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COMMON COUPLED FIXED POINT THEOREM FOR A PAIR OF HYBRID MAPS IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we obtain a common coupled fixed point theorem for a pair of hybrid maps in complex valued metric spaces. Also we give an example to illustrate our main theorem.

Keywords: complex valued metric ; *w*-compatible maps ; hybrid pair ; g.l.b property.

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1. Introduction and Preliminaries

Azam et al.[1] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive conditions. Later on several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, we refer the readers to [3,4,5,12,14,16,17,18,20,21,22,25,27]. Recently, Azam et al. [2] and Ahmad et al. [6] obtained some new fixed point results for multi-valued mappings in the setting of complex valued metric spaces.

The purpose of this paper is to study the common coupled fixed points for a pair of hybrid mappings satisfying a rational inequality in the frame work of a complex valued metric space. We also give an example to illustrate our

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main theorem. The proved result generalizes and extends the Theorem 2.1 of [23].

To begin with, we recall some basic definitions, notations and results.

Throughout this paper $\mathscr{R}^+$, $\mathscr{N}$ and $\mathbb{C}$ denote the set of all non-negative real numbers, positive integers and complex numbers respectively. First we refer the following preliminaries.

Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2)$.

Thus $z_1 \precsim z_2$ if one of the following holds:

(1). $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2). $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3). $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(4). $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Clearly $z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$.

We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied. Also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1. One can easily check the following statements :

(i) if $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$;

(ii) if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2.([1]) Let X be a non empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$.

The pair $(X,d)$is called a complex valued metric space.

Remark 1.3. Let (X,d) be a complex valued metric space. Then

(i) $|d(x,y)|\, or\, |d(u,v)| < |1 + d(x,y) + d(u,v)|, \forall x, y, u, v \in X$.

(ii) $|d(x,y)| > 0$,if $x \neq y$.

Definition 1.4.([1]) Let (X,d) be a complex valued metric space.

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) \cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each point of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is in B.

(v) The family $F = \{B(x,r) : x \in X \text{ and } 0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \preceq c$ there is $n_0 \in \mathscr{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathscr{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathscr{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d) then (X,d) is called a complete complex valued metric space. We require the follwing lemmas.

Lemma 1.5.([1]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 1.6.([1]) Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n,m \to \infty$.

One can easily prove the following lemma.

Lemma 1.7. Let (X,d) be a complex valued metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y respectively. Then $|d(x_n,y_n)| \to |d(x,y)|$ as $n \to \infty$.

Now, we follow the notations and definitions given in [6]. Let (X,d) be a complex valued metric space. We denote the family of nonempty, closed and bounded subsets of a complex valued metric space X by $CB(X)$. From now onwards, we denote for $z_1 \in \mathbb{C}, s(z_1) = \{z \in \mathbb{C} : z_1 \precsim z\}$ and for $a \in X$ and $B \in CB(X), s(a,B) = \bigcup_{b\in B} s(d(a,b)) = \bigcup_{b\in B} \{z \in \mathbb{C} : d(a,b) \precsim z\}$.

For $A,B \in CB(X)$, we denote $s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \cap \left(\bigcap_{b\in B} s(b,A)\right)$.

Remark 1.8.([6]): Let (X,d) be a complex valued metric space and $T : X \to CB(X)$ be a multivalued map. For $x \in X$ and $A \in CB(X)$, define $W_x(A) = \{d(x,a) : a \in A\}$.

Thus, for $x,y \in X$, we have $W_x(Ty) = \{d(x,u) : u \in Ty\}$.

Remark 1.9.([6]): Let (X,d) be a complex valued metric space. If $\mathbb{C} = \mathscr{R}$ then (X,d) is a metric space. Moreover, for $A,B \in CB(X)$, $H(A,B) = inf\{s(A,B)\}$ is the Hausdorff distance induced by d.

Definition 1.10.([6]): Let (X,d) be a complex valued metric space.A nonempty subset A of X is called bounded from below if there exists some $z \in \mathbb{C}$ such that $z \precsim a$ for all $a \in A$.

A multivalued mapping $F : X \to 2^{\mathbb{C}}$ is called bounded from below if for each $x \in X$ there exists $z_x \in \mathbb{C}$ such that $z_x \precsim u$ for all $u \in Fx$.

