=1}^{N} F(T_i)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by

\[
\begin{align*}
  y_n &= (1 - \beta_n)x_n + \beta_n u_n, \quad u_n \in P_{T_n}x_n, \\
  z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \quad w_n \in P_{T_n}y_n, \\
  x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
\]

(30)

where $u_n \in P_{T_n}x_n, w_n \in P_{T_n}y_n$ such that $\|u_n - w_n\| \leq 2D(P_{T_n}x_n, P_{T_n}y_n)$ and $T_n := T_{n(mod N)+1}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1, \forall n \geq 1$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1}}$, $\forall n \geq 1$ for $L := \max\{L_i : 1, 2, \ldots, N\}$.

Then, $\{x_n\}$ converges strongly to some point $p \in F$ nearest to $w$.

If, in Theorem 3.1, we assume that $T_i, i = 1, \ldots, N$, are $k$-strictly pseudocontractive-type mappings then by Proposition 2.1, $T_i$ are Lipschitz with $L_i = \frac{1 + \sqrt{k_i}}{1 - \sqrt{k_i}}, i = 1, \ldots, N$. Hence, we have the following theorem.

**Theorem 3.2.** Let $H$ be a real Hilbert space and $K$ be a non-empty, closed and convex subset of $H$. Let $T_i : K \to CB(K), i = 1, 2, \ldots, N$, be a finite family of $k$-strictly pseudocontractive-type mappings. Assume that $F = \cap_{i=1}^{N} F(T_i)$ is non-empty with $T_i(p) = \{p\}$, $\forall p \in F(T)$ and for each $i = 1, 2, \ldots, N$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by

\[
\begin{align*}
  y_n &= (1 - \beta_n)x_n + \beta_n u_n, \quad u_n \in T_{n}x_n, \\
  z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \quad w_n \in T_{n}y_n, \\
  x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
\]

(31)
where \( u_n \in T_n x_n, w_n \in T_n y_n \) such that \( ||u_n - w_n|| \leq 2D(T_n x_n, T_n y_n) \) and \( T_n := T_n(\text{mod} \ N)+1 \). \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \subset (0, 1) \) satisfy the following conditions:

i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \).

ii. \( 0 < \alpha \leq \gamma_n \leq \beta \leq \frac{1}{\sqrt{4L^2 + 1} + 1}, \forall n \geq 1 \) for \( L := \max \{ \frac{1 + \sqrt{k_i}}{1 - \sqrt{k_i}}, i = 1, \ldots, N \} \).

Then, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w \).

The following follows from Theorem 3.2. For the detail we refer the reader to [31]

**Corollary 3.3.** Let \( H \) be a real Hilbert space and \( K \) be a non-empty, closed and convex subset of \( H \). Let \( T_i : K \to CB(K), i = 1, 2, \ldots, N, \) be a finite family of nonexpansive-type mappings. Assume that \( F = \cap_{i=1}^{N} F(T_i) \) is non-empty with \( T_i(p) = \{ p \}, \forall p \in F(T) \) and for each \( i = 1, 2, \ldots, N \). Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 = w \in K \) by

\[
\begin{align*}
  y_n &= (1 - \beta_n)x_n + \beta_n u_n, \quad u_n \in T_n x_n, \\
  z_n &= \gamma_n w_n + (1 - \gamma_n)x_n, \quad w_n \in T_n y_n, \\
  x_{n+1} &= \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{align*}
\]

where \( u_n \in T_n x_n, w_n \in T_n y_n \) such that \( ||u_n - w_n|| \leq 2D(T_n x_n, T_n y_n) \) and \( T_n := T_n(\text{mod} \ N)+1 \). \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \subset (0, 1) \) satisfy the following conditions:

i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \).

ii. \( 0 < \alpha \leq \gamma_n \leq \beta \leq \frac{1}{\sqrt{5} + 1}, \forall n \geq 1 \).

Then, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w \).

