FIXED POINT THEOREMS FOR SELF MAP UNDER SOME CONTRACTIVE CONDITIONS RELATED TO $\Phi$ -MAP

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Abstract. In this paper, we prove some fixed point theorems for uniqueness of fixed points for self-map $T : X \rightarrow X$ under different contractive conditions related to $\Phi$ -map.

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1. Introduction

Many authors tried to give generalization of metric spaces in several ways and obtained many results [1-12]. Gähler [6] and Dhage [1] introduced the concepts of 2-metric spaces and $D$-metric spaces respectively. Mustafa and Sims [8] introduced a new structure of generalized metric spaces which are called $G$-metric spaces. Sedghi et al. [4] introduced the concept of $D^*$-metric spaces which was modification of the definition of $D$-metric spaces. Recently, Sedghi et al. [5] have introduced the notion of $S$-metric spaces and have proved some fixed point theorems in $S$-metric spaces. In this paper, we consider $\phi$ as a $\Phi$ -map and prove some fixed point theorems for self-map $T : X \rightarrow X$ under different contractive conditions related to $\phi$.

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2. Preliminaries

The following definitions and results will be used in the sequel:

**Definition 2.1** [4]. Let \( X \) be a non-empty set. An \( S \)-metric on \( X \) is a function 
\[ S : X \times X \times X \rightarrow [0, \infty) \]
that satisfies the following conditions, for each \( x, y, z, a \in X \),

- \((S_1)\) \( S(x, y, z) \geq 0 \),
- \((S_2)\) \( S(x, y, z) = 0 \) iff \( x = y = z \),
- \((S_3)\) \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \).

The pair \( (X, S) \) is called an \( S \)-metric space.

**Definition 2.2** [4]. Let \( (X, S) \) be an \( S \)-metric space.

(i) A sequence \( \{x_n\} \) in \( X \) converges to \( x \in X \) if \( S(x_n, x_n, x) \to 0 \) as \( n \to \infty \). That is, for each \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), we have \( S(x_n, x_n, x) < \varepsilon \). We write it as \( \lim_{n \to \infty} x_n = x \).

(ii) A sequence \( \{x_n\} \) in \( X \) is a Cauchy sequence if \( S(x_n, x_n, x_m) \to 0 \) as \( n, m \to \infty \). That is, for each \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for all \( n, m \geq n_0 \), we have \( S(x_n, x_n, x_m) < \varepsilon \).

(iii) The \( S \)-metric space \( (X, S) \) is complete if every Cauchy sequence in \( X \) converges to a point of \( X \).

We use the following results to prove our main result:

**Lemma 2.3** [4]. In an \( S \)-metric space, we have 
\[ S(x, x, y) = S(y, y, x) \], for all \( x, y \in X \).

**Lemma 2.4** [4]. Let \( (X, S) \) be an \( S \)-metric space. If \( x_n \to x \) and \( y_n \to y \), then 
\[ S(x_n, x_n, y_n) \to S(x, x, y) \).

Following to Matkowski [2], let \( \Phi \) be the set of all functions 
\[ \phi : [0, \infty) \to [0, \infty) \], \] where \( \phi \) is a non-decreasing function with
\[ \lim_{n \to \infty} \phi^n(t) = 0 \], for all \( t \in (0, \infty) \). If \( \phi \in \Phi \), then \( \phi \) is called a \( \Phi \)-map.

If \( \phi \) is a \( \Phi \)-map, then

(i) \( \phi(t) < t \), for all \( t \in (0, \infty) \),
(ii) \( \phi(0) = 0 \).

In our further discussion \( \phi \) will be considered as a \( \Phi \)-map.

3. Main Results

**Theorem 3.1.** Let \( X \) be a complete \( S \)-metric space. Suppose that the mapping \( T : X \to X \) satisfies the condition:

\[
S(Tx, Ty, Tz) \leq \phi(S(x, y, z)),
\]

for all \( x, y, z \in X \). Then \( T \) has a unique fixed point.

**Proof.** For arbitrary point \( x_0 \in X \), construct a sequence \( x_n = Tx_{n-1}, \ n \in \mathbb{N} \). Assume \( x_n \neq x_{n-1} \), for each \( n \in \mathbb{N} \). We claim \( \{x_n\} \) is a Cauchy sequence in \( X \).

