Available online at http://scik.org
Adv. Fixed Point Theory, 6 (2016), No. 2, 194-206
ISSN: 1927-6303

WEAK AND STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR LIPSCHITZ PSEUDO-CONTRACTIVE MAPS IN HILBERT SPACES

V. E. INGBIANFAM, F. A. TSAV*, I. S. IORNUMBE

Department of Mathematics and Computer Science, Benue State University, Makurdi, Nigeria

Abstract. In this paper, let $\mathscr{K}$ be a closed convex subset of a real Hilbert space $\mathscr{H}$ and $T:\mathscr{K}\to\mathscr{K}$ a Lipschitz pseudo-contractive map such that $F(T)\neq\emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$. For $x_1\in\mathscr{K}$, let $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = P_k[(1-\alpha_n-\gamma_n)x_n+\gamma_n T y_n], \\ y_n = (1-\beta_n)x_n+\beta_n T x_n, n\geq 1. \end{cases}$$

Under some mild conditions on parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, we prove that our new iterative algorithm converges strongly to a fixed point of T. No compactness assumption is imposed on T and no further requirement is imposed on $F(T)$.

Keywords: Fixed point; Lipschitz Pseudo-contractive map; weak convergence; strong convergence; Ishikawa algorithm; Mann iteration.

2010 AMS Subject Classification: 47H10.

1. Introduction

*Corresponding author
E-mail address: tfaondo@gmail.com
Received December 11, 2015

Let $\mathscr{K}$ be a nonempty subset of a real Hilbert space $\mathscr{H}$. Throughout this paper, $\mathscr{H}$ shall denote a real Hilbert space. The map $T:\mathscr{K}\to\mathscr{H}$ is said to be Lipschitz or Lipschitz continuous if there exists a constant $L\geq 0$ such that

$$\|Tx-Ty\|\leq L\|x-y\|,\quad \forall x,y\in\mathscr{K}. \tag{1}$$

If $L=1$, then T is said to be nonexpansive; and if $L<1$, then T is said to be a contraction. It is easy to see from (1) that every contraction is nonexpansive and every nonexpansive map is Lipschitz.

A map $T:\mathscr{K}\to\mathscr{H}$ is called pseudo-contractive if

$$\langle Tx-Ty,x-y\rangle\leq\|x-y\|^2,\ \forall x,y\in\mathscr{K}. \tag{2}$$

It is clear that (2) is equivalent to

$$\|Tx-Ty\|^2\leq\|x-y\|^2+\|(I-T)x-(I-T)y\|^2,\quad \forall x,y\in\mathscr{K}. \tag{3}$$

An important subclass of the class of pseudo-contractive maps is the class of $\lambda-$strictly pseudo-contractive maps. T is said to be $\lambda-$strictly pseudo-contractive (see for example [1]) if there exists $\lambda\in[0,1)$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+\lambda\|(I-T)x-(I-T)y\|^2,\ \forall\, x,y\in\mathscr{K}. \tag{4}$$

It is well known that if T is $\lambda-$strictly pseudo-contractive, then T is Lipschitz with Lipschitz constant $L=\dfrac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}$. We use $F(T)$ to denote the set of fixed points of T.

The Mann iteration scheme $\{x_n\}_1^\infty$ generated from arbitrary $x_1\in\mathscr{K}$ by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_nTx_n,\ n\geq 1, \tag{5}$$

where the control sequence $\{x_n\}_{n=1}^\infty$ in $[0,1]$ satisfying some appropriate conditions has been successfully employed in approximating fixed points (when they exist) of nonexpansive maps. This success has not carried over to the more general class of pseudo-contractions. If $\mathscr{K}$ is a compact convex subset of a Hilbert space $\mathscr{H}$ and $T:\mathscr{K}\to\mathscr{K}$ is Lipschitz , then, by Schauder fixed point theorem, T has a fixed point in $\mathscr{K}$. All efforts to approximate such a fixed point by means of the Mann sequence when T is also assumed to be pseudo-contractive proved to be abortive.

