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COMPLEX VALUED B-METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. In this note we establish some results on common fixed point for a pair of mappings satisfying more general contraction conditions represented by rational expressions having point dependent control functions as coefficients in complex valued b-metric spaces. The proved results generalize and extend the results of, Azam et al. [1], Bhatt et al. [3] and Mukheimer [8].

Keywords: common fixed point; complex valued b-metric space; Cauchy sequence.

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1. Introduction

In 1922, Banach [2] proved contraction principle which provides a technique for solving existence problems in many branches of mathematics such as mathematical analysis, computer sciences and engineering. Subsequently Banach contraction principle was generalized, extended and improved by many authors in different directions. In 1998, Czerwik [4] introduced the concept of b- metric space. In 2011, Azam et al.[1] introduced the notion of complex valued

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metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. After the establishment of complex valued metric spaces, many researchers have contributed with their work in this space. Rouzkard and Imdad [11] generalized Azam et al.[1]. Subsequently Sintunavarat et al. ([15],[16]) obtained common fixed point results by replacing the constant of contractive condition to control functions. On the other hand Sitthikul et al.[17] established some fixed point results by generalizing the contractive conditions in the context of complex valued metric spaces. Many researchers have contributed with different concepts in this space. One can see in [1]-[3],[9],[12],[14],[15]. In 2013, Rao et al. [10] developed the notion of complex valued b- metric spaces and proved fixed point results. After that Mukheimer [8] and Singh et al.[13] proved fixed point theorems for contractive type conditions satisfying rational inequalities in this spaces.
The aim of this paper is to study of a class of mappings satisfying a rational expression in the setting of complex value b-metric spaces.

2. Preliminaries

In what follows, we recall some definitions and notations that will be used in our note.
Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;

(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C2)$, $(C3)$ and $(C4)$ is satisfied and we write $z_1 \prec z_2$ if only $(C4)$ is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition is developed by Azam *et al.* [1].

Definition 2.1. [1] Let X be a nonempty set. A mapping $d: X \times X \to C$ satisfies the following conditions:

(CM1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(CM2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(CM3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.1. Let $X = C$ be a set of complex number. Define $d: C \times C \to C$. By

$$d(z_1,z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C,d) is a complex valued metric space.

Definition 2.2. [10] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d: X \times X \to C$ satisfies the following conditions:

(CVBM1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(CVBM2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(CVBM3) $d(x,y) \precsim s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

Then d is called a complex valued b-metric on X and (X,d) is called a complex valued b-metric space.

Example 2.2. Let $X = [0,1]$. Define the mapping $d: X \times X \to C$ by

$$d(x,y) = |x-y|^2 + i|x-y|^2,\ for\ all\ x,y \in X.$$

Then (X,d) is a complex valued b- metric space with $s = 2$.

Definition 2.3. [10] Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in C$ such that $B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever for every $0 \prec r \in C$ such that $B(x,r) \cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each limit point of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is belong to B.

(v) The family $F = \{B(x,r) : x \in X \ \ and \ \ 0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.4. [10] Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be converges to x and x is a limit point of $\{x_n\}$. We denote this by $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

(2) If for every $c \in C$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n,x_{n+m}) \prec c$ where $m \in N$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued b- metric space.

Lemma 2.1. [10] *Let (X,d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2.1. [10] *Let (X,d) be a complex valued b- metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.*

3. Main Results

We start to this section with the following observation.

Proposition 3.1. Let (X,d) be a complex valued b-metric space with $s \geq 1$ and $S,T : X \to X$. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, \qquad x_{2n+2} = Tx_{2n+1}, \ \forall\, n = 0,1,2,.... \tag{1}$$

Assume that $\exists$ a mapping $\lambda : X \times X \times X \to [0,1)$ such that
$\lambda(TSx,y,a) \leq \lambda(x,y,a)$ and $\lambda(x,STy,a) \leq \lambda(x,y,a)$;$\forall x,y \in X, n = 0,1,2$ where a is fixed element of X.
Then $\lambda(x_{2n},y,a) \leq \lambda(x_0,y,a)$ and $\lambda(x,x_{2n+1},a) \leq \lambda(x,x_1,a)$.