Definition 1.11.([6]) The multivalued mapping $T : X \to CB(X)$ is said to have the lower bound proerty(l.b.Property) on (X,d) if for any $x \in X$, the multi-valued mapping $F_x : X \to 2^{\mathbb{C}}$ defined by $F_x(y) = W_x(Ty)$ is bounded from below.That is, for $x,y \in X$, there exists an element $l_x(Ty) \in \mathbb{C}$ such that $l_x(Ty) \precsim u$, for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called a lower bound of T associated with (x,y).

Definition 1.12.([6]) Let (X,d) be a complex valued metric space.The multivalued mapping $T : X \to CB(X)$ is said to have the greatest lower bound proerty (g.l.b.Property) on (X,d) if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x,y \in X$. We denote $d(x,Ty)$ by the g.l.b.Property of $W_x(Ty)$. That is $d(x,Ty) = inf\{d(x,u) : u \in Ty\}$.

Bhaskar and Lakshmikantham [24] introduced the concept of coupled fixed points and Lakshmikantham and Ciric [26] defined the common coupled fixed points. Later on several authors obtained coupled fixed point theorems in various spaces, for example,[7,8,9,10,11,13] and the references therein.

Definition 1.13.(Hussain and Alotaibi,[19]) Let the mappings $F : X \times X \to CB(X)$ and $f : X \to X$ be given. An element $(x,y) \in X \times X$ is called

(i) a coupled coincidence point of a pair (F, f) if $fx \in F(x,y)$ and $fy \in F(y,x)$,

(ii) a common coupled fixed point of a pair (F, f) if $x = fx \in F(x,y)$ and $y = fy \in F(y,x)$.

Definition 1.14.(Abbas et al.,[15]) Let $F : X \times X \to 2^X$ be a multivalued map and f be a self map on X. The hybrid pair (F, f) is called w-compatible if $f(F(x,y)) \subseteq F(fx, fy)$ whenever (x,y) is a coupled coincidence point of F and f.

Now we prove our main result.

2. Main result

Theorem 2.1. *Let (X,d) be a complex valued metric space and $F : X \times X \to CB(X)$ be a multi-valued mapping with g.l.b property and $f : X \to X$ be a mapping satisfying the following*

(2.1.1) $F(X \times X) \subseteq f(X)$,

(2.1.2) $f(X)$ *is a complete subspace of X and*

(2.1.3.)
$$\alpha\, d(fx, fu) + \beta\, d(fy, fv) + \gamma\, \frac{d(fx,F(x,y))\; d(fu,F(u,v))}{1+d(fx,fu)+d(fy,fv)} + \delta\, \frac{d(fx,F(u,v))\; d(fu,F(x,y))}{1+d(fx,fu)+d(fy,fv)} \in s(F(x,y),F(u,v))$$

for all $x,y,u,v \in X$, where $\alpha,\beta,\gamma,\delta$ are non-negative real numbers such that $\alpha+\beta+\gamma+\delta < 1$.

Then F and f have a coupled coincidence point in $X \times X$.

(2.1.4) *Further if we assume that (F, f) is w-compatible and there exist $p,q \in X$ such that $\lim_{n\to\infty} f^n x = p$ and $\lim_{n\to\infty} f^n y = q$, whenever (x,y) is a coupled coincidence point of F and f and f is continuous at p and q, then F and f have a common coupled fixed point.*

Proof. Let $(x_0,y_0) \in X \times X$. From (2.1.1), we can choose $fx_1 \in F(x_0,y_0)$ and $fy_1 \in F(y_0,x_0)$ for some $x_1,y_1 \in X$. From (2.1.3), we have

$$\alpha d(fx_0, fx_1) + \beta d(fy_0, fy_1) + \gamma\, \frac{d(fx_0,F(x_0,y_0))\; d(fx_1,F(x_1,y_1))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

$$+\delta\, \frac{d(fx_0,F(x_1,y_1))\; d(fx_1,F(x_0,y_0))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)} \in s(F(x_0,y_0),F(x_1,y_1))$$

i.e.,

$$\alpha d(fx_0,fx_1)+\beta d(fy_0,fy_1)+\gamma \tfrac{d(fx_0,F(x_0,y_0))\ d(fx_1,F(x_1,y_1))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

$$+\delta \tfrac{d(fx_0,F(x_1,y_1))\ d(fx_1,F(x_0,y_0))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)} \in \bigcap_{x\in F(x_0,y_0)} s(x,F(x_1,y_1)).$$