For \( n \in \mathbb{N} \), we have

\[
S(x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq \phi(S(x_{n-1}, x_{n-1}, x_n)) \leq \phi^n(S(x_0, x_0, x_1)).
\]

Given \( \varepsilon > 0 \), since \( \lim_{n \to \infty} \phi^n(S(x_0, x_0, x_1)) = 0 \) and \( \phi(\varepsilon) < \varepsilon \), there is an integer \( n_0 \) such that

\[
\phi^n(S(x_0, x_0, x_1)) < \frac{\varepsilon}{2} - \frac{\phi(\varepsilon)}{2}, \text{ for all } n \geq n_0.
\]

This implies

\[
S(x_n, x_{n+1}) \leq \frac{\varepsilon}{2} - \frac{\phi(\varepsilon)}{2}, \text{ for all } n \geq n_0.
\]

For \( m, n \in \mathbb{N} \) with \( m > n \), we claim that

\[
S(x_n, x_m) < \varepsilon \text{ for all } m > n \geq n_0.
\]

We prove inequality (3.5) by induction on \( m \).

Inequality (3.5) holds for \( m = n+1 \) by using inequality (3.4) and the fact that \( \varepsilon - \phi(\varepsilon) < \varepsilon \).

Assume inequality (3.5) holds for \( m = k \).

For \( m = k + 1 \), we have

\[
S(x_n, x_{k+1}) \leq S(x_n, x_k, x_{k+1}) + S(x_k, x_{k+1}, x_{k+1}) + S(x_{k+1}, x_{k+1}, x_{n+1})
\]
Using condition (3.1), equation (3.4) and Lemma 2.3, we get
\[ S(x_n, x_n, x_{k+1}) \leq \varepsilon - \phi(\varepsilon) + \phi(S(x_k, x_k, x_n)) \]
\[ \leq \varepsilon - \phi(\varepsilon) + \phi(S(x_n, x_n, x_k)) \]
\[ < \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) \]
\[ = \varepsilon. \]

By induction on \( m \), we conclude that inequality (3.5) holds for all \( m > n \geq n_0 \). So \( \{x_n\} \) is a Cauchy sequence in complete \( S \)-metric space and hence \( \{x_n\} \) converges to some \( w \in X \).

For \( n \in N \), we have
\[ S(w, w, Tw) \leq S(w, w, x_{n+1}) + S(w, w, x_{n+1}) + S(Tw, Tw, x_{n+1}) \]
\[ \leq S(w, w, x_{n+1}) + S(w, w, x_{n+1}) + \phi(S(w, w, x_n)) \]
Since \( \phi \) is a \( \Phi \)-map, we have
\[ S(w, w, Tw) < S(w, w, x_{n+1}) + S(w, w, x_{n+1}) + S(w, w, x_n). \]

Letting \( n \to \infty \) and using the fact that \( S \) is continuous in its variables, we get that \( S(w, w, Tw) = 0 \). Hence \( T(w) = w \). So \( w \) is a fixed point of \( T \). Now, let \( v \) be another fixed point of \( T \) with \( v \neq w \). Since \( \phi \) is a \( \Phi \)-map, we have
\[ S(w, w, v) = S(Tw, Tw, Tv) \]
\[ \leq \phi(S(w, w, v)) \]
\[ < S(w, w, v), \]
which is not possible. So \( v = w \) and hence \( T \) has a unique fixed point.

**Corollary 3.2.** Let \( X \) be a complete \( S \)-metric space. Suppose that the mapping \( T : X \to X \) satisfies the condition:
\[ S(T^m x, T^m y, T^m z) \leq \phi(S(x, y, z)), \]
for all \( x, y, z \in X \) and \( m \in N \). Then \( T \) has a unique fixed point.

**Proof.** From Theorem 3.1, we obtain that \( T^m \) has a unique fixed point say \( w \).

Since \( T^m(Tw) = T^{m+1}w = T(T^m w) = Tw \), we get that \( Tw \) is also a fixed point of \( T^m \). But \( w \) is a unique fixed point of \( T^m \), so we have
Tw = w.