Proof. Let $x,y \in X$ and $n = 0,1,2,....$ we see that

$$\begin{aligned}\lambda(x_{2n},y,a) =&\lambda(TSx_{2n-2},y,a) \leq \lambda(x_{2n-2},y,a)\\ =&\lambda(TSx_{2n-4},y,a) \leq ... \leq \lambda(x_0,y,a).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\lambda(x,x_{2n+1},a) =&\lambda(x,STx_{2n-1},a) \leq \lambda(x,x_{2n-1},a)\\ =&\lambda(x,STx_{2n-3},a) \leq ... \leq \lambda(x,x_1,a).\end{aligned}$$

Theorem 3.1. Let (X,d) be a complete complex valued b- metric space with $s \geq 1$ and S,T : $X \to X$. If $\exists$ mappings $\lambda,\mu,\delta : X \times X \times X \to [0,1)$ such that $\forall x,y \in X$,

(a) $\lambda(TSx,y,a) \leq \lambda(x,y,a)$ and $\lambda(x,STy,a) \leq \lambda(x,y,a)$,
$\mu(TSx,y,a) \leq \mu(x,y,a)$ and $\mu(x,STy,a) \leq \mu(x,y,a)$,
$\delta(TSx,y,a) \leq \delta(x,y,a)$ and $\delta(x,STy,a) \leq \delta(x,y,a)$;

(b)

$$\begin{aligned}d(Sx,Ty) \precsim \lambda(x,y,a)d(x,y) + \mu(x,y,a)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}+\\ \delta(x,y,a)\{\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}\};\end{aligned} \tag{2}$$

(c)

$$s\{\lambda(x,y,a)+\mu(x,y,a)+\delta(x,y,a)\} < 1.$$

Then S and T have a unique common fixed point.

Proof. Let $x,y \in X$, from (2), we have

$$\begin{aligned}d(Sx,TSx) \precsim& \lambda(x,Sx,a)d(x,Sx) + \mu(x,Sx,a)\frac{d(x,Sx)d(Sx,TSx)}{1+d(x,Sx)}+\\ &\delta(x,Sx,a)\{\frac{d(x,Sx)d(x,TSx)+d(Sx,TSx)d(Sx,Sx)}{1+d(x,TSx)+d(Sx,Sx)}\}\\ =& \lambda(x,Sx,a)d(x,Sx) + \mu(x,Sx,a)\frac{d(x,Sx)d(Sx,TSx)}{1+d(x,Sx)}+\\ &\delta(x,Sx,a)\frac{d(x,Sx)d(x,TSx)}{d(x,TSx)}.\end{aligned}$$

$$Or \quad |d(Sx_1,TSx)| \leq \lambda(x,Sx,a)|d(x,Sx)| + \mu(x,Sx,a)|\frac{d(x,Sx)d(Sx,TSx)}{1+d(x,Sx)}|+ \delta(x,Sx,a)|d(x,Sx)| \tag{3}$$

$$\leq \lambda(x,Sx,a)|d(x,Sx)| + \mu(x,Sx,a)|\frac{d(x,Sx)}{1+d(x,Sx)}|\,|d(Sx,TSx)|+ \delta(x,Sx,a)|d(x,Sx)|.$$

$$\Rightarrow |\, d(Sx,TSx)\,| \leq \lambda(x,Sx,a)\,|\,d(x,Sx)\,| + \mu(x,Sx,a)\,|\,d(Sx,TSx)\,| + \delta(x,Sx,a)\,|\,d(x,Sx)\,|\,. \tag{4}$$

Similarly, from (2), we have

$$d(STy,Ty) \precsim \lambda(Ty,y,a)d(Ty,y) + \mu(Ty,y,a)\frac{d(Ty,STy)d(y,Ty)}{1+d(y,Ty)} + \delta(Ty,y,a)\{\frac{d(Ty,STy)d(Ty,Ty)+d(y,Ty)d(y,STy)}{1+d(Ty,Ty)+d(y,STy)}\}.$$

With the same treatment as above following yields,

$$|\, d(STy,Ty)\,| \leq \lambda(Ty,y,a)\,|d(Ty,y)| + \mu(Ty,y,a)\,|\,d(Ty,STy)\,| + \delta(Ty,y,a)\,|\,d(y,Ty)\,|\,. \tag{5}$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (1). We show that $\{x_n\}$ is a Cauchy sequence. From Proposition 3.1 and inequalities (4), (5) and for all $k = 0,1,2,...$, we have

