ABSORBING MAPS AND COMMON FIXED POINT THEOREM IN Menger SPACE

ANUJA P. CHOUSHAN, * ABHAY S. RANADIVE

1Department of Mathematics, Govt. Dr. W.W. Patankar PG Girls College, Durg (C. G.) INDIA
2Department of Pure and Applied Mathematics, Guru Ghasidas Central University, Bilaspur, (C.G.), INDIA

Abstract. In this paper, we prove a common fixed point theorem, using newly defined absorbing maps in Menger space. Our result generalizes the result of Razani et al [11].

Keywords: Menger space, absorbing map, reciprocal continuous, semi-compatible mapping.

2000 AMS Subject Classification: 47H10; 54H25.

1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [6]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic function analysis. It is a probabilistic generalization in which we assign a distribution function $F_{x,y}$ to any two points $x$ and $y$. Sehgal et al [14] initiated the study of fixed points in probabilistic metric spaces. Moreover, this theory is studied in Menger probabilistic metric space by many authors such as Schweizer and Sklar [16], Razani et-al [11] and etc. It is observed by many authors that contraction condition in metric space may be exactly translated into PM-space endowed with min
norms. Mishra [7] introduced the concept of compatible self-maps in Menger space and proved the existence of a common fixed point of a pair of compatible maps using a contractive condition. Subsequently, Singh et al [17] introduced the concept of semi-compatible mapping in Menger space and proved a fixed point theorem using semi-compatibility. Ranadive et-al [12] introduced the concept of absorbing mapping in metric space and prove a common fixed point theorem in this space. Moreover they [12] observe that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non-compatible maps. In [9], Ranadive et-al introduced absorbing maps in fuzzy metric space and prove a common fixed point theorem in this spaces. Recently we [13] introduce absorbing maps in Menger space and prove a fixed point theorem in this space.

In this paper we prove, a common fixed point theorem using reciprocally continuity and employing absorbing mapping with semi-compatibility. In order to do this, we recall some definitions, Lemmas, prepositions and known results from [7], [17], and [18].

2. Preliminaries

Definition 2.1. A mapping $F : R \to R^+$ is called a distribution if it is non decreasing left-continuous with $\inf\{F(t) : t < R = 0\}$ and $\sup\{F(t) : t < R = 1\}$. We shall denote by $L$ the set of all distribution functions while $H$ will always denote the specific distribution function denoted by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. A probabilistic metric space (PM-space) is an ordered pair $(X, F)$ where $X$ is a nonempty set and $F : X \times X \to L$ is defined by $(p, q) \mapsto F_{p,q}$ where L is the set of all distribution function, i.e., $L = \{F_{p,q} : p, q \in X\}$, where the functions $F_{p,q}$ satisfy:

1. $F_{p,q}(x) = 1$ for all $x > 0$, if and only if $p = q$,
2. $F_{p,q}(0) = 0$,
3. $F_{p,q} = F_{q,p}$,
4. if $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x+y) = 1$, for all $p, q, r \in X$ $x, y \geq 0$. 

**Definition 2.3.** A mapping $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t-norm if the following conditions are satisfied:

1. $t(a, 1) = a$ for every $a \in [0, 1]$,
2. $t(a, b) = t(b, a)$ for every $a, b \in [0, 1]$,
3. $t(c, d) \geq t(a, b)$ for $c \geq a, d \geq b$ $a, b, c, d \in [0, 1]$;
4. $t(t(a, b), c) = t(a, t(b, c))$ $a, b, c \in [0, 1]$.

**Definition 2.4.** A Menger probabilistic metric space is a triplet $(X, F, t)$, where $(X, F)$ is PM-space and $t$ is a t-norm and the following inequality holds:

$$F_{p,r}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y))$$

for all $p, q, r \in X$ and for all $x, y \geq 0$.

**Definition 2.5.** Let $(X, F, t)$ be a Menger space with t-norm

1. A sequence $\{x_n\}$ in $X$ is said to convergent to $x$ in $X$ (written as $x_n \rightarrow x$) if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.
2. The sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $\lambda > 0$, there is a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) \geq 1 - \lambda$, whenever $n, m \geq N$.
3. A Menger space $(X, F, t)$ is said to be complete if every Cauchy sequence in $X$ if each Cauchy sequence in $X$ is convergent to some point in $X$.