Hicks and Kubicek [17], gave an example of a discontinuous pseudo-contraction with unique fixed point for which the Mann iteration does not always converge. Borwein and Borwein [18], gave an example of a Lipschitz map (which is not pseudo-contractive) with a unique fixed point for which the Mann sequence fails to converge. For Lipschitz pseudo-contractive maps, the Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in \mathscr{K}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions
(i) $0 \leq \alpha_n \leq \beta_n < 1$;
(ii) $\lim\limits_{n\to\infty} \beta_n = 0$;
(iii) $\Sigma_{n\geq 0} \alpha_n \beta_n = \infty$
is usually applicable.
In real Hilbert spaces, one of the most general well known convergence theorems using the Mann iteration algorithm for the class of $\lambda-$strictly pseudo-contractive maps is the following:

Theorem 1.1. [7] *For $\mathscr{K}$ a nonempty closed convex subset of the Hilbert space $\mathscr{H}$, let $T : \mathscr{K} \to \mathscr{K}$ be a λ-strictly pseudo-contractive map with a nonempty fixed point set $F(T)$ and let $\{x_n\}_{n=1}^{\infty}$ be a real sequence in $(0, 1-\lambda)$ satisfying the conditions*
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$
(ii) $\Sigma_{n=1}^{\infty} \alpha_n (1-\alpha_n-\lambda) = \infty$.
Then the Mann iteration algorithm $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.

If $\lambda = 0$ in Theorem 1.1., we obtain weak convergence theorem for nonexpansive maps. To obtain strong convergence of Mann to a fixed point of a $\lambda-$strictly pseudo-contractive map or even a nonexpansive map in the setting of Theorem 1.1., additional conditions are usually required on T or the subset $\mathscr{K}$. (see for example [1] to [6]).

Recently, Yao and Li [16] studied a modified Mann iteration algorithm and proved strong convergence of the modified algorithm to a fixed point of a $\lambda-$strictly pseudo-contractive map in real Hilbert spaces. They proved the following:

Theorem 1.2. *Let $\mathscr{H}$ be a real Hilbert space and $T : \mathscr{H} \to \mathscr{H}$ be a $\lambda-$strictly pseudo-contractive map such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$. Assume* *that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$*;*

(iii) $\beta_n \in [\varepsilon, (1-\lambda)(1-\alpha_n))$ *for some* $\varepsilon > 0$.

Then the sequence $\{x_n\}$ *generated by* $x_{n+1} = (1-\alpha_n-\beta_n)x_n+\beta_nTx_n$, $n \geq 0$ *strongly converges to a fixed point of* T.

Clearly, the modified Mann iteration algorithm reduces to the normal Mann iteration algorithm when $\alpha_n = 0$. For $L-$Lipschitzian pseudo-contractive maps for which the Ishikawa algorithm rather than the Mann algorithm has been applicable, Ishikawa [9] first proved the following:

Theorem 1.3. *For* $\mathscr{K}$ *a nonempty convex compact subset of a Hilbert space* $\mathscr{H}$ *and* $T : \mathscr{K} \to \mathscr{K}$ *an* $L-$*Lipschitzian pseudo-contractive map, let* $\{x_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *be real sequences satisfying the conditions:*

(i) $0 \leq \alpha_n \leq \beta_n < 1$*;*

(ii) $\lim_{n\to\infty} \beta_n = 0$*;*

(iii) $\Sigma_{n=1}^{\infty}\alpha_n\beta_n = \infty$.

Then the Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ *generated from an arbitrary* $x_1 \in K$ *by* $x_{n+1} = (1-\alpha_n)x_n+\alpha_nT[(1-\beta_n)x_n+\beta_nTx_n]$, $n \geq 1$ *converges to a fixed point of* T.

Since the appearance of Theorem 1.3., many authors have extended it in various forms (see for example [8] to [12]). However, strong convergence has not been achieved without compactness assumption on T or $\mathscr{K}$; or other requirements on the set of fixed point $F(T)$; or complete modification of the scheme to a hybrid algorithm (see [8] to [12]).

It is our purpose in this paper to complement Yao and Li [16] by introducing a modified Ishikawa algorithm analogous to the modified Mann iteration algorithm studied in [16]. We further prove that our modified Ishikawa algorithm converges strongly to a fixed point of a Lipschitz pseudo-contractive map in real Hilbert spaces.