Hence w is a unique fixed point of T.

**Corollary 3.3.** Let X be a complete S-metric space. Suppose that the mapping \( T : X \rightarrow X \) satisfies the condition:

\[
S(Tx, Tx, Tz) \leq \phi(S(x, x, z)),
\]

for all \( x, z \in X \). Then T has a unique fixed point.

**Proof.** We obtain the result by taking \( y = x \) in Theorem 3.1.

**Corollary 3.4.** Let X be a complete S-metric space. Suppose there is \( k \in [0,1) \) such that the mapping \( T : X \rightarrow X \) satisfies the condition:

\[
(3.7) \quad S(Tx, Ty, Tz) \leq kS(x, y, z),
\]

for all \( x, y, z \in X \). Then T has a unique fixed point.

**Proof.** Define \( \phi : [0, \infty) \rightarrow [0, \infty) \) by \( \phi(t) = kt \). Then clearly \( \phi \) is a non-decreasing function with

\[
\lim_{n \to \infty} \phi^n(t) = 0, \text{ for all } t > 0.
\]

Using condition (3.7) and by virtue of \( \phi \), we have

\[
S(Tx, Ty, Tz) \leq \phi(S(x, y, z)), \text{ for all } x, y, z \in X.
\]

Now the result follows from Theorem 3.1.

**Corollary 3.5.** Let X be a complete S-metric space and suppose the mapping \( T : X \rightarrow X \) satisfies the condition:

\[
(3.8) \quad S(Tx, Ty, Tz) \leq \frac{S(x, y, z)}{1 + S(x, y, z)},
\]

for all \( x, y, z \in X \). Then T has a unique fixed point.

**Proof.** Define \( \phi : [0, \infty) \rightarrow [0, \infty) \) by \( \phi(w) = \frac{w}{1 + w} \).

Then clearly \( \phi \) is non-decreasing function with \( \lim_{n \to \infty} \phi^n(t) = 0 \), for all \( t > 0 \).

Using condition (3.8) and by virtue of \( \phi \), we have

\[
S(Tx, Ty, Tz) \leq \phi(S(x, y, z)), \text{ for all } x, y, z \in X.
\]

Now the result follows from Theorem 3.1.
Theorem 3.6. Let $X$ be a complete $S$-metric space. Suppose that the mapping $T : X \to X$ satisfies the condition:
\[
S(Tx, Ty, Tz) \leq \phi(\max\{S(x, y, z), S(Tx, Tx, x), S(Ty, Ty, y), S(Tz, Tz, x)\}),
\]
for all $x, y, z \in X$. Then $T$ has a unique fixed point.

**Proof.** For arbitrary point $x_0 \in X$, construct a sequence $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$.

Assume $x_n \neq x_{n-1}$, for each $n \in \mathbb{N}$. Thus for $n \in \mathbb{N}$, we have
\[
S(x_{n+1}, x_{n+1}, x_n) = S(Tx_n, Tx_n, Tx_{n-1}) \leq \phi(\max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n+1}), S(x_n, x_n, x_n)\}),
\]
\[
\leq \phi(\max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}).
\]
If $\max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} = S(x_{n+1}, x_{n+1}, x_n)$, then
\[
S(x_{n+1}, x_{n+1}, x_n) \leq \phi(\max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}) < S(x_{n+1}, x_{n+1}, x_n),
\]
which is impossible.

So $\max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} = S(x_n, x_n, x_{n-1})$.

Thus for $n \in \mathbb{N}$, we have
\[
S(x_{n+1}, x_{n+1}, x_n) \leq \phi(S(x_n, x_n, x_{n-1})) \leq \phi^2(S(x_{n-1}, x_{n-1}, x_{n-2})) \leq \cdots \leq \phi^n(S(x_1, x_1, x_0)).
\]
This implies
\[
S(x_{n+1}, x_{n+1}, x_n) \leq \phi^n(S(x_1, x_1, x_0)).
\]
Using Lemma 2.3, we get
\[
S(x_n, x_n, x_{n+1}) \leq \phi^n(S(x_0, x_0, x_1)).
\]
By similar arguments as in Theorem 3.1, we get \( \{x_n\} \) is a Cauchy sequence in complete \( S \)-metric space. So \( \{x_n\} \) converges to some \( w \in X \).