**4. Convergence results for finite family of lipschitz pseudocontractive multi-valued mappings**

**Theorem 4.1** Let \( H \) be a real Hilbert space and \( K \) be a non-empty, closed and convex subset of \( H \). Let \( T_i : K \to CB(K), i = 1, 2, \ldots, N, \) be a finite family of Lipschitz pseudocontractive multi-valued mappings with Lipschitz constants \( L_i, i = 1, 2, \ldots, N, \) respectively. Assume that \( \mathcal{F} = \cap_{i=1}^{N} F(T_i) \) is non-empty. Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 = w \in K \)
Thus, let \( u_n \in T_n x_n, w_n \in T_n y_n \) such that \( \| u_n - w_n \| \leq 2D(T_n x_n, T_n y_n) \) and \( T_n := T_n(\text{mod } N)+1 \). \( \{ \alpha_n \} \), \( \{ \beta_n \} \), \( \{ \gamma_n \} \subset (0, 1) \) satisfy the following conditions:

(i) \( 0 \leq \alpha_n \leq c < 1 \), \( \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \)

(ii) \( 0 < \alpha \leq \gamma_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1} \), \( \forall n \geq 1 \) for \( L := \max\{ L_i : i = 1, 2, \ldots, N \} \).

Then, \( \{ x_n \} \) converges strongly to some point \( p \in F \) nearest to \( w \).

**Proof.** Let \( p = P_F(w) \). Now, using Lemma 2.1 we get that

\[
\| x_{n+1} - p \|^2 = \| \alpha_n (w - p) + (1 - \alpha_n) (z_n - p) \|^2 \\
\leq \alpha_n \| w - p \|^2 + (1 - \alpha_n) \| z_n - p \|^2 \\
= \alpha_n \| w - p \|^2 + (1 - \alpha_n) \| \gamma_n (w_n - p) + (1 - \gamma_n) (x_n - p) \|^2 \\
= \alpha_n \| w - p \|^2 + (1 - \alpha_n) \| \gamma_n \| w_n - p \| \|^2 + (1 - \alpha_n) (1 - \gamma_n) \\
\times \| x_n - p \|^2 - (1 - \alpha_n) \gamma_n (1 - \gamma_n) \| w_n - x_n \|^2 \\
\leq \alpha_n \| w - p \|^2 + (1 - \alpha_n) (1 - \gamma_n) \| x_n - p \|^2 + (1 - \alpha_n) \gamma_n \\
\| y_n - p \|^2 + \| y_n - p - (w_n - p) \|^2 \\
- (1 - \alpha_n) \gamma_n (1 - \gamma_n) \| w_n - x_n \|^2 \\
= \alpha_n \| w - p \|^2 + (1 - \alpha_n) (1 - \gamma_n) \| x_n - p \|^2 + (1 - \alpha_n) \gamma_n \\
\| y_n - p \|^2 + \| y_n - w_n \|^2 - (1 - \alpha_n) \gamma_n (1 - \gamma_n) \| w_n - x_n \|^2.
\]

Thus,

\[
\| x_{n+1} - p \|^2 \leq \alpha_n \| w - p \|^2 + (1 - \alpha_n) (1 - \gamma_n) \| x_n - p \|^2 + (1 - \alpha_n) \gamma_n \\
\times \| y_n - p \|^2 + (1 - \alpha_n) \gamma_n \| y_n - w_n \|^2 - (1 - \alpha_n) \gamma_n (1 - \gamma_n) \| w_n - x_n \|^2.
\]
On the other hand, using (33), the assumption that \( \|u_n - w_n\| \leq 2D(T_n x_n, T_n y_n) \), Lemma 2.1 and \( T_n \) is Lipschitz,

\[
\|y_n - w_n\|^2 = \|(1 - \beta_n)(x_n - w_n) + \beta_n(u_n - w_n)\|^2
\]

\[
= (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2
\]

\[
\leq (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n4D(T_n x_n, T_n y_n)^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2
\]

\[
\leq (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n4L^2\|x_n - y_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2
\]

\[
= (1 - \beta_n)\|x_n - w_n\|^2 + 4\beta_n^2 L^2\|x_n - u_n\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2.
\]

Hence,

(35) \[
\|y_n - w_n\|^2 \leq (1 - \beta_n)\|x_n - w_n\|^2 - \beta_n(1 - \beta_n - 4L^2 \beta_n^2)\|x_n - u_n\|^2
\]