$$\begin{aligned}
|\, d(x_{2k+1},x_{2k})\,| &= |\, d(STx_{2k-1},Tx_{2k-1})\,| \\
&\leq \lambda(Tx_{2k-1},x_{2k-1},a)\ |\, d(Tx_{2k-1},x_{2k-1})\,| + \mu(Tx_{2k-1},x_{2k-1},a) \\
&\quad |\, d(Tx_{2k-1},STx_{2k-1})\,| + \delta(Tx_{2k-1},x_{2k-1},a)\ |\, d(Tx_{2k-1},x_{2k-1})\,| \\
&= \lambda(x_{2k},x_{2k-1},a)|d(x_{2k-1},x_{2k})| + \mu(x_{2k},x_{2k-1},a)\,|\, d(x_{2k},x_{2k+1})\,| + \\
&\quad \delta(x_{2k},x_{2k-1},a)\,|\, d(x_{2k-1},x_{2k})\,| \\
&\leq \lambda(x_0,x_{2k-1},a)\ |\, d(x_{2k-1},x_{2k})\,| + \mu(x_0,x_{2k-1},a)\,|\, d(x_{2k+1},x_{2k})\,| + \\
&\quad \delta(x_0,x_{2k-1},a)\,|\, d(x_{2k-1},x_{2k})\,| \\
&\leq \lambda(x_0,x_1,a)\,|\, d(x_{2k-1},x_{2k})\,| + \mu(x_0,x_1,a)\,|\, d(x_{2k+1},x_{2k})\,| + \\
&\quad \delta(x_0,x_1,a)\,|\, d(x_{2k-1},x_{2k})\,|\,.
\end{aligned}$$

Which implies that

$$| d(x_{2k+1}, x_{2k}) | \leq \frac{\{\lambda(x_0, x_1, a) + \delta(x_0, x_1, a)\}}{1 - \mu(x_0, x_1, a)} |d(x_{2k-1}, x_{2k})|.$$

Similarly, one can obtain

$$| d(x_{2k+2}, x_{2k+1}) | \leq \frac{\{\lambda(x_0, x_1, a) + \delta(x_0, x_1, a)\}}{1 - \mu(x_0, x_1, a)} |d(x_{2k}, x_{2k+1})|.$$

Let $p = \frac{\lambda(x_0,x_1,a)+\delta(x_0,x_1,a)}{1-\mu(x_0,x_1,a)}$.

Thus we have,

$$| d(x_{2k+2}, x_{2k+1}) | \leq p \mid d(x_{2k}, x_{2k+1}) \mid$$

or in general

$$\begin{aligned}
| d(x_{n+1}, x_n) | &\leq p \mid d(x_{n-1}, x_n) \mid, \forall n \in N \\
&\leq p^2 \mid d(x_{n-2}, x_{n-1}) \mid \\
&\leq \ldots \\
&\leq p^n \mid d(x_0, x_1) \mid .
\end{aligned}$$

Thus for any $m > n, \forall\, m, n \in N$, we have

$$\begin{aligned}
| d(x_n, x_m) | &\leq s \mid d(x_n, x_{n+1}) \mid + s \mid d(x_{n+1}, x_m) \mid \\
&\leq s \mid d(x_n, x_{n+1}) \mid + s^2 \mid d(x_{n+1}, x_{n+2}) \mid + s^2 \mid d(x_{n+2}, x_m) \mid \\
&\leq s \mid d(x_n, x_{n+1}) \mid + s^2 \mid d(x_{n+1}, x_{n+2}) \mid + s^3 \mid d(x_{n+2}, x_{n+3}) \mid + \ldots + s^{m-n} \mid d(x_{m-1}, x_m) \mid \\
&\leq s p^n \mid d(x_0, x_1) \mid + s^2 p^{n+1} \mid d(x_0, x_1) \mid + s^3 p^{n+2} \mid d(x_0, x_1) \mid + \ldots + s^{m-n} p^{m-1} \mid d(x_0, x_1) \mid \\
&= \sum_{i=1}^{m-n} s^i p^{i+n-1} \mid d(x_0, x_1) \mid \\
&\leq \sum_{i=1}^{m-n} s^{i+n-1} p^{i+n-1} \mid d(x_0, x_1) \mid \\
&= \sum_{t=n}^{m-1} (sp)^t \mid d(x_0, x_1) \mid, t = i + n - 1 \\
&\leq \sum_{t=n}^{\infty} (sp)^t \mid d(x_0, x_1) \mid \\
&= \frac{(sp)^n}{1 - sp} \mid d(x_0, x_1) \mid \to 0, as\ \ m, n \to \infty (since\ \ sp < 1)
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in (X,d).

By completness of $X, \exists, z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now we show that z is a fixed point of S.