**Definition 2.6.** Two self maps $A$ and $S$ of a Menger space $(X, F, t)$ are said to be reciprocal continuous if and only if $\lim_{n \rightarrow \infty} ASx_n = Az$ and $\lim_{n \rightarrow \infty} SAx_n = Sz$, whenever there exists a subsequence in $X$ such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$, for some $z$ in $X$. It is well known that if $A$ and $S$ are both continuous then obviously they are reciprocal continuous but converse is not true.

**Definition 2.7.** Self maps $A$ and $S$ of a Menger space $(X, F, t)$ are said to be weakly compatible if they commute at their coincidence points, i.e. if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$. 
Definition 2.8. Self maps $A$ and $S$ of a Menger space $(X, F, t)$ are said to be compatible if $F_{A s_{n}, s_{n}}(x) \to 1$ for all $x > 0$, whenever $\{p_{n}\}$ is a sequence in $X$ such that $A p_{n}, s_{n}(x) \to z$, for some $z$ in $X$, as $n \to \infty$.

Definition 2.9. Self maps $A$ and $S$ of a Menger space $(X, F, t)$ are said to be semi-compatible if $F_{A s_{n}, s_{n}}(x) \to 1$ for all $x > 0$, whenever $\{p_{n}\}$ is a sequence in $X$ such that $A p_{n}, s_{n} \to z$ for some $z$ in $X$, as $n \to \infty$. It follows that if $(A, S)$ is semi-compatible and $A y = S y$, then $A S y = S A y$. Thus if the pair $(A, S)$ is semi-compatible, then it is weak-compatible. But the converse is not true.

Recently we [13] define a new notion of mappings called absorbing mapping in Menger space as follows.

Definition 2.10. Let $A$ and $S$ be two self maps on a Menger space $(X, F, t)$, then $A$ is called $S$-absorbing if there exist some real number $R > 0$ such that $F_{s_{n}, s_{n}}(t) \geq F_{s_{n}, s_{n}}(t)$ for all $x$ in $X$. Similarly $S$ is called $A$-absorbing if there exist some real number $R > 0$ such that $F_{A s_{n}, s_{n}}(t) \geq F_{A s_{n}, s_{n}}(t)$ for all $x$ in $X$. Thus we see that, the notion of absorbing maps is different from other generalizations of commutativity.

Lemma 2.11. Let $\{p_{n}\}$ be a sequence in a Menger space $(X, F, t)$ with continuous $t$-norm and $t(x, x) \geq x$. Suppose for all $x \in [0, 1]$, $\exists k \in (0, 1)$ such that for all $x > 0$ and $n \in N$,

$$F_{p_{n}, p_{n+1}}(k x) \geq F_{p_{n-1}, p_{n}}(x)$$

Then $\{p_{n}\}$ is a Cauchy sequence in $X$.

Lemma 2.12. Let $(X, F, t)$ be a Menger space if there exists $k \in (0, 1)$ such that for $p, q \in X$ and $x > 0$

$$F_{p, q}(k x) \geq F_{p, q}(x).$$

Then $p = q$.

The following theorem is proved by Razani, and Shirdaryazdi [11].

Theorem RS. Let $P_{1}, P_{2}, \ldots, P_{2n}, Q_{0}$ and $Q_{1}$ are self maps on a complete Menger space $(X, F, \Delta)$ with continuous $t$-norm with $\Delta(x, x) \geq x$ for all $x \in [0, 1]$, satisfying conditions:

1. $Q_{0}(X) \subseteq P_{1}P_{3} \ldots P_{2n-1}(X), Q_{1}(X) \subseteq P_{2}P_{4} \ldots P_{2n}(X), \ldots$
\( P_2(P_4...P_{2n}) = (P_4...P_{2n})P_2, \)
\[ P_2P_4(P_6...P_{2n}) = (P_6...P_{2n})P_2P_4, \]
\[ \vdots \]
\[ P_2...P_{2n-2}(P_{2n}) = (P_{2n})P_2...P_{2n-2}, \]
\[ Q_0(P_4...P_{2n}) = (P_4...P_{2n})Q_0, \]
\[ Q_0(P_6...P_{2n}) = (P_6...P_{2n})Q_0, \]
\[ \vdots \]
\[ Q_0P_{2n} = P_{2n}Q_0, \]
\[ P_1(P_3...P_{2n-1}) = (P_3...P_{2n-1})P_1, \]
\[ P_1P_3(P_5...P_{2n-1}) = (P_5...P_{2n-1})P_1P_3, \]
\[ \vdots \]
\[ P_1...P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1...P_{2n-3}, \]
\[ Q_1(P_3...P_{2n-1}) = (P_3...P_{2n-1})Q_1, \]
\[ Q_1(P_5...P_{2n-1}) = (P_5...P_{2n-1})Q_1, \]
\[ \vdots \]
\[ Q_1P_{2n-1} = P_{2n-1}Q_1, \]
\( P_2...P_{2n} \) or \( Q_0 \) is continuous,
\( Q_0, P_2...P_{2n} \) is compatible and \( Q_1, P_1...P_{2n-1} \) is weakly compatible,
\(\) There exists \( \alpha \in (0, 1) \) such that
\[ F_{Q_0u,Q_1v}(\alpha x) \geq \min\{F_{P_2P_4...P_{2n}u,Q_0v}(x), F_{P_1P_3...P_{2n-1}u,Q_1v}(x), F_{P_1P_3...P_{2n-1}v,Q_0u}(\beta x), F_{P_2P_4...P_{2n}u,Q_1v((2-\beta)x), F_{P_2P_4...P_{2n}u,P_1P_3...P_{2n-1}v}(x)\}, \]
for all \( u, v \in X, \beta \in (0, 2) \) and \( x > 0 \). Then \( P_1, P_2, ...P_{2n}, Q_0 \) and \( Q_1 \) have a unique common fixed point in \( X \).

3. Main results

In this paper, we prove a fixed point theorem using reciprocally continuity and employing absorbing mappings with semi-compatibility.
Theorem 3.1. Let $R_1, R_2, \ldots, R_{2n}$, $S_0$ and $S_1$ are self maps on a complete Menger space $(X, F, \Delta)$ with continuous t-norm with $\Delta(x, x) \geq x$ for all $x \in [0, 1]$, satisfying conditions:

1. $S_0(X) \subseteq R_1R_3\ldots R_{2n-1}(X), S_1(X) \subseteq R_2R_4\ldots R_{2n}(X)$,
2. $R_2(R_4\ldots R_{2n}) = (R_4\ldots R_{2n})R_2$,
   $R_2R_4(R_6\ldots R_{2n}) = (R_6\ldots R_{2n})R_2R_4$,
   $\vdots$
   $R_2\ldots R_{2n-2}(R_{2n}) = R_{2n}(R_2\ldots R_{2n-2})$,
3. $S_0(R_4\ldots R_{2n}) = (R_4\ldots R_{2n})S_0$,
   $S_0(R_6\ldots R_{2n}) = (R_6\ldots R_{2n})S_0$,
   $\vdots$
   $S_0R_{2n} = R_{2n}S_0$,
4. $R_1(R_3\ldots R_{2n-1}) = (R_3\ldots R_{2n-1})R_1$,
   $R_1R_3(R_5\ldots R_{2n-1}) = (R_5\ldots R_{2n-1})R_1R_3$,
   $\vdots$
   $R_1\ldots R_{2n-3}(R_{2n-1}) = R_{2n-1}(R_1\ldots R_{2n-3})$,
5. $S_1(R_3\ldots R_{2n-1}) = (R_3\ldots R_{2n-1})S_1$,
   $S_1(R_5\ldots R_{2n-1}) = (R_5\ldots R_{2n-1})S_1$,
   $\vdots$
   $S_1R_{2n-1} = R_{2n-1}S_1$;
6. $S_1$ is $(R_1\ldots R_{2n-1})$ absorbing;
7. There exists $k \in (0, 1)$ such that
   $F_{S_0p, S_1q}(kx) \geq \min\{F_{R_2R_4\ldots R_{2n}p, S_1q}(2-\beta)x, F_{R_2R_4\ldots R_{2n}p, S_0p}(x), F_{R_1R_3\ldots R_{2n-1}q, S_1q}(x), F_{R_2R_4\ldots R_{2n}p, R_1R_3\ldots R_{2n-1}q}(x)\}$
   for all $p, q \in X, \beta \in (0, 2)$ and $x > 0$. If $(S_0, R_2\ldots R_{2n})$ is reciprocal continuous, semi-compatible maps. Then $R_1, R_2, \ldots R_{2n}, S_0$ and $S_1$ have a unique common fixed point in $X$.