Again, using the assumption that \( T_n \) is pseudocontractive,

\[
\|y_n - p\|^2 = \|(1 - \beta_n)x_n + \beta_n u_n - p\|^2
\]

\[
= \|(1 - \beta_n)(x_n - p) + \beta_n(u_n - p)\|^2
\]

\[
= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2
\]

\[
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n[\|x_n - p\|^2 + \|x_n - u_n\|^2]
\]

\[
- \beta_n(1 - \beta_n)\|x_n - u_n\|^2
\]

\[
= \|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2.
\]

Thus,

(36) \[
\|y_n - p\|^2 \leq \|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2.
\]

Now, substituting (35), (36) into (34),

\[
\|x_{n+1} - p\|^2 \leq \alpha_n\|w - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 + (1 - \alpha_n)\gamma_n\|x_n - p\|^2
\]

\[
+ (1 - \alpha_n)\gamma_n\beta_n^2\|x_n - u_n\|^2 + (1 - \alpha_n)\gamma_n(1 - \beta_n)\|x_n - w_n\|^2
\]

\[
- \beta_n(1 - \alpha_n)\gamma_n(1 - \beta_n - 4L^2 \beta_n^2)\|u_n - x_n\|^2
\]

\[
- (1 - \alpha_n)\gamma_n(1 - \gamma_n)\|w_n - x_n\|^2,
\]
which reduces to

\[
\|x_{n+1} - p\|^2 \leq \alpha_n \|w - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \beta_n (1 - \alpha_n) \\
\times \gamma_n (1 - 2\beta_n - 4L^2\beta_n^2) \|u_n - x_n\|^2 + (1 - \alpha_n) \gamma_n (y_n - \beta_n) \|x_n - w_n\|^2.
\]

From hypothesis (ii) in (33) we have that

\[
1 - 2\beta_n - 4L^2\beta_n^2 \geq 1 - 2\beta - 4L^2\beta^2
\]

\[
\gamma_n \leq \beta_n.
\]

Using (38) and (39) in (37) we get that

\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n \|w - p\|^2.
\]

Thus, by induction, we have

\[
\|x_{n+1} - p\|^2 \leq \max\{\|x_1 - p\|^2, \|w - p\|^2\}, \forall n \geq 1.
\]

This implies that \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) are all bounded. Furthermore, from (33), Lemma 2.8 and (37) we get that

\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n) (\gamma_n w_n + (1 - \gamma_n)x_n) + \alpha_n w - p)\|^2
\]

\[
= \|(1 - \alpha_n) ((\gamma_n w_n + (1 - \gamma_n)x_n) - p) + \alpha_n (w - p)\|^2
\]

\[
\leq (1 - \alpha_n) \|\gamma_n w_n + (1 - \gamma_n)x_n - p\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle
\]

\[
= (1 - \alpha_n) [\gamma_n \|w_n - p\|^2 + (1 - \gamma_n) \|x_n - p\|^2
\]

\[-\gamma_n (1 - \gamma_n) \|x_n - w_n\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle]
\]

\[
\leq (1 - \alpha_n) [\gamma_n \|y_n - p\|^2 + \|y_n - w_n\|^2 + (1 - \gamma_n) \|x_n - p\|^2
\]

\[-\gamma_n (1 - \gamma_n) \|x_n - w_n\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle]
\]

\[
\leq (1 - \alpha_n) \gamma_n \|y_n - p\|^2 + (1 - \alpha_n) \gamma_n \|y_n - w_n\|^2
\]

\[(1 - \alpha_n) (1 - \gamma_n) \|x_n - p\|^2 - (1 - \alpha_n) \gamma_n (1 - \gamma_n) \|x_n - w_n\|^2
\]

\[+ 2\alpha_n \langle w - p, x_{n+1} - p \rangle,
\]
which implies that
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\gamma_n\|x_n - p\|^2 + (1 - \alpha_n)\alpha^n \beta_n \|x_n - u_n\|^2 + (1 - \alpha_n)\gamma_n \\
\times \left[ (1 - \beta_n)\|x_n - w_n\|^2 - \beta_n(1 - \beta_n - 4L^2 \beta_n^2)\|x_n - u_n\|^2 \right] \\
+ (1 - \alpha_n)(1 - \gamma_n)\|x_n - p\|^2 - (1 - \alpha_n)\gamma_n(1 - \gamma_n)\|w_n - x_n\|^2 \\
+ 2\alpha_n\langle w - p, x_{n+1} - p \rangle \\
\leq (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\gamma_n \beta_n(1 - 2\beta_n - 4L^2 \beta_n^2)\|x_n - u_n\|^2 \\
+ 2\alpha_n\langle w - p, x_{n+1} - p \rangle + (1 - \alpha_n)\gamma_n(\gamma_n - \beta_n)\|x_n - w_n\|^2.
\]