Since $fx_1 \in F(x_0,y_0)$, we have

$$\alpha d(fx_0,fx_1)+\beta d(fy_0,fy_1)+\gamma \tfrac{d(fx_0,F(x_0,y_0))\ d(fx_1,F(x_1,y_1))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

$$+\delta \tfrac{d(fx_0,F(x_1,y_1))\ d(fx_1,F(x_0,y_0))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)} \in s(fx_1,F(x_1,y_1)) = \bigcup_{y\in F(x_1,y_1)} s(d(fx_1,y)).$$

From (2.1.1), there exists $x_2 \in X$ with $fx_2 \in F(x_1,y_1)$ such that

$$\alpha d(fx_0,fx_1)+\beta d(fy_0,fy_1)+\gamma \tfrac{d(fx_0,F(x_0,y_0))\ d(fx_1,F(x_1,y_1))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

$$+\delta \tfrac{d(fx_0,F(x_1,y_1))\ d(fx_1,F(x_0,y_0))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)} \in s(d(fx_1,fx_2)).$$

Hence

$$d(fx_1,fx_2) \precsim \alpha d(fx_0,fx_1)+\beta d(fy_0,fy_1)+\gamma \tfrac{d(fx_0,F(x_0,y_0))\ d(fx_1,F(x_1,y_1))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

$$+\delta \tfrac{d(fx_0,F(x_1,y_1))\ d(fx_1,F(x_0,y_0))}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}.$$

By using the g.l.b property of F, we get

$$d(fx_1,fx_2) \precsim \alpha d(fx_0,fx_1)+\beta d(fy_0,fy_1)+\gamma \tfrac{d(fx_0,fx_1)\ d(fx_1,fx_2)}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$
$$+\delta \tfrac{d(fx_0,f(x_2)\ d(fx_1,fx_1)}{1+d(fx_0,fx_1)+d(fy_0,fy_1)}$$

which implies that

$$\begin{aligned}&|d(fx_1,fx_2)|\\ &\leq \alpha\,|d(fx_0,fx_1)|+\beta\,|d(fy_0,fy_1)|+\gamma\,\tfrac{|d(fx_0,fx_1)|\ |d(fx_1,fx_2)|}{|1+d(fx_0,fx_1)+d(fy_0,fy_1)|}\\ &< \alpha\,|d(fx_0,fx_1)|+\beta\,|d(fy_0,fy_1)|+\gamma\,|d(fx_1,fx_2)|,\ \textit{from Remark } 1.3(i).\end{aligned}$$

Thus

$$(1)\quad |d(fx_1,fx_2)| \leq \tfrac{\alpha+\beta}{1-\gamma}\max\{|d(fx_0,fx_1)|,|d(fy_0,fy_1)|\}$$

Similarly, we can show that

$$(2)\quad |d(fy_1,fy_2)| \leq \tfrac{\alpha+\beta}{1-\gamma}\max\{|d(fx_0,fx_1)|,|d(fy_0,fy_1)|\}$$

From (1) and (2), we have

$$\max\left\{\begin{array}{c}|d(fx_1,fx_2)|,\\ |d(fy_1,fy_2)|\end{array}\right\}\leq\lambda\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}, where\ \lambda=\frac{\alpha+\beta}{1-\gamma}<1.$$

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$(3)\ \max\left\{\begin{array}{c}|d(fx_n,fx_{n+1})|,\\ |d(fy_n,fy_{n+1})|\end{array}\right\}\leq\lambda\max\left\{\begin{array}{c}|d(fx_{n-1},fx_n)|,\\ |d(fy_{n-1},fy_n)|\end{array}\right\}$$
$$\vdots$$
$$\leq\lambda^n\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}.$$