Proof Let $x_0 \in X$, from condition (1) there exists $x_1, x_2 \in X$ such that $S_0x_0 = R_1R_3\ldots R_{2n-1}x_1 = y_0$ and $S_1x_1 = R_2R_4\ldots R_{2n}x_2 = y_1$, in general we can construct $\{x_n\}$ and $\{y_n\}$ in $X$ such that $S_0x_{2n} = R_1R_3\ldots R_{2n-1}x_{2n+1} = y_{2n}$ or $S_0x_{2n} = R_1R_3\ldots R_{2n-1}x_{2n+1} = y_{2n}$. If $(S_0, R_2\ldots R_{2n})$ is reciprocal continuous, semi-compatible maps. Then $R_1, R_2, \ldots R_{2n}, S_0$ and $S_1$ have a unique common fixed point in $X$. 
Putting $p = x_2n, q = x_{2n+1}, x > 0$ and $\beta = 1 - \alpha$ with $\alpha \in (0, 1)$ in contractive condition, we have

\[ F_{S_{2}x_{2n}, S_{1}x_{2n+1}}(kx) \geq \min\{F_{R_{1}^x x_{2n}, S_{1}x_{2n+1}}((2 - (1 - \alpha)x), F_{R_{1}^x x_{2n}, S_{0}x_{2n+2}}(x), F_{R_{2}^x x_{2n+1}, S_{1}x_{2n+1}}(x), F_{R_{2}^x x_{2n+1}, R_{2}^x x_{2n+1}}(x)\}\]

\[ F_{y_{2n}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n-1}, y_{2n+1}}((1 + \alpha)x), F_{y_{2n-1}, y_{2n+1}}(x), F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n-1}, y_{2n+1}}(x)\}\]

As t-norm is continuous, letting $\alpha \to 1$, we have,

\[ F_{y_{2n}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n-1}, y_{2n+1}}(x), F_{y_{2n}, y_{2n+1}}(x)\}. \]

Again $p = x_{2n+2}, q = x_{2n+1}$, in contractive condition, gives

\[ F_{S_{0}x_{2n+2}, S_{1}x_{2n+1}}(kx) \geq \min\{F_{R_{1}^x x_{2n+2}, S_{1}x_{2n+1}}((1 + \alpha)x), F_{R_{1}^x x_{2n+2}, S_{0}x_{2n+2}}(x), F_{R_{2}^x x_{2n+1}, S_{1}x_{2n+1}}(x), F_{R_{2}^x x_{2n+1}, R_{2}^x x_{2n+1}}(x)\}\]

\[ F_{y_{2n+2}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n+1}, y_{2n+1}}((1 + \alpha)x), F_{y_{2n+1}, y_{2n+2}}(x), F_{y_{2n}, y_{2n+1}}(x), F_{y_{2n+1}, y_{2n}}(x)\}\]

\[ F_{y_{2n+2}, y_{2n+1}}(kx) \geq \min\{F_{y_{2n+1}, y_{2n+2}}(x), F_{y_{2n}, y_{2n+1}}(x)\}. \]

Consequently; for all $n$ we have

\[ F_{y_{n}, y_{n+1}}(kx) \geq \min\{F_{y_{n-1}, y_{n}}(k^{-1}x), F_{y_{n}, y_{n+1}}(k^{-m}x)\}. \]

So, $F_{y_{n}, y_{n+1}}(k^{-m}x) \to 1$ as $m \to \infty$ for any $t > 0$, it follows that

\[ F_{y_{n}, y_{n+1}}(\alpha x) \geq F_{y_{n-1}, y_{n}}(x), \]
for all \( n \in N \) and \( x > 0 \). Therefore, by Lemma 2.11 \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, therefore \( \{y_n\} \to z \) in \( X \) and its subsequences \( \{S_{1x_{2n+1}}\}, \{R_1R_3...R_{2n-1}x_{2n+1}\}, \{S_0x_{2n}\} \) and \( \{R_2R_4...R_{2n}x_{2n}\} \to z \)

Case (I): By the reciprocally continuity and Semi-compatibility of maps \((S_0, R_2R_4...R_{2n})\), we have

\[
\lim_{n \to \infty} S_0(R_2R_4...R_{2n}x_{2n}) = S_0z,
\]

and

\[
\lim_{n \to \infty} R_2R_4...R_{2n}(S_0x_{2n}) = R_2R_4...R_{2n}z = R'_1z,
\]

which implies that \( S_0z = R'_1z \). We claim \( S_0z = R'_1z = z \).