This implies that,
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\gamma_n \beta_n(1 - 2\beta_n - 4L^2 \beta_n^2) \\
\times \|x_n - u_n\|^2 + 2\alpha_n\langle w - p, x_{n+1} - p \rangle;
\]
and hence by (i) and (ii) we have
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\gamma_n \beta_n(1 - 2\beta_n - 4L^2 \beta_n^2)\|x_n - u_n\|^2 \\
+ 2\alpha_n\langle w - p, x_{n+1} - p \rangle.
\]

Now we consider the following two cases:

**Case 1.** Suppose that there exists \(n_0 \in \mathbb{N}\) such that \(\{\|x_n - p\|\}\) is non-increasing, \(\forall n \geq n_0\). Then, we get that \(\{\|x_n - p\|\}\) is convergent. So, from (42) we have that
\[
(1 - c)\alpha^n(1 - 2\beta_n - 4L^2 \beta_n^2)\|x_n - u_n\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
+ 2\alpha_n\langle w - p, x_{n+1} - p \rangle.
\]

Thus, from the fact that \(\alpha_n \to 0\), we get
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0.
\]

Now, from (33) we obtain that
\[
y_n - x_n = \beta_n(u_n - x_n) \to 0,
\]
and hence we get that

\[ \|z_n - x_n\| = \gamma_n \|w_n - x_n\| = \gamma_n \|w_n - u_n + u_n - x_n\| \]
\[ \leq \gamma_n \|w_n - u_n\| + \gamma_n \|u_n - x_n\| \]
\[ \leq 2\gamma_n D(T_n y_n, T_n x_n) + \gamma_n \|u_n - x_n\| \]
\[ \leq 2\gamma_n L \|y_n - x_n\| + \gamma_n \|u_n - x_n\| \to 0. \tag{44} \]

Furthermore, from (33), (44), the fact that \(\|w - z_n\|\) is bounded and \(\alpha_n \to 0\), we obtain

\[ \|x_{n+1} - x_n\| = \|x_{n+1} - z_n + z_n - x_n\| \]
\[ \leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \]
\[ = \alpha_n \|w - z_n\| + \|z_n - x_n\| \to 0. \tag{45} \]

But then, since, \(\|x_{n+i} - x_n\| \leq \|x_{n+i} - x_{n+i-1}\| + \ldots + \|x_{n+1} - x_n\|\), we get that

\[ \|x_{n+i} - x_n\| \to 0, \forall i = 1, 2, \ldots, N. \tag{46} \]

Thus, from (43) and (46) we obtain that

\[ \|u_{n+i} - x_n\| \leq \|u_{n+i} - x_{n+i}\| + \|x_{n+i} - x_n\| \to 0, \forall i = 1, 2, \ldots, N. \tag{47} \]

Now we show that for \(i \in \{1, 2, \ldots, N\}\), \(\lim_{n \to \infty} d(x_n, T_{n+i} x_n) = 0\). But from (46), Lemma 2.10, (47) and Lipschitz property of \(T_i\) for each \(i \in \{1, 2, \ldots, N\}\) we get that

\[ d(x_n, T_{n+i} x_n) = d(x_n, T_{n+i} x_{n+i}) + D(T_{n+i} x_n, T_{n+i} x_{n+i}) \]
\[ \leq \|x_n - u_{n+i}\| + L \|x_n - x_{n+i}\| \to 0, \tag{48} \]

which is the required result. Now, since \(\{\|x_n - p\|\}\) converges, there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_{n+1}\}\) such that

\[ \limsup_{n \to \infty} \langle w - p, x_{n+1} - p \rangle = \lim_{j \to \infty} \langle w - p, x_{n_j+1} - p \rangle, \]

and \(x_{n_j} \to z\), for some \(z \in K\). Now, from (45) we get \(x_{n_j} \to z\). Hence, from (48) and the fact that \(T_i, \forall i = 1, \ldots, N\) are demiclosed by Lemma 2.13 , we get that \(z \in F(T_i), \forall i = 1, \ldots, N\). i.e.,
\[ z \in \mathcal{F}. \] Therefore, by Lemma 2.4 we obtain that
\[
\limsup_{n \to \infty} \langle w - p, x_{n+1} - p \rangle = \lim_{j \to \infty} \langle w - p, x_{n_j+1} - p \rangle = \langle w - p, z - p \rangle \leq 0.
\] (49)

