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FIXED POINT THEOREMS FOR GENERALIZED WEAKLY CONTRACTIVE MAPPINGS IN QUASI-METRIC SPACE

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Abstract. In this paper we establish a fixed point result of generalized weakly contractive mapping and generalized altering distance on a complete quasi-metric space. We support our results by an examples.

Keywords: fixed point; complete quasi-metric spaces; generalized weak contraction; generalized altering distance; partially ordered set.

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1. Introduction

Banach's contraction principle is one of very important theorems has been generalized in various directions. The concept of weak contraction has introduced by guerre delabre in hilbert space [1], Rhoeds extend this concept to metric space[2]. Weakly contractive mapping used in a several work $[3-7]$ to show a fixed point theorem (for a self mapping and a common fixed point

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result for two self-mapping defined on a complete metric space). In [9] Binayak Choudhury proposed the definition of generalized altering distance function. he proved a common fixed point for two self-mapping satisfying a contractive inequality which involves two generalized altering distance. Many mathematics researchers obtained some results of fixed point in quasi-metric space. In$[8-10]$the authors obtained the existence and uniqueness of a fixed point in quasi-metric space for some type of weakly contractive-mapping.

The purpose of this work is to show some fixed point results in quasi-metric space, firstly for generalized weakly contractive mapping, secondly for generalized altering distance mapping.

2. Preliminaries

In 2010, Binayak and all [10] have established the following result.

Theorem 2.1. *Let (X,d) be a complete metric space, T a self-mapping of X. such that for all $x,y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max\{d(x,y),d(y,Ty)\})$$

where

$$m(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\},$$

$\psi,\phi : [0,+\infty) \to [0,+\infty)$ are a continuous function with ψ is monotone increasing and ($\psi(t) = \phi(t) = 0$ if and only if $t = 0$).Then T has a unique fixed point.

Binayak choudhury,[9] has introduced a notion of generalization altering distances to a three-variable function, and has established the following result.

Theorem 2.2. *Let (X,d) be a complete metric space, T and S be a self mappings of X such that, for all $x,y \in X$,*

$$\phi_1(d(Sx,Ty)) \leq \psi_1(d(x,y),d(x,Sx),d(y,Ty)) - \psi_2(d(x,y),d(x,Sx),d(y,Ty))$$

where $\psi_1,\psi_2 : [0,+\infty)^3 \to [0,+\infty)$ are a continuous functions with ψ_1 is monotone increasing in all the three variables and ($\psi_1(x,y,z) = \psi_2(x,y,z) = 0$ if and only if $x = y = z = 0$). and $\phi_1 : x \mapsto \psi_1(x,x,x)$.

Then, T and S has a unique common fixed point.

Our propose here is to prove the previous theorems without symmetry(quasi-metric space), we add a new condition for all $x,y \in X$ $d^{-1}(x,y) \leq d^{-1}(x,T^2y)$, without this condition we can't prove our results. We have change $m(x,y)$ of theorem 2.1 by $\max\{d(x,y),d(x,Tx),d(y,Ty)\}$. and we show theorem 2.2 under our new condition for one application.

Definition 2.2. Let X be a nonempty set and let $d : X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying following conditions :
(i) $d(x,y) = 0 \Leftrightarrow x = y$
(ii) $d(x,y) \leq d(x,z) + d(z,y)$
Then d is called a quasi-metric on X.

Definition 2.3. Let (X,d) be a quasi-metric space, $(x_n)_n$ be a sequence in X,
and $x \in X$. The sequence $(x_n)_n$ converges to x if and only if $\lim_{n\to+\infty} d(x_n,x) = \lim_{n\to+\infty} d(x,x_n) = 0$.

Definition 2.4. Let (X,d) be a quasi-metric space and $(x_n)_n$ be a sequence in X. We say that $(x_n)_n$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$, for all $n > m \geq N$.

Definition 2.5. Let (X,d) be a quasi-metric space and $(x_n)_n$ be a sequence in X. We say that $(x_n)_n$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$, for all $m > n \geq N$.