By (2) and Proposition 3.1, we have

$$\begin{aligned}
d(z,Sz) &\precsim s[d(z,Tx_{2n+1}) + d(Tx_{2n+1},Sz)] \\
&= s[d(z,x_{2n+2}) + d(Sz,Tx_{2n+1})] \\
&\precsim s[d(z,x_{2n+2}) + \lambda(z,x_{2n+1},a)d(z,x_{2n+1}) + \mu(z,x_{2n+1},a)\frac{d(z,Sz)d(x_{2n+1},Tx_{2n+1})}{1+d(z,x_{2n+1})} \\
&+ \delta(z,x_{2n+1},a)\{\frac{d(z,Sz)d(z,Tx_{2n+1}) + d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Sz)}{1+d(z,Tx_{2n+1})+d(x_{2n+1},Sz)}\}] \\
&\precsim s[d(z,x_{2n+2}) + \lambda(z,x_1,a)d(z,x_{2n+1}) + \mu(z,x,a)\frac{d(z,Sz)d(x_{2n+1},x_{2n+2})}{1+d(z,x_{2n+1})} \\
&+ \delta(z,x,a)\{\frac{d(z,Sz)d(z,x_{2n+2}) + d(x_{2n+1},x_{2n+2})d(x_{2n+1},Sz)}{1+d(z,x_{2n+2})+d(x_{2n+1},Sz)}\}]
\end{aligned}$$

employing $n \to \infty$, gives

$d(z,Sz) = 0 \Rightarrow Sz = z$.

Next we will show that z is the fixed point of T. By (2) we have

$$\begin{aligned}
d(z,Tz) &\precsim d(z,Sx_{2n}) + d(Sx_{2n},Tz) \\
&\precsim d(z,x_{2n+1}) + \lambda(x_{2n},z,a)d(x_{2n},z) + \mu(x_{2n},z,a)\frac{d(x_{2n},Sx_{2n})d(z,Tz)}{1+d(x_{2n},z)} \\
&+ \delta(x_{2n},z,a)\{\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tz) + d(z,Tz)d(z,Sx_{2n})}{d(x_{2n},Tz)+d(z,Sx_{2n})}\} \\
&\precsim d(z,x_{2n+1}) + \lambda(x_0,z,a)d(x_{2n},z) + \mu(x_0,z,a)\frac{d(x_{2n},x_{2n+1})d(z,Tz)}{1+d(x_{2n},z)} \\
&+ \delta(x_0,z,a)\{\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tz) + d(z,Tz)d(z,x_{2n+1})}{d(x_{2n},Tz)+d(z,x_{2n+1})}\}
\end{aligned}$$

Taking limit $n \to \infty$, we get

$d(z,Tz) = 0$ and hence $Tz = z$.

$\Rightarrow z$ is a common fixed point of S and T.

Next the uniqueness of z is established.

Suppose there exists $w \in X$ such that $Sw = Tw = w$ and $w \neq z$.

Then

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\precsim \lambda(z,w,a)d(z,w)+\mu(z,w,a)\frac{d(z,Sz)d(w,Tw)}{1+d(z,w)} \\ &+\delta(z,w,a)\{\frac{d(z,Sz)d(z,Tw)+d(w,Tw)d(w,Sz)}{d(z,Tw)+d(w,Sz)}\} \\ &= \lambda(z,w,a)d(z,w). \end{aligned}$$

Now we have

$$| d(z,w) | \leq \lambda(z,w,a) \mid d(z,w) |$$

$$\Rightarrow | d(z,w) | \leq \lambda(z,w,a) \mid d(z,w) | .$$

Leads to a contraction, as

$$s\{\lambda(x,y,a)+\mu(x,y,a)\}+\delta(x,y,a)<1, i.e. \lambda(z,w,a)<1 \Rightarrow | d(z,w) | = 0 \Rightarrow z=w.$$

Thus S and T have a unique common fixed point.

By choosing $\mu=0, \delta=0$ in Theorem 3.1, following corollaries are deduced.

Corollary 3.1. Let (X,d) be a complete complex valued b-metric space with $s \geq 1$
$S,T;X \to X$, If $\exists$ a mapping $\lambda : X \times X \times X \to [0,1)$
such that
$\lambda(TSx,y,a) \leq \lambda(x,y,a), \lambda(x,STy,a) \leq \lambda(x,y,a)$, and $s\lambda(x,y,a)<1$
satisfying

$$d(Sx,Ty) \precsim \lambda(x,y,a)d(x,y).$$

Then S and T have a unique common fixed point.