Now, from (42) we have that
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p \rangle.
\] (50)

It then follows from (50), (49) and Lemma 2.9 that
\[ \|x_n - p\| \to 0 \text{ i.e., } x_n \to p. \]

**Case 2.** Suppose there exists a subsequence \( \{n_k\} \) of \( \{n\} \) such that
\[ \|x_{n_k} - p\| < \|x_{n_k+1} - p\|, \quad \forall k \in \mathbb{N}. \]

Thus, by Lemma 2.5, there is a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty, \|x_{m_k} - p\| \leq \|x_{m_k+1} - p\| \) and \( \|x_k - p\| \leq \|x_{m_k+1} - p\|, \quad \forall k \in \mathbb{N}. \) Now, from (42) and the fact that \( \alpha_n \to 0 \)
we get that \( x_{m_k} - u_{m_k} \to 0 \), when \( u_{m_k} \in T_i x_{m_k}, \forall i = 1, \ldots, N. \) Hence as in Case 1, \( x_{m_k+1} - x_{m_k} \to 0 \)
and that
\[
\limsup_{k \to \infty} \langle w - p, x_{m_k+1} - p \rangle \leq 0.
\] (51)

From (42) we have that
\[
\|x_{m_k+1} - p\|^2 \leq (1 - \alpha_{m_k})\|x_{m_k} - p\|^2 + 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle
\] (52)

and since \( \|x_{m_k} - p\| \leq \|x_{m_k+1} - p\|, \) (52) implies that
\[
\alpha_{m_k} \|x_{m_k} - p\|^2 \leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle
\]
\[
\leq 2\alpha_{m_k} \langle w - p, x_{m_k+1} - p \rangle,
\]

which implies that
\[
\|x_{m_k} - p\|^2 \leq 2 \langle w - p, x_{m_k+1} - p \rangle.
\]

So, from (51) we get that \( \|x_{m_k} - p\|^2 \to 0 \) and hence this with (52) give that \( \|x_{m_k+1} - p\| \to 0. \)

But, \( \|x_k - p\| \leq \|x_{m_k+1} - p\|, \forall k \in \mathbb{N}. \) Thus, \( x_k \to p. \) Therefore, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w. \)
Remark 4.1. We note that, since every Lipschitz $k$-strongly pseudocontractive mapping is Lipschitz pseudocontractive mapping the above theorem holds for a finite family of Lipschitz $k$-strongly pseudocontractive mappings.

If, in Theorem 4.1 we consider a single Lipschitz pseudocontractive mapping we get the following corollary.

Corollary 4.1. Let $H$ be a real Hilbert space and $K$ be a non-empty, closed and convex subset of $H$. Let $T : K \rightarrow CB(K)$, be Lipschitz pseudocontractive multi-valued mapping with Lipschitz constant $L$. Assume that $F(T)$ is non-empty and that $T(p) = \{p\}$, $\forall p \in F(T)$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by

$$
\begin{cases}
  y_n = (1 - \beta_n) x_n + \beta_n u_n, \\
  z_n = \gamma_n w_n + (1 - \gamma_n) x_n, \\
  x_{n+1} = \alpha_n w + (1 - \alpha_n) z_n, \quad n \geq 1,
\end{cases}
$$

(53)

where $u_n \in Tx_n, w_n \in Ty_n$ such that $\|u_n - w_n\| \leq 2D(Tx_n, Ty_n)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

1. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

2. $0 < \alpha \leq \gamma_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ converges strongly to some point $p \in \mathcal{F}$ nearest to $w$.