For $m>n$ and using (3) we get

$$\begin{aligned}|d(fx_n,fx_m)|&\leq|d(fx_n,fx_{n+1})|+|d(fx_{n+1},fx_{n+2})|+.....+|d(fx_{m-1},fx_m)|\\&\leq(\lambda^n+\lambda^{n+1}+.....+\lambda^{m-1})\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}\\&\leq\tfrac{\lambda^n}{1-\lambda}\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}\\&\to 0\ as\ n,m\to\infty.\end{aligned}$$

Thus $\{fx_n\}$ is a Cauchy sequence in X. Similarly, we can show that $\{fy_n\}$ is also a Cauchy sequence in X. Since $f(X)$ is a complete subspace of X, there exist $u,v\in X$ such that $fx_n\to fu$ and $fy_n\to fv$ as $n\to\infty$. From (2.1.3), we have

$$\alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\tfrac{d(fx_n,F(x_n,y_n))\ d(fu,F(u,v))}{1+d(fx_n,fu)+d(fy_n,fv)}+\delta\,\tfrac{d(fx_n,F(u,v))\ d(fu,F(x_n,y_n))}{1+d(fx_n,fu)+d(fy_n,fv)}\in s(F(x_n,y_n),F(u,v)).$$

$$\alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\tfrac{d(fx_n,F(x_n,y_n))\ d(fu,F(u,v))}{1+d(fx_n,fu)+d(fy_n,fv)}+\delta\,\tfrac{d(fx_n,F(u,v))\ d(fu,F(x_n,y_n))}{1+d(fx_n,fu)+d(fy_n,fv)}\in\bigcap_{x\in F(x_n,y_n)}s(x,F(u,v)).$$

Since $fx_{n+1}\in F(x_n,y_n)$, we have

$$\alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\tfrac{d(fx_n,F(x_n,y_n))\ d(fu,F(u,v))}{1+d(fx_n,fu)+d(fy_n,fv)}+\delta\,\tfrac{d(fx_n,F(u,v))\ d(fu,F(x_n,y_n))}{1+d(fx_n,fu)+d(fy_n,fv)}\in s(fx_{n+1},F(u,v))=\bigcup_{u'\in F(u,v)}s(d(fx_{n+1},u')).$$

From (2.1.1), there exists $u_n\in X$ with $fu_n\in F(u,v)$ such that

$$\alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\tfrac{d(fx_n,F(x_n,y_n))\ d(fu,F(u,v))}{1+d(fx_n,fu)+d(fy_n,fv)}+\delta\,\tfrac{d(fx_n,F(u,v))\ d(fu,F(x_n,y_n))}{1+d(fx_n,fu)+d(fy_n,fv)}\in s(d(fx_{n+1},fu_n)).$$

Thus

$$d(fx_{n+1},fu_n)\precsim\alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\tfrac{d(fx_n,F(x_n,y_n))\ d(fu,F(u,v))}{1+d(fx_n,fu)+d(fy_n,fv)}+\delta\,\tfrac{d(fx_n,F(u,v))\ d(fu,F(x_n,y_n))}{1+d(fx_n,fu)+d(fy_n,fv)}.$$

From g.l.b property of F, we get

$$d(fx_{n+1},fu_n)\precsim \alpha d(fx_n,fu)+\beta d(fy_n,fv)+\gamma\,\frac{d(fx_n,fx_{n+1})\;d(fu,fu_n)}{1+d(fx_n,fu)+d(fy_n,fv)} +\delta\,\frac{d(fx_n,fu_n))\;d(fu,fx_{n+1})}{1+d(fx_n,fu)+d(fy_n,fv)}$$

which implies that

$$|d(fx_{n+1},fu_n)|\leq \alpha\,|d(fx_n,fu)|+\beta\,|d(fy_n,fv)|+\gamma\,\frac{|d(fx_n,fx_{n+1})|\;|d(fu,fu_n)|}{|1+d(fx_n,fu)+d(fy_n,fv)|} +\delta\,\frac{|d(fx_n,fu_n)|\;|d(fu,fx_{n+1})|}{|1+d(fx_n,fu)+d(fy_n,fv)|}.$$