To be specific when $\mu=0$ in Theorem 3.1, one gets the following result as corollary.

Corollary 3.2. Let(X,d) be a complete complex valued b-metric space with $s \geq 1$
$S,T;X \to X$, If; $\exists$ a mapping $\lambda, \delta : X \times X \times X \to [0,1)$
such that

$$\lambda(TSx,y,a) \leq \lambda(x,y,a) \textit{ and } \lambda(x,STy,a) \leq \lambda(x,y,a),$$

$$\delta(TSx,y,a) \leq \delta(x,y,a) \text{ and } \delta(x,STy,a) \leq \delta(x,y,a)$$

$$\text{and } s\{\lambda(x,y,a)+\delta(x,y,a)\} < 1,$$

satisfying

$$d(Sx,Ty) \precsim \lambda(x,y,a)d(x,y)+\delta(x,y,a)\frac{\{d(x,Sx)d(y,Ty)+d(y,Ty)d(y,Sx)\}}{d(x,Ty)+d(y,Sx)}.$$

Then S and T have a unique common fixed point.

In the Theorem 3.1 choosing $\delta = 0$, one gets the following result as corollary.

Corollary 3.3. Let (X,d) be a complete complex valued b-metric space with $s \geq 1$ and $S,T : X \to X$. If $\exists$ mapping $\lambda, \mu : X \times X \times X \to [0,1)$,
such that $s\{\lambda(x,y,a)+\mu(x,y,a) < 1\}$ and

$$\lambda(TSx,y,a) \leq \lambda(x,y,a) \text{ and } \lambda(x,STy,a) \leq \lambda(x,y,a),$$

$$\mu(TSx,y,a) \leq \mu(x,y,a) \text{ and } \mu(x,STy,a) \leq \mu(x,y,a);$$

also satisfying

$$d(Sx,Ty) \precsim \lambda(x,y,a)d(x,y)+\mu(x,y,a)\frac{\{d(x,Sx)d(y,Ty)\}}{1+d(x,y)}.$$

Then S and T have a unique common fixed point.

Remark 3.1. In the Corollary 3.3, if we replace $\lambda, \mu : X \times X \times X \to [0,1)$ with
$\lambda(x,y,a) = \lambda$ and $\mu(x,y,a) = \mu$,where $s(\mu+\lambda) < 1$ and

$$d(Sx,Ty) \precsim \lambda d(x,y)+\mu\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

Then S and T have a unique common fixed point.
Since $s(\lambda+\mu) < 1 \Rightarrow s\lambda+\mu < 1$, with this fact from above we obtain Theorem 15 of Mukheimer [8].

Remark 3.2. Replacing $S = T$ and $\lambda(x,y,a) = \lambda, \mu(x,y,a) = \mu$ and
$\delta(x,y,a) = 0$ in Theorem 3.1, Corollary 16 of Mukheimer [8] is obtained.

Note if $\lambda = \mu = 0$ in Theorem 3.1, following corollary is found.

Corollary 3.4. Let (X,d) be a complete complex valued b-metric space with $s \geq 1$ and $S,T : X \to X$. If $\delta : X \times X \times X \to [0,1)$ $s.t.$ $\forall x,y \in X$, $\delta(TSx,y,a) \leq \delta(x,y,a)$ and $\delta(x,STy,a) \leq \delta(x,y,a)$ and $s\delta(x,y,a) < 1$ satisfying

$$d(Sx,Ty) \leq \delta(x,y,a)\frac{\{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)\}}{d(x,Ty)+d(y,Sx)}$$

Then S and T have a unique common fixed point.

Remark 3.3. In the above Corollary 3.4, replace $\delta : X \times X \times X \to [0,1)$ as $\delta(x,y,a) = a$, where $sa < 1$, and

$$d(Sx,Ty) \precsim a\frac{\{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)\}}{d(x,Ty)+d(y,Sx)}$$

Then S and T have a unique common fixed point .This result is Theorem 19 of Mukheimer [8].

Remark 3.4. If we take $s = 1$ in Remark 3.3, above result reduces due to Bhatt et al. [3]. On the other hand on taking $S = T$ and $\lambda(x,y,a) = 0, \mu(x,y,a) = 0$ and $\delta(x,y,a) = a$ in our Theorem 3.1, One gets Corollary 20 of Mukheimer [8].

Remark 3.5. Choosing $s = 1$, in Remark 3.1, Azam et al. [1] is obtained.

