Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 1, 59-67
https://doi.org/10.28919/afpt/2663
ISSN: 1927-6303

FIXED POINT THEOREMS FOR SELF MAP UNDER SOME CONTRACTIVE CONDITIONS RELATED TO Φ-MAP

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Abstract. In this paper, we prove some fixed point theorems for uniqueness of fixed points for self-map $T : X \to X$ under different contractive conditions related to Φ-map.

Keywords: S-metric; Φ-map; *S*-Cauchy sequence; *S*-convergent sequence.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Many authors tried to give generalization of metric spaces in several ways and obtained many results [1-12]. Gähler [6] and Dhage [1] introduced the concepts of 2-metric spaces and *D*-metric spaces respectively. Mustafa and Sims [8] introduced a new structure of generalized metric spaces which are called *G*-metric spaces. Sedghi *et al.* [4] introduced the concept of D^*-metric spaces which was modification of the definition of *D*-metric spaces. Recently, Sedghi *et al.* [5] have introduced the notion of S-metric spaces and have proved some fixed point theorems in S-metric spaces. In this paper, we consider ϕ as a Φ-map and prove some fixed point theorems for self-map $T : X \to X$ under different contractive conditions related to ϕ.

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Received January 30, 2016

2. Preliminaries

The following definitions and results will be used in the sequel:

Definition 2.1 [4]. Let X be a non-empty set. An S-metric on X is a function $S: X\times X\times X \to [0,\infty)$ that satisfies the following conditions, for each $x,y,z,a\in X$,

(S1) $S(x,y,z)\geq 0$,

(S2) $S(x,y,z)=0$ iff $x=y=z$,

(S3) $S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a)$.

The pair (X,S) is called an *S*-metric space.

Definition 2.2 [4]. Let (X,S) be an S-metric space.

(i) A sequence $\{x_n\}$ in *X* converges to $x\in X$ if $S(x_n,x_n,x)\to 0$ as $n\to\infty$. That is, for each $\varepsilon>0$, there exists $n_0\in N$ such that for all $n\geq n_0$, we have $S(x_n,x_n,x)<\varepsilon$. We write it as $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in *X* is a Cauchy sequence if $S(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$. That is, for each $\varepsilon>0$, there exists $n_0\in N$ such that for all $n,m\geq n_0$, we have $S(x_n,x_n,x_m)<\varepsilon$.

(iii)The S-metric space (X,S) is complete if every Cauchy sequence in *X* converges to a point of *X*.

We use the following results to prove our main result:

Lemma 2.3 [4]. In an *S*-metric space, we have

$$S(x,x,y)=S(y,y,x), \text{ for all } x,y\in X.$$

Lemma 2.4 [4]. Let (X,S) be an S-metric space. If $x_n\to x$ and $y_n\to y$, then

$$S(x_n,x_n,y_n)\to S(x,x,y).$$

Following to Matkowski [2], let Φ be the set of all functions $\phi:[0,\infty)\to[0,\infty)$, where ϕ is a non-decreasing function with $\lim_{n\to\infty}\phi^n(t)=0$, for all $t\in(0,\infty)$. If $\phi\in\Phi$, then ϕ is called a Φ-map.

If ϕ is a Φ-map, then

(i) $\phi(t)<t$, for all $t\in(0,\infty)$,

(ii) $\phi(0)=0.$

In our further discussion ϕ will be considered as a Φ-map.

3. Main Results

Theorem 3.1. Let X be a complete S-metric space. Suppose that the mapping $T: X \to X$ satisfies the condition:

(3.1) $S(Tx,Ty,Tz) \le \phi(S(x,y,z))$,

for all $x,y,z \in X$. Then T has a unique fixed point.

Proof. For arbitrary point $x_0 \in X$, construct a sequence $x_n = Tx_{n-1}$, $n \in N$. Assume $x_n \ne x_{n-1}$, for each $n \in N$. We claim $\{x_n\}$ is a Cauchy sequence in X.

For $n \in N$, we have

(3.2) $$S(x_n,x_n,x_{n+1}) = S(Tx_{n-1},Tx_{n-1},Tx_n) \le \phi(S(x_{n-1},x_{n-1},x_n)) \\ \vdots \\ \le \phi^n(S(x_0,x_0,x_1)).$$

Given $\varepsilon > 0$, since $\lim_{n\to\infty} \phi^n(S(x_0,x_0,x_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there is an integer n_0 such that

(3.3) $$\phi^n(S(x_0,x_0,x_1)) < \frac{\varepsilon}{2} - \frac{\phi(\varepsilon)}{2}$$, for all $n \ge n_0$.

This implies

(3.4) $$S(x_n,x_n,x_{n+1}) < \frac{\varepsilon}{2} - \frac{\phi(\varepsilon)}{2}$$, for all $n \ge n_0$.