2. Preliminaries

We shall make use of the following lemmas in section three.

Lemma 2.1. [9] *Let $\mathscr{H}$ be a Hilbert space. Then for all $x,y \in \mathscr{H}$, $\alpha \in [0,1]$ the following equality holds:*

$\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2.$

Lemma 2.2. *Let $\mathscr{H}$ be a real Hilbert space. Then there holds the following well known results:*

(i) $\|x-y\|^2 = \|x\|^2-2\langle x,y\rangle+\|y\|^2 \quad \forall x,y \in \mathscr{H}$;

(ii) $\|x+y\|^2 \leq \|x\|^2+2\langle y,x+y\rangle \quad \forall x,y \in \mathscr{H}$.

Lemma 2.3. [14] *Assume $\{a_n\}$ is a sequence of non negative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n+\gamma_n\delta_n, n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\Re$, the reals, such that*

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

The following lemmas can be found in [15], [19].

Lemma 2.4. (Demi-closed Principle) *Let $\mathscr{H}$ be a Hilbert space, $\mathscr{K}$ a closed convex subset of $\mathscr{H}$ and $T : \mathscr{K} \to \mathscr{K}$ a continuous pseudo-contractive map, then*

(i) $F(T)$ is a closed convex subset of $\mathscr{K}$.

(ii) $I-T$ is demi-closed at zero, i.e., if $\{x_n\}$ is a sequence in $\mathscr{K}$ such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.

Lemma 2.5. [15] *Let $T : \mathscr{K} \to \mathscr{H}$ be non-expansive and $y \in \mathscr{K}$ be a weak cluster point of a sequence $\{x_n\}_{n=0}^{\infty}$. If $\|Tx_n - x_n\| \to 0$, then $y \in F(T)$.*

Lemma 2.6. [20] *Let $\mathscr{H}$ be a real Hilbert space. If $\{x_n\}$ is a sequence in $\mathscr{H}$ weakly convergent to z, then* $\limsup_{n\to\infty}\|x_n-y\|^2 = \limsup_{n\to\infty}\|x_n-z\|^2+\|z-y\|^2 \;\forall y \in \mathscr{H}$.

3. Main result

Theorem 3.1. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{K}$ be a closed convex subset of $\mathscr{H}$. Let $T : \mathscr{K} \to \mathscr{K}$ be a L-Lipschitz pseudo-contractve map such that $F(T) \neq \emptyset$. For $x_1 \in \mathscr{K}$, let $\{x_n\}$*

be generated iteratively by

$$(7) \qquad \begin{cases} x_{n+1} = P_k[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n], \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, n \geq 1. \end{cases}$$

Assume the sequences $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\} \in (0,1)$ *satisfy*

(i) $\beta_n(1-\alpha_n) > \gamma_n \forall n \geq 1$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma\alpha_n = \infty$*;*

(iii) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{[\sqrt{1+L^2}+1]}$ *for all* $n \geq 1$*.*

Then the sequence $\{x_n\}$ *generated by (7) strongly converges to a fixed point of* T*.*

Proof. Since $F(T) \neq \emptyset$, we can take $p \in F(T)$. From (7) we have

$$\begin{aligned} \|x_{n+1}-p\| &= \|P_k[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] - p\| \\ &\leq \|(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n - p\| \\ &= \|(1-\alpha_n-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p) - \alpha_n p\| \\ &\leq \|(1-\alpha_n-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p)\| + \alpha_n\|p.\| \qquad (8) \end{aligned}$$