Finally setting $S = T$ in Theorem 3.1, we get

Corollary 3.5. Let (X,d) be a complete complex valued b-metric space and $S : X \to X$, If $\exists$ a mapping $\lambda, \mu, \delta : X \times X \times X \to [0,1)$ such that

(a) $\lambda(S^2x,y,a) \leq \lambda(x,y,a)$ and $\lambda(x,S^2y,a) \leq \lambda(x,y,a)$,
$\mu(S^2x,y,a) \leq \mu(x,y,a)$ and $\mu(x,S^2y,a) \leq \mu(x,y,a)$,
$\delta(S^2x,y,a) \leq \delta(x,y,a)$ and $\delta(x,S^2y,a) \leq \delta(x,y,a)$;

(b) $\lambda(x,y,a) + \mu(x,y,a) + \delta(x,y,a) < 1$; ;

(c)

$$d(Sx,Sy) \precsim \lambda(x,y,a)d(x,y) + \mu(x,y,a)\frac{d(x,Sx)d(y,Sy)}{1+d(x,y)} +$$
$$\delta(x,y,a)\{\frac{d(x,Sx)d(x,Sy)+d(y,Sy)d(y,Sx)}{d(x,Sy)+d(y,Sx)}\};$$

Then S has a unique fixed point.

Author's contribution

All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numerical Functional Analysis and Optimization 32 (2011), No.3, 243-253.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fundamenta Mathematicae 3 (1922), 133-181.

[3] S. Bhatt, S. Chaukiyal, and R. C. Dimri, Common fixed point of mapping rational inequality in complex valued metric spaces, International Journal of pure and Applied Mathematics, 73 (2011), No.2, 159-164.

[4] S. Czerwik, Nonlinear set-valued contraction mapping in b- metric spaces, Atti Sem. Mat. Univ. Modena 46 (1998), 263-276.

[5] B.K. Dass, S. Gupta, An extension of banach Contraction principle through rational expression. Indian J Pure App. Math 6(12)(1975), 1455-1458.

[6] S.M. Kang, M. Kumar, P. Kumar, S. Kumar, Coupled Fixed Point theorems in Complex Valued Metric Spaces, Int. Journal of Math. Analysis, 7(46)(2013), 2269-2277.

[7] M. A. Kutbi, A. Azam, A. Jamshaid, C. Bari, Some coupled fixed points results for Generalized contraction in complex valued metric spaces, J of appl. Mathematics, 2013 (2013) Article ID 352927.

[8] A. A. Mukheimer, Some common fixed point theorems in complex valued b- metric spaces, The Scientific World Journal 2014 (2014), Article ID 587825.

[9] H.K. Nashine, M. Imdad, M. Hassan, Common fixed point theorems under rational contractions in complex valued metric spaces, J. Nonlinear Sci. Appl. 7(2014), 42-50.

[10] K. P. R. Rao, P. Ranga Swamy, J. Rajendra Prasad, A common fixed point theorem in complex valued b-metric spaces, Bull. Math. Stat. Res. 1 (2013), Issue 1.

[11] F. Rouzkard, M. Imdad, Some common fixed point theorems on complex valued metric spaces, Computer and Mathematics with Applications 64 (2012), 1866-1874.

[12] N. Singh D. Singh, A. Badal, V. Joshi, Fixed point theorems in complex valued metric spaces, Journal of the Egyptian Mathematical Society 24 (2016), Issue 3, 402-409.

[13] D. Singh, O. P. Chauhan, N. Singh, V. Joshi, Common fixed point theorems in complex valued b-metric spaces, J. Math. Comput. Sci. 5 (2015), No. 3, 412-429.

[14] D. Singh, O. P. Chauhan, N. Singh, V. Joshi, Complex valued rectangular metric spaces and Common fixed point theorems, Bulletin of Mathematical Analysis and Applications, 7 (2015), Pages 1-13.

[15] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex Valued metric spaces and applications. J Inequal Appl, 2012(2012), Article ID 84.

[16] W. Sintunavarat, J. C. Yeol and P. Kumam, Urysohn integral equations approach by common fixed points in complex-valued metric spaces, Advances in Difference Equations, 2013 (2013), Article ID 49.

[17] K. Sitthikul, S. Saejung, Some fixed point theorems in complex valued metric space,Fixed Point Theory and Applications, 2012 (2012), Article ID 189.

[18] R. K. Verma and H. K. Pathak, Common fixed point theorems using property (E.A) in complex-vauled metric spaces, Thai J. of Mathematics, 11 (1013) No.2, 347-355.