Step (i): By using contractive condition, with \( p = z, q = x_{2n+1} \) and \( \beta = 1 \), \( R'_1 = R_2R_4...R_{2n}, R'_2 = R_1R_3...R_{2n-1} \) we have

\[
F_{S_0z, s_1x_{2n+1}}(kx) \geq \min\{F_{R'_1z, s_1x_{2n+1}}(x), F_{R'_1z, s_0z}(x), F_{R'_2x_{2n+1}, s_1x_{2n+1}}(x), F_{R'_1z, R'_2x_{2n+1}}(x)\},
\]

Letting \( n \to \infty \), we get

\[
F_{S_0z,z}(kx) \geq \min\{F_{S_0z,z}(x), F_{S_0z,s_0z}(x), F_{z,z}(x), F_{S_0z,z}(x)\}.
\]

Thus by Lemma 2.12, we have

\[
S_0z = R'_1z = z.
\]

Step (ii): Putting \( p = R_4...R_{2n}z, q = x_{2n+1}, \alpha = 1 \) with \( R'_1 = R_2R_4...R_{2n}, R'_2 = R_1R_3...R_{2n-1} \), in contractive condition, we have

\[
F_{S_0R_4...R_{2n}z, s_1x_{2n+1}}(kx) \geq \min\{F_{R'_1R_4...R_{2n}z, s_1x_{2n+1}}(x), F_{R'_1R_4...R_{2n}z, s_0R_4...R_{2n}z}(x),
\]

\[
F_{R'_2x_{2n+1}, s_1x_{2n+1}}(x), F_{R'_1R_4...R_{2n}z, R'_2x_{2n+1}}(x)\},
\]

Letting \( n \to \infty \), we see that

\[
F_{R_4...R_{2n}z,z}(kx) \geq \min\{F_{R_4...R_{2n}z,z}(x), F_{R_4...R_{2n}z,R_4...R_{2n}z}(x), F_{z,z}(x), F_{R_4...R_{2n}z,z}(x)\},
\]

\[
F_{R_4...R_{2n}z,z}(kx) \geq F_{R_4...R_{2n}z,z}(x).
\]
Because \( R_2(R_4...R_{2n}) = (R_4...R_{2n})R_2 \) and \( S_0(R_4...R_{2n}) = (R_4...R_{2n})S_0 \), in (2); So \( R_4...R_{2n}z = z \). Therefore \( R_2z = z \). Continuing this procedure, we can obtain the following result:

\[
S_0z = z = R_2z = R_4z = ... = R_{2n}z = z.
\]

Since \( S_0(X) \subseteq R_1R_3...R_{2n-1}(X) \), there exists \( u \in X \), such that

\[
z = S_0z = R_1R_3...R_{2n-1}u \text{ or } z = S_0z = R_2'u.
\]

Step (iii): Putting \( p = x_{2n}, q = u, R_1' = R_2R_4...R_{2n} \) and \( R_2' = R_1R_3...R_{2n-1} \) with \( \beta = 1 \) in contractive condition, we have

\[
F_{S_0x_{2n}, s_{1u}(kx)} \geq \min\{F_{R_1'x_{2n}, s_{1u}(x)}, F_{R_1'x_{2n}, S_0x_{2n}(x)}, F_{R_2'u, s_{1u}(x)}, F_{R_2'u, x_{2n}z}, F_{R_2'u}(x)\},
\]

Letting \( n \to \infty \), we obtain that

\[
F_{z,s_{1u}(kx)} \geq \min\{F_{z,s_{1u}(x)}, F_{z,z}(x), F_{z,s_{1u}(x)}, F_{z,z}(x)\}.
\]

Therefore \( z = S_{1u} \). Hence \( z = S_1u = R_2'u = R_1R_3...R_{2n-1}u \).

Since \( S_1 \) is \( R_1R_3...R_{2n-1} \)-absorbing, we have,

\[
F_{R_1R_3...R_{2n-1}u, R_1R_3...R_{2n-1}S_{1u}(x)} \geq F_{R_1R_3...R_{2n-1}u, S_{1u}(x/R)} \geq 1.
\]

\[
\Rightarrow R_1R_3...R_{2n-1}u = R_2'u = R_2'S_{1u}.
\]

Therefore

\[
z = R_2'z
\]