Now, consider

$$\begin{aligned} &(1-\alpha_n-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p)\|^2 \\ &= \|(1-\alpha_n)[(1-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p)] + \alpha_n[-\gamma_n x_n + \gamma_n T y_n]\|^2. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} \|(1-\alpha_n-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p)\|^2 &= (1-\alpha_n)\|(1-\gamma_n)(x_n-p) + \gamma_n(Ty_n-p)\|^2 \\ &\quad +\alpha_n\|\gamma_n(Ty_n-x_n)\|^2 - \alpha_n(1-\alpha_n)\|x_n-p\|^2 \\ &= (1-\alpha_n)[(1-\gamma_n)\|x_n-p\|^2 + \gamma_n\|Ty_n-p\|^2 \\ &\quad -\gamma_n(1-\gamma_n)\|x_n-Ty_n\|^2] + \alpha_n\gamma_n^2\|Ty_n-x_n\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n-p\|^2 \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\|x_n-p\|^2-\gamma_n(1-\alpha_n)\|x_n-p\|^2 \\
&\quad +\gamma_n(1-\alpha_n)\|Ty_n-p\|^2 \\
&\quad -\gamma_n(1-\gamma_n)(1-\alpha_n)\|x_n-Ty_n\|^2 \\
&\quad +\alpha_n\gamma_n^2\|Ty_n-x_n\|^2-\alpha_n(1-\alpha_n)\|x_n-p\|^2 \\
&\leq (1-\alpha_n)^2\|x_n-p\|^2-\gamma_n(1-\alpha_n)\|x_n-p\|^2 \\
&\quad +\gamma_n(1-\alpha_n)[\|y_n-p\|^2+\|y_n-Ty_n\|^2] \\
&\quad -\gamma_n(1-\alpha_n-\gamma_n)\|Ty_n-x_n\|^2. \qquad (9)
\end{aligned}
$$

But

$$
\begin{aligned}
\|y_n-p\|^2 &= \|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\|^2 \\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\|Tx_n-p\|^2-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\beta_n\|x_n-Tx_n\|^2-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\
&= \|x_n-p\|^2+\beta_n^2\|x_n-Tx_n\|^2. \qquad (10)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|y_n-Ty_n\|^2 &= \|(1-\beta_n)(x_n-Ty_n)+\beta_n(x_n-Ty_n\|^2 \\
&= (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n\|Tx_n-Ty_n\|^2-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n-Ty_n\|^2+\beta_nL^2\|x_n-y_n\|^2-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\
&\leq (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n^3L^2\|x_n-Tx_n\|^2-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\
&= (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n[L^2\beta_n^2+\beta_n-1]\|x_n-Tx_n\|^2. \qquad (11)
\end{aligned}
$$

Substituting (10) and (11) in (9), we have

$$
\begin{aligned}
&\|(1-\alpha_n-\gamma_n)(x_n-p)+\gamma_n(Ty_n-p)\|^2\\
&=(1-\alpha_n)^2\|x_n-p\|^2-\gamma_n(1-\alpha_n)\|x_n-p\|^2\\
&+\gamma_n(1-\alpha_n)[\|x_n-p\|^2+\beta_n^2\|x_n-Tx_n\|^2]\\
&+(1-\beta_n)\|x_n-Ty_n\|^2+\beta_n(L^2\beta_n^2+\beta_n-1)\|x_n-Tx_n\|^2]\\
&-\gamma_n(1-\gamma_n)(1-\alpha_n)\|x_n-Ty_n\|^2+\alpha_n\gamma_n^2\|Ty_n-x_n\|^2\\
&=(1-\alpha_n)^2\|x_n-p\|^2-\gamma_n[\beta_n(1-\alpha_n)-\gamma_n]\|x_n-Ty_n\|^2+\gamma_n\beta_n(1-\alpha_n)[L^2\beta_n^2+2\beta_n-1]\|x_n-Tx_n\|^2\\
&=(1-\alpha_n)^2\|x_n-p\|^2-\gamma_n[\beta_n(1-\alpha_n)-\gamma_n]\|x_n-Ty_n\|^2-\gamma_n\beta_n(1-\alpha_n)[1-L^2\beta_n^2-2\beta_n]\|x_n-Tx_n\|^2.
\end{aligned}
$$

Therefore, by conditions (iii) and (i), we have that $1-2\beta_n-L^2\beta_n^2>0$ and $\beta_n(1-\alpha_n)>\gamma_n\Rightarrow$

$$\|(1-\alpha_n-\gamma_n)(x_n-p)+\gamma_nTy_n\|^2\leq(1-\alpha_n)^2\|x_n-p\|^2\ \leq(1-\alpha_n)\|x_n-p\|. \tag{12}$$