For $m,n \in N$ with $m > n$, we claim that

(3.5) $$S(x_n,x_n,x_m) < \varepsilon$$ for all $m > n \ge n_0$.

We prove inequality (3.5) by induction on m.

Inequality (3.5) holds for $m = n+1$ by using inequality (3.4) and the fact that $\varepsilon - \phi(\varepsilon) < \varepsilon$.

Assume inequality (3.5) holds for $m = k$.

For $m = k+1$, we have

(3.6) $$S(x_n,x_n,x_{k+1}) \le S(x_n,x_n,x_{n+1}) + S(x_n,x_n,x_{n+1}) + S(x_{k+1},x_{k+1},x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(Tx_k, Tx_k, Tx_n).$$

Using condition (3.1), equation (3.4) and Lemma 2.3, we get

$$\begin{aligned} S(x_n, x_n, x_{k+1}) &\le \varepsilon - \phi(\varepsilon) + \phi(S(x_k, x_k, x_n)) \\ &\le \varepsilon - \phi(\varepsilon) + \phi(S(x_n, x_n, x_k)) \\ &< \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

By induction on m, we conclude that inequality (3.5) holds for all $m > n \ge n_0$. So $\{x_n\}$ is a Cauchy sequence in complete S-metric space and hence $\{x_n\}$ converges to some $w \in X$.

For $n \in N$, we have

$$\begin{aligned} S(w,w,Tw) &\le S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,x_{n+1}) \\ &\le S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(S(w,w,x_n)). \end{aligned}$$

Since ϕ is a Φ-map, we have

$$S(w,w,Tw) < S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(w,w,x_n).$$

Letting $n \to \infty$ and using the fact that S is continuous in its variables, we get that $S(w,w,Tw) = 0$. Hence $T(w) = w$. So w is a fixed point of T. Now, let v be another fixed point of T with $v \ne w$. Since ϕ is a Φ-map, we have

$$\begin{aligned} S(w,w,v) &= S(Tw,Tw,Tv) \\ &\le \phi(S(w,w,v)) \\ &< S(w,w,v), \end{aligned}$$

which is not possible. So $v = w$ and hence T has a unique fixed point.

Corollary 3.2. Let X be a complete S-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S(T^m x, T^m y, T^m z) \le \phi(S(x,y,z)),$$

for all $x, y, z \in X$ and $m \in N$. Then T has a unique fixed point.

Proof. From Theorem 3.1, we obtain that T^m has a unique fixed point say w.

Since $T^m(Tw) = T^{m+1}w = T(T^m w) = Tw,$ we get that Tw is also a fixed point of T^m. But w is a unique fixed point of T^m, so we have

$$Tw = w.$$

Hence w is a unique fixed point of *T*.

Corollary 3.3. Let *X* be a complete S-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S(Tx,Tx,Tz) \leq \phi(S(x,x,z)),$$

for all $x,z \in X$. Then *T* has a unique fixed point.

Proof. We obtain the result by taking $y = x$ in Theorem 3.1.

Corollary 3.4. Let *X* be a complete S-metric space. Suppose there is $k \in [0,1)$ such that the mapping $T : X \to X$ satisfies the condition:

(3.7) $$S(Tx,Ty,Tz) \leq kS(x,y,z),$$

for all $x,y,z \in X$. Then *T* has a unique fixed point.

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = kt$. Then clearly ϕ is a non-decreasing function with

$$\lim_{n\to\infty} \phi^n(t) = 0, \text{ for all } t > 0.$$

Using condition (3.7) and by virtue of ϕ,we have

$$S(Tx,Ty,Tz) \leq \phi(S(x,y,z)), \text{ for all } x,y,z \in X.$$

Now the result follows from Theorem 3.1.

Corollary 3.5. Let *X* be a complete S-metric space and suppose the mapping $T : X \to X$ satisfies the condition:

(3.8) $$S(Tx,Ty,Tz) \leq \frac{S(x,y,z)}{1 + S(x,y,z)},$$

for all $x,y,z \in X$. Then *T* has a unique fixed point.

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(w) = \frac{w}{1+w}$.

Then clearly ϕ is non-decreasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for all $t > 0$.

Using condition (3.8) and by virtue of ϕ, we have

$$S(Tx,Ty,Tz) \leq \phi(S(x,y,z)), \text{ for all } x,y,z \in X.$$

Now the result follows from Theorem 3.1.

Theorem 3.6. Let X be a complete S-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:

$$S(Tx,Ty,Tz) \le \phi(\max\{S(x,y,z),S(Tx,Tx,x),$$
$$S(Ty,Ty,y),S(Tz,Tz,x)\}),$$

for all $x,y,z \in X$. Then T has a unique fixed point.

Proof. For arbitrary point $x_0 \in X$, construct a sequence $x_n = Tx_{n-1}$, for all $n \in N$.