Step(iv): Putting \( p = x_{2n}, q = z \) with \( \beta = 1 \) and \( R_1' = R_2R_4...R_{2n} \) and \( R_2' = R_1R_3...R_{2n-1} \) in contractive condition, we have

\[
F_{S_0x_{2n}, s_{1z}(kx)} \geq \min\{F_{R_1'x_{2n}, s_{1z}(x)}, F_{R_1'x_{2n}, S_0x_{2n}(x)}, F_{R_2'z, s_{1z}(x)}, F_{R_2'z, x_{2n}z}, F_{R_2'z}(x)\}.
\]

Letting \( n \to \infty \), we get

\[
F_{z, s_{1z}(kx)} \geq \min\{F_{z, s_{1z}(x)}, F_{z, z(x)}, F_{z, s_{1z}(x)}, F_{z, z(x)}\},
\]

Therefore

\[
z = S_{1z} = R_2'z
\]
Step (v): Putting \( p = x_{2n}, q = R_{3...2n-1} \) with \( \beta = 1 \) and \( R'_1 = R_2R_4...R_{2n} \) and \( R'_2 = R_1R_3...R_{2n-1} \) in contractive condition, we have

\[
F_{S_0x_{2n}, s_1R_3...2n-1z}(kx) \geq \min\{F_{R'_1x_{2n}, s_1R_3...2n-1z}(x), F_{R'_1x_{2n}, s_0x_{2n}}(x),
F_{R'_2R_3...2n-1z, s_1R_3...2n-1z}(x), F_{R'_2R_3...2n-1z, s_0x_{2n}}(x)\},
\]

Again letting \( n \to \infty \), so that

\[
F_{z, R_3...2n-1z}(kx) \geq \min\{F_{z, R_3...2n-1z}(x), F_{z, z}(x), F_{R_3...2n-1z, R_3...2n-1z}(x), F_{z, R_3...2n-1z}(x)\},
\]

Therefore

\[
z = R_3...2n-1z.
\]

Because \( R_1\{R_3...2n-1\} = \{R_3...2n-1\}R_1 \) and \( S_1\{R_3...2n-1\} = \{R_3...2n-1\}S_1 \), we obtain \( R_3...2n-1z = z \). Therefore \( R_1z = z \). Continuing this procedure, we obtain the following results; \( S_1 = R_1z = R_3z = ... = R_{2n-1}z \). So,

\[
S_0z = S_1z = R_1z = R_2z = ... = R_{2n-1} = R_{2n}z = z.
\]

**Uniqueness:** Let \( w \) be another common fixed point of \( S_0, S_1, R_1R_3...2n-1 \) and \( R_2R_4...R_{2n} \) putting \( p = z \) and \( q = w \) with \( \beta = 1 \), \( R'_1 = R_2R_4...R_{2n} \), \( R'_2 = R_1R_3...R_{2n-1} \) in contractive condition, we have

\[
F_{S_0z, s_1w}(kx) \geq \min\{F_{R'_1z, s_1w}(x), F_{R'_1z, s_0w}(x), F_{R'_2w, s_1w}(x), F_{R'_2z, s_0w}(x)\},
\]

\[
F_{z,w}(kx) \geq \min\{F_{z,w}(x), F_{z,z}(x), F_{w,w}(x), F_{z,w}(x)\},
\]

i.e. \( z = w \). Hence \( z \) is unique common fixed point of maps.

**Example:** Let \((X,d)\) be a metric space with the usual metric \( d \) where \( X = [0,1] \) and \((X,F,\ast)\) be the induced Menger space with \( F_{x,y}(t) = H(t - d(x,y)) \) for all \( x, y \in X, t > 0 \). Clearly \((X,F,\ast)\) is complete Menger space where t-norm \( \ast \) is defined by \( a \ast b = \min\{a, b\} \) for all \( a, b \in [0,1] \).

Let \( S_0, S_1, R'_1 \) and \( R'_2 \) be maps from \( X \) into it self defined as \( S_0(X) = x/6, S_1(X) = \)
$0, R'_1(X) = x/3, R'_2(X) = x/2 \forall x \in X.$ Then $S_0(X) = [0,1/6] \subseteq [0,1/2] = R'_2(X)$ and $S_1(X) = \{0\} \subseteq [0,1/3] = R'_1(X).$ Clearly all conditions of main Theorem are satisfied if $\lim_{n \to \infty} x_n = 0,$ where $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} S_0 x_n = \lim_{n \to \infty} R'_2 x_n = 0$ and $\lim_{n \to \infty} = S_1 x_n = \lim_{n \to \infty} R'_1 x_n = 0$ for some $0 \in X.$ Thus all condition of the main Theorem are satisfied. This completes the proof.

References