Now consider

$$\begin{aligned}|d(fu,fu_n)| &\leq |d(fu,fx_{n+1})|+|d(fx_{n+1},fu_n)|\\ &\leq |d(fu,fx_{n+1})|+\alpha\,|d(fx_n,fu)|+\beta\,|d(fy_n,fv)|\\ &\quad+\gamma\frac{|d(fx_n,fx_{n+1})|\;|d(fu,fu_n)|}{|1+d(fx_n,fu)+d(fy_n,fv)|}+\delta\,\frac{|d(fx_n,fu_n)|\;|d(fu,fx_{n+1})|}{|1+d(fx_n,fu)+d(fy_n,fv)|}.\end{aligned}$$

Letting $n\to\infty$, we obtain

$\lim_{n\to\infty}|d(fu,fu_n)|\leq 0$ so that $fu_n\to fu$ as $n\to\infty$. Since $F(u,v)$ is closed and $fu_n\in F(u,v)$, it follows that $fu\in F(u,v)$.

Similarly, we have $fv\in F(v,u)$. Thus (u,v) is a coupled coincidence point of F and f.

Since (u,v) is a coupled coincidence point of F and f, from (2.1.4) there exist $p,q\in X$ such that $\lim_{n\to\infty}f^nu = p$ and $\lim_{n\to\infty}f^nv=q$.

Since f is continuous at p and q, we have $fp=p$ and $fq=q$.

Since the pair (F,f) is w-compatible, $fu\in F(u,v)$ and $fv\in F(v,u)$, we have $f^2u\in f(F(u,v)\subseteq F(fu,fv)$ and $f^2v\in f(F(v,u)\subseteq F(fv,fu)$. Thus (fu,fv) is a coupled coincidence point of F and f.

Continuing in this way,we can show that (f^nu,f^nv) is a coupled coincidence point of F and f.

It is also clear that $f^nu\in F(f^{n-1}u,f^{n-1}v)$ and $f^nv\in F(f^{n-1}v,f^{n-1}u)$.

From (2.1.3), we have

$$\alpha d(f^nu,fp)+\beta d(f^nv,fq)+\gamma\,\frac{d(f^nu,F(f^{n-1}u,f^{n-1}v))\;d(fp,F(p,q))}{1+d(f^nu,fp)+d(f^nv,fq)} +\delta\,\frac{d(f^nu,F(p,q))\;d(fp,F(f^{n-1}u,f^{n-1}v))}{1+d(f^nu,fp)+d(f^nv,fq)}\in s(F(f^{n-1}u,f^{n-1}v),F(p,q))$$

$$\alpha d(f^nu,fp)+\beta d(f^nv,fq)+\gamma\,\frac{d(f^nu,F(f^{n-1}u,f^{n-1}v))\;d(fp,F(p,q))}{1+d(f^nu,fp)+d(f^nv,fq)} +\delta\,\frac{d(f^nu,F(p,q))\;d(fp,F(f^{n-1}u,f^{n-1}v))}{1+d(f^nu,fp)+d(f^nv,fq)}\in \bigcap_{x\in F(f^{n-1}u,f^{n-1}v)} s(x,F(p,q)).$$

Since $f^nu\in F(f^{n-1}u,f^{n-1}v)$, we have

$$\alpha d(f^nu,fp)+\beta d(f^nv,fq)+\gamma\,\frac{d(f^nu,F(f^{n-1}u,f^{n-1}v))\;d(fp,F(p,q))}{1+d(f^nu,fp)+d(f^nv,fq)} +\delta\,\frac{d(f^nu,F(p,q))\;d(fp,F(f^{n-1}u,f^{n-1}v))}{1+d(f^nu,fp)+d(f^nv,fq)}\in s(f^nu,F(p,q))=\bigcup_{z\in F(p,q)} s(d(f^nu,z)).$$

From (2.1.1), there exists $fz_n\in F(p,q)$ such that

$$\alpha d(f^nu,fp)+\beta d(f^nv,fq)+\gamma\,\frac{d(f^nu,F(f^{n-1}u,f^{n-1}v))\;d(fp,F(p,q))}{1+d(f^nu,fp)+d(f^nv,fq)} +\delta\,\frac{d(f^nu,F(p,q))\;d(fp,F(f^{n-1}u,f^{n-1}v))}{1+d(f^nu,fp)+d(f^nv,fq)}\in s(d(f^nu,fz_n)).$$