Definition 2.6. Let (X,d) be a quasi-metric space and $(x_n)_n$ be a sequence in X. We say that (X,d) is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$, for all $m,n \geq N$.

Definition 2.7. Let (X,d) be a quasi-metric space. We say that

(1) (X,d) is left-complete if and only if each left-Cauchy sequence in X is convergent.
(2) (X,d) is right-complete if and only if each right-Cauchy sequence in X is convergent.
(3) (X,d) is complete if and only if each Cauchy sequence in X is convergent.

Remark 2.8.

- A sequence $(x_n)_n$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

- Any metric space is quasi-metric, but the converse is not true in general.
- The function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$, for all $x,y \in X$, is also a quasi-metric on X.
- The base of the topology τ_d is open balls $\{B_d(x,\varepsilon)\ ;\ x \in X,\ \varepsilon > 0\}$, where for all $x \in X$ and $\varepsilon > 0$, $B_d(x,\varepsilon) = \{y \in X\ ;\ d(x,y) < \varepsilon\}$.

3. Main results

We consider two functions $\phi,\ \psi\ : [0,+\infty[\to [0,+\infty[$ satisfied :

(1) ϕ continuous,

(2) ψ is monotone nondecreasing and continuous,

(3) $\psi(t) = 0$ (resp. $\phi(t) = 0$) if and only if $t = 0$.

Theorem 3.1. *Let (X,d) be a complete quasi-metric space and T a self mapping of X such that for all $x,y \in X$,*

$$\psi(d(Tx,Ty)) \le \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty))) \tag{3.1}$$

where

$$m(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and

$$d^{-1}(x,y) \le d^{-1}(x,T^2y)$$

Then, T has a unique fixed point.

Proof. First step. Let $x_0 \in X$, we define a sequence $(x_n)_n$ in X such that $x_{n+1} = Tx_n$, for all integer $n \in \mathbb{N}$.

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T.

Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (3.1), we obtain :

$$\psi(d(Tx_n,Tx_{n+1})) \le \psi(m(x_n,x_{n+1})) - \phi(\max\{d(x_n,x_{n+1}),d(x_{n+1},Tx_{n+1})\}) \tag{3.2}$$

$$\psi(d(x_{n+1},x_{n+2})) \le \psi(m(x_n,x_{n+1})) - \phi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))\}),$$

we have

$$m(x_n,x_{n+1}) = \max\{d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}$$

So,

$$\psi(d(x_{n+1},x_{n+2})) \leq \psi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) - \phi(\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\})$$

Suppose that $d(x_n,x_{n+1}) \leq d(x_{n+1},x_{n+2})$ for some positive integer n, we have :

$$\psi(d(x_{n+1},x_{n+2})) \leq \psi(d(x_{n+1},x_{n+2})) - \phi(d(x_{n+1},x_{n+2}))$$

That is $\phi(d(x_{n+1},x_{n+2})) \leq 0$ which implies $d(x_{n+1},x_{n+2}) = 0$ i.e. $x_{n+1} = x_{n+2}$, contradicting our assumption that $x_{n+1} \neq x_{n+2}$ for each $n \in \mathbb{N}$.
Then, $(d(x_n,x_{n+1}))_n$ is monotone decreasing sequence of non negative real numbers.

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}), \text{ for all } n \in \mathbb{N}$$

Substituting $x = x_{n+1}$ and $y = x_n$ in (3.1)

$$\psi(d(x_{n+2},x_{n+1})) \leq \psi(m(x_{n+1},x_n)) - \phi(max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\}),$$

we have :

$$m(x_{n+1},x_n) = max\{d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\}$$

$$\psi(d(x_{n+2},x_{n+1})) \leq \psi(\max\{d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})\}) - \phi(\max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\})$$

Since $(d(x_n,x_{n+1}))_n$ is monotone decreasing sequence of non negative real numbers,

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}), \text{ for all } n \in \mathbb{N},$$

so

$$\psi(d(x_{n+2},x_{n+1})) \leq \psi(\max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\}) - \phi(\max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\})$$