Combining (12) and (8), we have

$$\|x_{n+1}-p\|\leq(1-\alpha_n)\|x_n-p\|+\alpha_n\|p\|\leq max\{\|x_n-p\|,\|p\|\}. \tag{13}$$

By induction, we have $\|x_{n+1}-p\|\leq max\{\|x_0-p\|,\|p\|\}$, which implies that $\{x_n\}$ is bounded. Furthermore, from (7), Lemma 2.4., and following the methods above, we get that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_k[(1-\alpha_n-\gamma_n)x_n+\gamma_nTy_n]-p\|\\
&\leq \|x_n-p-\gamma_n(x_n-Ty_n)-\alpha_nx_n\|^2\\
&\leq \|x_n-p-\gamma_n(x_n-Ty_n)\|^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&= \|(1-\gamma_n)x_n+\gamma_nTy_n-p\|^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle. \qquad (14)
\end{aligned}
$$

Now, consider

$$\|\gamma_nTy_n+(1-\gamma_n)x_n-p\|^2 = \gamma_n\|Ty_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2-\gamma_n(1-\gamma_n)\|Ty_n-x_n\|^2.$$

From (10) and (11), we have

$$\begin{aligned}\|\gamma_n Ty_n+(1-\gamma_n)x_n-p\|^2 &\leq \gamma_n[\|y_n-p\|^2+\|y_n-Ty_n\|^2]+(1-\gamma_n)\|x_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)\|Ty_n-x_n\|^2\\ &\leq \gamma_n[\|x_n-p\|^2+\beta_n^2\|x_n-Tx_n\|^2]+\gamma_n[(1-\beta_n)\|x_n-Ty_n\|^2\\ &\quad +\beta_n(L^2\beta_n^2+\beta_n-1)\|x_n-Tx_n\|^2]+(1-\gamma_n)\|x_n-p\|^2\\ &\quad -\gamma_n(1-\gamma_n)\|Ty_n-x_n\|^2\\ &= \|x_n-p\|^2-\gamma_n\beta_n[1-2\beta_n-L^2\beta_n^2]\|x_n-Tx_n\|^2\\ &\quad +\gamma_n(\gamma_n-\beta_n)\|x_n-Ty_n\|^2.\end{aligned}$$

From (iii), we have that $(\gamma_n-\beta_n)\leq 0$ and $1-2\beta_n-L^2\beta_n^2>0$, $\forall n\geq 1$, we have that

$$\|\gamma_n Ty_n+(1-\gamma_n)x_n-p\|^2 \leq \|x_n-p\|^2-\gamma_n\beta_n[1-2\beta_n-L^2\beta_n^2]\|x_n-Tx_n\|^2. \tag{15}$$

Substituting (15) in (14) we have that

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2-\gamma_n\beta_n[1-2\beta_n-L^2\beta_n^2]\|x_n-Tx_n\|^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle. \tag{16}$$

Since $\{x_n\}$ is bounded, there exists a constant $M\geq 0$ *such that* $-2\langle x_n,x_{n+1}\rangle\leq M\ \forall\ n\geq 0$. Consequently, from (16), we get

$$\|x_{n+1}-p\|^2-\|x_n-p\|^2+\gamma_n\beta_n[1-2\beta_n-L^2\beta_n^2]\|x_n-Tx_n\|^2\leq M\alpha_n. \tag{17}$$

We now consider the following two cases:

Case 1: Suppose that there exists $n_0\in N$ such that $\{\|x_n-p\|\}$ is non increasing. Then, we have that $\{\|x_n-p\|\}$ is convergent. Clearly we have that $\|x_{n+1}-p\|^2-\|x_n-p\|^2\to 0$. This, together with (ii) and (17), imply that

$$\|x_n-Tx_n\|\to 0. \tag{18}$$

By Lemma 2.5. and (18), it is easy to see that $\omega_\omega(x_n)\subset F(T)$, where $\omega_\omega(x_n)=\{x:\exists x_{ni}\rightharpoonup x\}$ is the weak $\omega-$limit set of $\{x_n\}$. This implies that $\{x_n\}$ converges weakly to a fixed point x^* of T. Indeed, if we take $x^*,\bar{x}\in\omega_\omega(x_n)$ and let $\{x_{ni}\}$ and $\{x_{mj}\}$ be sequences in $\{x_n\}$ such that