Hence

$$d(f^n u, fz_n) \precsim \alpha d(f^n u, fp) + \beta d(f^n v, fq) + \gamma \frac{d(f^n u, F(f^{n-1}u, f^{n-1}v))\ d(fp, F(p,q))}{1+d(f^n u, fp)+d(f^n v, fq)}$$
$$+\delta \frac{d(f^n u, F(p,q))\ d(fp, F(f^{n-1}u, f^{n-1}v))}{1+d(f^n u, fp)+d(f^n v, fq)}.$$

By g.l.b property of F, we obtain

$$\begin{aligned} d(f^n u, fz_n) &\precsim \alpha d(f^n u, fp) + \beta d(f^n v, fq) + \gamma \frac{d(f^n u, f^n u)\ d(fp, fz_n))}{1+d(f^n u, fp)+d(f^n v, fq)} \\ &\quad +\delta \frac{d(f^n u, fz_n))\ d(fp, f^n u)}{1+d(f^n u, fp)+d(f^n v, fq)} \\ &= \alpha d(f^n u, fp) + \beta d(f^n v, fq) + \delta \frac{d(f^n u, fz_n))\ d(fp, f^n u)}{1+d(f^n u, fp)+d(f^n v, fq)}. \end{aligned}$$

which implies that

$$|d(f^n u, fz_n)| \leq \alpha\, |d(f^n u, fp)| + \beta\, |d(f^n v, fq)| + \delta\, \frac{|d(f^n u, fz_n))|\ |d(fp, f^n u)|}{|1+d(f^n u, fp)+d(f^n v, fq)|}.$$

Now,we consider

$$\begin{aligned} |d(fp, fz_n)| &\leq |d(fp, f^n u)| + |d(f^n u, fz_n)| \\ &\leq |d(fp, f^n u)| + \alpha\, |d(f^n u, fp)| + \beta\, |d(f^n v, fq)| \\ &\quad + \delta \frac{|d(f^n u, fz_n)|\ |d(fp, f^n u)|}{|1+d(f^n u, fp)+d(f^n v, fq)|} \end{aligned}$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} |d(fp, fz_n)| \leq 0$ so that $fz_n \to fp = p$ as $n \to \infty$. Since $F(p,q)$ is closed and $fz_n \in F(p,q)$, it follows that $p \in F(p,q)$. Similarly, we have $q \in F(q,p)$. Thus (p,q) is a common coupled fixed point of F and f. This completes the proof.

Now we give an example to support Theorem 2.1.

Example 2.2. *Let* $X = [0,1]$, $d(x,y) = |x-y|\, e^{i\theta}, \forall\, x,y \in X$, *where*
$\theta = tan^{-1} \frac{y}{x}$.
Define $F : X \times X \to X$ *by* $F(x,y) = [0, \frac{x+2y}{8}]$ *and* $f : X \to X$ *by* $fx = \frac{7x}{8}$.
Then we have $d(fx, fu) = \frac{7}{8}\, |x-u|\, e^{i\theta}, d(fy, fv) = \frac{7}{8}\, |y-v|\, e^{i\theta}$ *and*
$s(F(x,y), F(u,v)) = s(\left|\frac{(x+2y)-(u+2v)}{8}\right| e^{i\theta})$.
As, $\frac{1}{7} d(fx, fu) + \frac{2}{7} d(fy, fv)$
$= \frac{1}{8}\, |x-u|\, e^{i\theta} + \frac{1}{8}\, |2y-2v|\, e^{i\theta} \geq e^{i\theta} \left|\frac{x-u+2y-2v}{8}\right| = e^{i\theta} \left|\frac{x+2y-(u+2v)}{8}\right|$,
we have
$\alpha d(fx, fu) + \beta d(fy, fv) + \gamma \frac{d(fx, F(x,y))\ d(fu, F(u,v))}{1+d(fx, fu)+d(fy, fv)}$

$$+\delta \frac{d(fx, F(u,v))\ d(fu, F(x,y))}{1+d(fx, fu)+d(fy, fv)} \in s(F(x,y), F(u,v))$$

with $\alpha = \frac{1}{7}$, $\beta = \frac{2}{7}, \gamma = \delta = 0$. *One can easily verify the remaining conditions of Theorem* 2.1. *Clearly* $(0,0)$ *is a common coupled fixed point of* F *and* f. .

Conflict of Interests

The authors declare that there is no conflict of interests.

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