For \( n \in \mathbb{N} \), we have
\[
S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,x_{n+1})
\]
\[
= S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,Tx_n)
\]
\[
\leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(\max\{S(w,w,x_n),
S(Tw,Tw,w), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\})
\]
\[
= S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(\max\{S(w,w,x_n),
S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\}).
\]

Case I.

If \( \max \{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\} \)
\[
= S(w,w,x_n),
\]
then
\[
S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(S(w,w,x_n))
\]
\[
< S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(w,w,x_n).
\]

Letting \( n \to \infty \), we have \( Tw = w \).

Case II.

If \( \max \{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\} \)
\[
= S(Tw,Tw,w),
\]
then
\[
S(w,w,Tw) \leq S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + \phi(S(Tw,Tw,w))
\]
\[
< S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(Tw,Tw,w).
\]

Using Lemma 2.3, we get
\[
S(w,w,Tw) < S(w,w,x_{n+1}) + S(w,w,x_{n+1}) + S(w,w,Tw).
\]

Letting \( n \to \infty \), we get \( T(w) = w \).

Case III.

If \( \max\{S(w,w,x_n), S(Tw,Tw,w), S(x_{n+1},x_{n+1},w)\} \)
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\[ S(x_{n+1}, x_{n+1}, w) , \]

then

\[ S(w, w, Tw) < S(w, w, x_{n+1}) + S(w, w, x_{n+1}) + S(x_{n+1}, x_{n+1}, w) . \]

Letting \( n \to \infty \), we get \( Tw = w \).

Hence, we can say that \( w \) is a fixed point of \( T \).

If \( v \) is another fixed point of \( T \), then

\[ S(w, w, v) = S(Tw, Tw, Tv) \]

\[ \leq \phi(\max\{S(w, w, v), S(Tw, Tw, w), S(Tw, Tw, w), S(Tv, Tv, w)\}) \]

\[ \leq \phi(\max\{S(w, w, v), S(w, w, w), S(w, w, w), S(v, v, w)\}) \]

\[ \leq \phi(\max\{S(w, w, v), S(v, v, w)\}) \]

\[ = \phi(S(w, w, v)) \ (\because \text{by Lemma 2.3, } S(v, v, w) = S(w, w, v)) \]

\[ < S(w, w, v), \ (\because \phi \text{ is } \Phi \text{-map}) \]

which is not possible and hence \( w \) is a unique fixed point of \( T \).

**Corollary 3.7.** Let \( X \) be a complete \( S \)-metric space. Suppose there is \( k \in [0,1) \) such that the mapping \( T : X \to X \) satisfies the condition:

(3.9) \[ S(Tx, Ty, Tz) \leq k \max\{S(x, y, z), S(Tx, Tx, x), S(Ty, Ty, y), S(Tz, Tz, x)\} , \]

for all \( x, y, z \in X \). Then \( T \) has a unique fixed point.

**Proof.** Define \( \phi : [0, \infty) \to [0, \infty) \) by \( \phi(w) = kw \).

Then clearly \( \phi \) is non-decreasing function with

\[ \lim_{n \to \infty} \phi^n(t) = 0 , \text{ for all } t > 0. \]

Using condition (3.9) and by virtue of \( \phi \), we get

\[ S(Tx, Ty, Tz) \leq \phi(\max\{S(x, y, z), S(Tx, Tx, x), S(Ty, Ty, y), S(Tz, Tz, x)\}) , \]

for all \( x, y, z \in X \).

Now the result follows from Theorem 3.6.
Corollary 3.8. Let $X$ be a complete $S$-metric space and suppose the mapping $T : X \to X$ satisfies the condition:

$$S(Tx, Tx, Tz) \leq \phi(\max\{S(x, x, z), S(Tx, Tx, x), S(Tz, Tz, x)\}),$$

for all $x, z \in X$. Then $T$ has a unique fixed point.

**Proof.** We obtain the result by taking $y = x$ in Theorem 3.6.

Conflict of Interests

The authors declare that there is no conflict of interests.

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