$x_{ni} \rightharpoonup x^*$ and $x_{mj} \rightharpoonup \bar{x}$, respectively. Since $\lim_{n\to\infty} \|x_n - z\|$ exists for $z \in F(T)$, by Lemma 2.6., we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\|^2 &= \lim_{j\to\infty} \|x_{mj} - x^*\|^2 \\
&= \lim_{j\to\infty} \|x_{mj} - \bar{x}\|^2 + \|\bar{x} - x^*\|^2 \\
&= \lim_{i\to\infty} \|x_{ni} - x^*\|^2 + 2\|\bar{x} - x^*\|^2 \\
&= \lim_{n\to\infty} \|x_n - x^*\|^2 + 2\|\bar{x} - x^*\|^2.
\end{aligned}$$

Hence, $\bar{x} = x^*$.

Next, we prove that $\{x_n\}$ strongly converges to x^*. Let $z_n = \gamma_n T y_n + (1-\gamma_n)x_n$. Then from (7) we have $x_{n+1} = P_k[z_n - \alpha_n x_n], \forall n \geq 0$. It follows that

$$x_{n+1} = P_k[(1-\alpha_n)z_n + \alpha_n(z_n - x_n)]. \tag{19}$$

At the same time, we note that $\|z_n - x^*\|^2 = \|x_n - x^* - \gamma_n(x_n - Ty_n)\|^2$. Using the same approach as in (15) we have

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma_n\beta_n[1 - 2\beta_n - L^2\beta_n^2]\|x_n - Tx_n\|^2 \leq \|x_n - x^*\|^2. \tag{20}$$

Furthermore, from (7) and (18) we have $\|y_n - x_n\| = \beta_n\|Tx_n - x_n\| \leq \|Tx_n - x_n\| \to 0$ as $n \to \infty$. Also, T is Lipschitz implies that,

$$\begin{aligned}
\|z_n - x_n\| &= \|\gamma_n(Ty_n - x_n) + \gamma_n(Tx_n - Tx_n)\| \\
&\leq \gamma_n L\|y_n - x_n\| + \gamma_n\|Tx_n - x_n\| \\
&\leq \gamma_n L\|Tx_n - x_n\| + \gamma_n\|Tx_n - x_n\| \to 0. \qquad (21)
\end{aligned}$$

Applying Lemma 2.2. to (19), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|P_k[(1-\alpha_n)z_n+\alpha_n(z_n-x_n)]-x^*\|^2 \\
&\leq \|(1-\alpha_n)(z_n-x^*)+\alpha_n(z_n-x_n)-\alpha_n x^*\|^2 \\
&\leq \|(1-\alpha_n)(z_n-x^*)+\alpha_n(z_n-x_n)\|^2-2\alpha_n\langle x^*,x_{n+1}-x^*\rangle \\
&= (1-\alpha_n)\|z_n-x^*\|^2+\alpha_n\|z_n-x_n\|^2-\alpha_n(1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*,x_{n+1}-x^*\rangle \\
&\leq (1-\alpha_n)\|x_n-x^*\|^2+\|z_n-x_n\|^2-\alpha_n(1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*,x_{n+1}-x^*\rangle \\
&\leq (1-\alpha_n)^2\|x_n-x^*\|^2+\|z_n-x_n\|^2-2\alpha_n\langle x^*,x_{n+1}-x^*\rangle.
\end{aligned}$$

It is clear that $\|z_n-x_n\|^2 \leq (\gamma_n L\|Tx_n-x_n\|+\gamma_n\|Tx_n-x_n\|)^2 \to 0 \; as \; n\to\infty$ and $\lim_{n\to\infty}\langle x^*,x_{n+1}-x^*\rangle = 0 \Rightarrow$

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*,x_{n+1}-x^*\rangle. \tag{22}$$

Since $\lim_{n\to\infty}\langle x^*,x_{n+1}-x^*\rangle = 0 \Rightarrow \|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2$, applying Lemma 2.3. to (22) we immediately deduce that $x_n \to x^*$.