Assume $x_n \ne x_{n-1}$, for each $n \in N$. Thus for $n \in N$, we have

$$S(x_{n+1},x_{n+1},x_n) = S(Tx_n,Tx_n,Tx_{n-1})$$
$$\le \phi(\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n),S(x_{n+1},x_{n+1},x_n),$$
$$S(x_n,x_n,x_n)\})$$
$$\le \phi(\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n)\}).$$

If $\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n)\} = S(x_{n+1},x_{n+1},x_n)$,

then

$$S(x_{n+1},x_{n+1},x_n) \le \phi(S(x_{n+1},x_{n+1},x_n))$$
$$< S(x_{n+1},x_{n+1},x_n),$$

which is impossible.

So $\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n)\} = S(x_n,x_n,x_{n-1})$.

Thus for $n \in N$, we have

$$S(x_{n+1},x_{n+1},x_n) \le \phi(S(x_n,x_n,x_{n-1}))$$
$$\le \phi^2(S(x_{n-1},x_{n-1},x_{n-2}))$$
$$\vdots$$
$$\le \phi^n(S(x_1,x_1,x_0)).$$

This implies

$$S(x_{n+1},x_{n+1},x_n) \le \phi^n(S(x_1,x_1,x_0)).$$

Using Lemma 2.3, we get

$$S(x_n,x_n,x_{n+1}) \le \phi^n(S(x_0,x_0,x_1)).$$

By similar arguments as in Theorem 3.1, we get $\{x_n\}$ is a Cauchy sequence in complete S-metric space. So $\{x_n\}$ converges to some $w \in X$.

For $n \in N$, we have

$$S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,x_{n+1})$$
$$= S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,Tx_n)$$
$$\leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(\max\{S(w,w,x_n),$$
$$S(Tw,Tw,w), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\})$$
$$= S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(\max\{S(w,w,x_n),$$
$$S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\}).$$

Case I.

If max $\{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\}$

$$= S(w,w,x_n),$$

then

$$S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(S(w,w,x_n))$$
$$< S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(w,w,x_n).$$

Letting $n \to \infty$, we have $Tw = w.$

Case II.

If max $\{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\}$

$$= S(Tw,Tw,w),$$

then

$$S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(S(Tw,Tw,w))$$
$$< S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,w).$$

Using Lemma 2.3, we get

$$S(w,w,Tw) < S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(w,w,Tw).$$

Letting $n \to \infty$, we get $T(w) = w.$

Case III.

If $\max\{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\}$

$$= S(x_{n+1}, x_{n+1}, w),$$

then

$$S(w,w,Tw) < S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(x_{n+1}, x_{n+1}, w).$$

Letting $n \to \infty$, we get $Tw = w$.

Hence, we can say that w is a fixed point of *T*.

If v is another fixed point of *T*, then

$$\begin{aligned}
S(w,w,v) &= S(Tw,Tw,Tv) \\
&\le \phi(\max\{S(w,w,v), S(Tw,Tw,w), S(Tw,Tw,w), S(Tv,Tv,w)\}) \\
&\le \phi(\max\{S(w,w,v), S(w,w,w), S(w,w,w), S(v,v,w)\}) \\
&\le \phi(\max\{S(w,w,v), S(v,v,w)\}) \\
&= \phi(S(w,w,v)) \quad (\because \text{ by Lemma 2.3, } S(v,v,w) = S(w,w,v)) \\
&< S(w,w,v), (\because \ \phi \text{ is } \Phi\text{-map})
\end{aligned}$$

which is not possible and hence w is a unique fixed point of T.

Corollary 3.7. Let *X* be a complete S-metric space. Suppose there is $k \in [0,1)$ such that the mapping $T : X \to X$ satisfies the condition:

(3.9) $$S(Tx,Ty,Tz) \le k \max\{S(x,y,z), S(Tx,Tx,x), S(Ty,Ty,y), S(Tz,Tz,x)\},$$

for all $x, y, z \in X$. Then *T* has a unique fixed point.

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(w) = kw$.

Then clearly ϕ is non-decreasing function with

$$\lim_{n\to\infty} \phi^n(t) = 0, \text{ for all } t > 0.$$

Using condition (3.9) and by virtue of ϕ, we get

$$S(Tx,Ty,Tz) \le \phi(\max\{S(x,y,z), S(Tx,Tx,x), S(Ty,Ty,y), S(Tz,Tz,x)\}),$$

for all $x, y, z \in X$.

Now the result follows from Theorem 3.6.

Corollary 3.8. Let X be a complete S-metric space and suppose the mapping $T : X \to X$ satisfies the condition:

$$S(Tx,Tx,Tz) \leq \phi(\max\{S(x,x,z), S(Tx,Tx,x), S(Tz,Tz,x)\}),$$

for all $x,z \in X$. Then T has a unique fixed point.

Proof. We obtain the result by taking $y = x$ in Theorem 3.6.

Conflict of Interests

The authors declare that there is no conflict of interests.

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