Proof. Put $T_i := T$, $\forall i = 1, \ldots, N$ in (33) and the scheme reduces to (53). Now, as in (41) and (42),

$$
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n) \|x_n - p\|^2 - (1 - \alpha_n) \gamma_n \beta_n (1 - 2\beta_n - 4L^2 \beta_n^2)
\times \|x_n - u_n\|^2 + 2\alpha_n \langle w - p, x_n + 1 - p \rangle, \quad u_n \in Tx_n
\leq (1 - \alpha_n) \|x_n - p\|^2 - (1 - c) \alpha^2 (1 - 2\beta - 4L^2 \beta^2) \|x_n - u_n\|^2
+ 2\alpha_n \langle w - p, x_n + 1 - p \rangle
\leq (1 - \alpha_n) \|x_n - p\|^2 + 2\alpha_n \langle w - p, x_n + 1 - p \rangle
$$

The rest of the proof is as in Theorem 4.1.
If, in Theorem 4.1 we assume that $P_{T_{i}}, i = 1, \ldots, N$ are Lipschitz pseudocontractive mappings, then by Lemma 2.6, the requirement that $T_{i}(p) = \{p\}$ may not be needed. Thus, we get the following Corollary.

**Corollary 4.2.** Let $H$ be a real Hilbert space and $K$ be a non-empty, closed and convex subset of $H$. Let $T_{i} : K \to CB(K), i = 1, 2, \ldots, N$, be a finite family of multi-valued mappings. Let $P_{T_{i}}, i = 1, 2, \ldots, N$, be Lipschitz pseudocontractive mappings with Lipschitz constants $L_{i}, i = 1, 2, \ldots, N$, respectively. Suppose also that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_{i})$ is non-empty. Let $\{x_{n}\}$ be the sequence generated from an arbitrary $x_{1} = w \in K$ by

$$
\begin{align*}
    y_{n} &= (1 - \beta_{n})x_{n} + \beta_{n}u_{n}, \\
    z_{n} &= \gamma_{n}w_{n} + (1 - \gamma_{n})x_{n}, \\
    x_{n+1} &= \alpha_{n}w + (1 - \alpha_{n})z_{n}, \quad n \geq 1,
\end{align*}
$$

(54)

where $u_{n} \in P_{T_{i}}x_{n}, w_{n} \in P_{T_{i}y_{n}}$ such that $\|u_{n} - w_{n}\| \leq 2D(P_{T_{i}}x_{n}, P_{T_{i}y_{n}})$, and $T_{n} := T_{n(mod N)+1}$.  

$\{\alpha_{n}\}, \{\beta_{n}\}, \{\gamma_{n}\} \subset (0, 1)$ satisfy the following conditions:

i. $0 \leq \alpha_{n} \leq c < 1, \forall n \geq 1$ such that $\lim_{n \to \infty} \alpha_{n} = 0$ and $\sum_{n=1}^{\infty} \alpha_{n} = \infty$,

ii. $0 < \alpha \leq \gamma_{n} \leq \beta \leq \frac{1}{\sqrt{4L^{2} + 1}}$, $\forall n \geq 1$ for $L := \max\{L_{i} : 1, 2, \ldots, N\}$.

Then, $\{x_{n}\}$ converges strongly to some point $p \in \mathcal{F}$ nearest to $w$.

If, in Theorem 4.1 we assume that $P_{T_{i}} : K \to CBC(K), i = 1, \ldots, N$ are Lipschitz pseudocontractive mappings, then $P_{T_{i}}(x)$ is singleton and hence the following corollary follows.

**Corollary 4.3** Let $H$ be a real Hilbert space and $K$ be a non-empty, closed and convex subset of $H$. Let $T_{i} : K \to CBC(K), i = 1, 2, \ldots, N$, be a finite family of multi-valued mappings. Let $P_{T_{i}}, i = 1, 2, \ldots, N$, be Lipschitz pseudocontractive mappings with Lipschitz constants $L_{i}, i = 1, 2, \ldots, N$, respectively. Suppose also that $\mathcal{F} = \bigcap_{i=1}^{N} F(P_{T_{i}})$ is non-empty. Let $\{x_{n}\}$ be the sequence generated from an arbitrary $x_{1} = w \in K$ by

$$
\begin{align*}
    y_{n} &= (1 - \beta_{n})x_{n} + \beta_{n}P_{T_{i}}x_{n}, \\
    z_{n} &= \gamma_{n}P_{T_{i}y_{n}} + (1 - \gamma_{n})x_{n}, \\
    x_{n+1} &= \alpha_{n}w + (1 - \alpha_{n})z_{n}, \quad n \geq 1,
\end{align*}
$$

(55)

where $T_{n} := T_{n(mod N)+1}$ and $\{\alpha_{n}\}, \{\beta_{n}\}, \{\gamma_{n}\} \subset (0, 1)$ satisfy the following conditions:
i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

ii. \( 0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1}, \forall n \geq 1 \) for \( L := \max\{L_i : 1, 2, \ldots, N\} \).