Case 2. Assume that $\{\|x_n-p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n-p\|^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by $\tau(n) = max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}$. Clearly, τ is a non decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (17), it is easy to see that

$$\|x_{\tau(n)}-Tx_{\tau(n)}\|^2 = \frac{M\alpha_{\tau(n)}}{\gamma_{\tau(n)}\beta_{\tau(n)}[1-2\beta_{\tau(n)}-L^2\beta^2_{\tau(n)}]} \to 0.$$

Thus $\|x_{\tau(n)}-Tx_{\tau(n)}\| \to 0$. By the same similar argument as above in case 1, we conclude immediately that $x_{\tau(n)}$ weakly converges to x^* as $\tau(n)\to\infty$. At the same time we note that, for all $n_0 \geq n$, we have

$0 \leq \|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2 \leq \alpha_{\tau(n)}[2\langle x^*,x^*-x_{\tau(n)+1}\rangle-\|x_{\tau(n)}-x^*\|^2]$. Hence, we have that $\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 = 0$. Therefore, $\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_{\tau(n)} = \Gamma_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence we obtain, for all $n \geq n_0$, $0 \leq \Gamma_{\tau(n)} \leq max\{\Gamma_{\tau(n)},\Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty}\Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. This completes the proof.

Remark 3.2. A prototype example for our parameter is

$$\gamma_n = \frac{n^2-1}{4(n+1)^2(2+L^2)}, \quad \beta_n = \frac{n}{4(n+1)(2+L^2)}, \quad \alpha_n = \frac{1}{n+1}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197-228.

[2] S. A. Naimpally and K. L. Singh, Extensions of some fixed points theorems of Rhoades, J. Math. Anal. Appl. 96 (1983), 437-446.

[3] T. L. Hicks and J. D. Kubicek, On the Mann iteration process in a Hilbert spaces, J. Math. Anal. Appl. 59 (1977), 498-504.

[4] S. Maruster, Sur le calcul des zeros dun operateur discontinu par iteration, Canad. Math. Bull. 16 (1973), 541-544.

[5] S. Maruster, The solution by iteration of nonlinear equations in Hilbert spaces, Proc. Amer. Math. Soc. 63 (1977), 69-73.

[6] C. E. Chidume and S. A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc. 129 (2001), 2359-2363.

[7] G. Marino, H. K. Xu, Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336-346.

[8] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[9] M. O. Osilike and D. I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl.40 (2000), 559-567

[10] H. Zegeye, N. Shahzad, M. A. Alghamdi, Convergence of Ishikawas iterates method for pseudocontractive mappings, Nonlinear Anal. 74 (2011), 7304-7311.

[11] Q. Liu, The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings, J. Math. Anal. Appl. 148 (1990), 55-62.

[12] C. E. Chidume and C. Moore, Fixed point iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127 (1999), 1163-1170.

[13] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 33 (1998), 1-12.

[14] H.K. Xu, Iteration algorithms for nonlinear operators, J. London Math. Soc. 2 (2002), 240-256.

[15] G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudocontractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336-349.

[16] M. Li, Y. Yao, Strong Convergence of an iterative algorithm for λ-strictly pseudocontractive mappings in Hilbert spaces, An. St. Univ. Ovidius Constanta 18 (2010), 219-228.

[17] T.L. Hicks, J.R.kubicek, On the Mann iteration process in a Hilbert space, J. Math. Anal. Appl. 59 (1976), 498-504.

[18] D. Borwein and J.M. Bowein, Fixed point iterations for real functions, J. Math. Anal. Appl. 157 (1991), 112-126.

[19] H. Zhou, Convergence theorems of fixed points of $K-$strict pseudocontractions in Hilbert spaces, Nonlinear Anal. 69 (2008), 546-462.

[20] L. Liu, Ishikawa and Mann iteration processes with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194 (1995), 114-125.