Then, \( \{x_n\} \) converges strongly to some point \( p \in \mathcal{F} \) nearest to \( w \).

Next, we state and prove a convergence theorem for a common zero of a finite family of monotone mappings.

**Theorem 4.2** Let \( H \) be a real Hilbert space. Let \( A_i : H \to CB(H) \), \( i = 1, 2, \ldots, N \) be a family of Lipschitz monotone mappings with Lipschitz constants, \( 1 + L_i, i = 1, 2, \ldots, N, \) respectively. Assume \( \mathcal{F} := \bigcap_{i=1}^{N} N(A_i) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated from an arbitrary \( x_1 = w \in H \) by

\[
\begin{cases}
y_n = x_n - \beta_n u_n, \\
z_n = x_n - \gamma_n w_n, \\
x_{n+1} = \alpha_n w + (1 - \alpha_n)z_n, \quad n \geq 1,
\end{cases}
\]

(56)

where \( u_n \in A_n x_n, w_n \in A_n y_n \) such that \( ||u_n - w_n|| \leq 2D(x_n - A_n x_n, y_n - A_n y_n) + ||x_n - y_n|| \), and \( A_n := A_{n(mod N) + 1}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1) \) satisfy the following conditions:

i. \( 0 \leq \alpha_n \leq c < 1, \forall n \geq 1 \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

ii. \( 0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1}, \forall n \geq 1 \) for \( L := \max\{L_i, i = 1, \ldots, N\} \).

Then, \( \{x_n\} \) converges strongly to a common zero point of \( A_1, A_2, \ldots, A_n \) nearest to \( w \).

**Proof.** Let \( T_i x := (I - A_i)x, i = 1, 2, \ldots, N \). Then \( T_i, i = 1, 2, \ldots, N \) are Lipschitz pseudocontractive mappings with Lipschitz constants \( L_i := (1 + L_i) \) and

\[
\bigcap_{i=1}^{N} F(T_i) = \bigcap_{i=1}^{N} N(A_i) \neq \emptyset.
\]

Now replacing \( A_i \) with \( (I - T_i) \) for each \( i = 1, 2, \ldots, N \) in (56) we get the Scheme (33). Hence the result follows from Theorem 4.1 .

In Theorem 4.2, if we consider a single Lipschitz monotone mapping we obtain,

**Corollary 4.4.** Let \( H \) be a real Hilbert space. Let \( A : H \to CB(H) \) be a Lipschitz monotone mapping with Lipschitz constant, \( L \). Assume \( N(A) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated
from an arbitrary $x_1 = w \in H$ by

$$
\begin{align*}
    y_n &= x_n - \beta_n u_n, \\
    z_n &= x_n - \gamma_n w_n, \\
    x_{n+1} &= \alpha_n w + (1 - \alpha_n) z_n, \quad n \geq 1,
\end{align*}
$$

where, $u_n \in Ax_n, w_n \in Ay_n$ such that $||u_n - w_n|| \leq 2D(x_n - Ax_n, y_n - Ay_n) + ||x_n - y_n||$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2 + 1} + 1}$, $\forall n \geq 1$ for $L' := 1 + L$.

Then, $\{x_n\}$ converges strongly to a zero of $A$, nearest to $w$.

Remark 4.2. Our work improves Theorem 1 and Theorem 2 of Song and Wang [29] and Theorem 2.7 of Shahzad and Zegeye [27] and extends the work of Woldeamanuel et al. [32] for Lipschitz pseudocontractive multi-valued case. It also extends the work of Daman and Zegeye [6] for the multivalued case.

Conflict of Interests

The authors declare that there is no conflict of interests.

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References