Suppose that $d(x_{n+1},x_n) \leq d(x_{n+2},x_{n+1})$ for some positive integer n

Case 1 : $d(x_{n+1},x_n) \geq d(x_n,x_{n+1})$

$$\psi(d(x_{n+1},x_n)) \leq \psi(d(x_{n+2},x_{n+1})) \leq \psi(d(x_{n+1},x_n)) - \phi(d(x_{n+1},x_n))$$

Then

$$\phi(d(x_{n+1},x_n)) \leq 0$$

Imply $d(x_{n+1},x_n) = 0$ i.e. $x_n = x_{n+1}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

Case 2 : $d(x_n,x_{n+1}) > d(x_{n+1},x_n)$

$$\psi(d(x_{n+2},x_{n+1})) \leq \psi(d(x_n,x_{n+1})) - \phi(d(x_n,x_{n+1}))$$

Or, for each $x,y \in X$, $d(y,x) \leq d(T^2y,x)$, so $d(x_n,x_{n+1}) \leq d(x_{n+2},x_{n+1})$

$$\psi(d(x_n,x_{n+1})) \leq \psi(d(x_{n+2},x_{n+1})) \leq \psi(d(x_n,x_{n+1})) - \phi(d(x_n,x_{n+1}))$$

Then

$$\phi(d(x_n,x_{n+1})) \leq 0$$

Imply $d(x_n,x_{n+1}) = 0$ i.e. $x_n = x_{n+1}$, contradicting our assumption that $x_n \neq x_{n+1}$, for each $n \in \mathbb{N}$.

Hence, $d(x_{n+2},x_{n+1}) \leq d(x_{n+1},x_n)$, for each $n \in \mathbb{N}$.

$(d(x_{n+1},x_n))_n$ is monotone decreasing sequence of non negative real numbers.
Consequently, there exists $r > 0$ such that :

$$d(x_n,x_{n+1}) \longrightarrow r \ \ as \quad n \longrightarrow \ \infty,$$

we have :

$$\psi(d(x_{n+1},x_{n+2})) \leq \psi(d(x_n,x_{n+1})) - \phi(d(x_n,x_{n+1})) \tag{3.3}$$

Letting $n \longrightarrow \infty$ in (3.3) we obtain :

$$\psi(r) \leq \psi(r) - \phi(r)$$

So $\phi(r) \leq 0$ i.e. $r = 0$.

$$d(x_n,x_{n+1}) \longrightarrow 0 \ \ as \quad n \longrightarrow \ \infty$$

Also, there exists $r' > 0$, such that :

$$d(x_{n+1},x_n) \longrightarrow r' \ \ as \quad n \longrightarrow \ \infty,$$

we have :

$$\psi(d(x_{n+2},x_{n+1})) \leq \psi(\max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\}) - \phi(\max\{d(x_{n+1},x_n),d(x_n,x_{n+1})\}) \tag{3.4}$$

Letting $n \longrightarrow \infty$ in (3.4) we obtain :

$$\psi(r') \leq \psi(\max\{r',0\}) - \phi(\max\{r',0\})$$

So $\phi(r') \leq 0$ i.e. $r' = 0$.

$$d(x_{n+1},x_n) \longrightarrow 0 \ \ as \quad n \longrightarrow \ \infty$$

Second step. Next we show that $(x_n)_n$is a Cauchy sequence.

Firstly we show $(x_n)_n$ is a right-Cauchy sequence, if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $(m(k))_k$ and $(n(k))_k$ such that, for all positive integers k, $n(k) > m(k) > k$,

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon$$

and

$$d(x_{m(k)},x_{n(k)-1}) < \varepsilon$$

we have :

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$$

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq \varepsilon + d(x_{n(k)-1}, x_{n(k)})$$

Taking the limit as $k \to \infty$

$$d(x_{m(k)}, x_{n(k)}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

Again

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})$$

So,

$$d(x_{m(k)+1}, x_{n(k)+1}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

Setting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (3.1), we obtain :

$$\psi(d(x_{m(k)+1}, x_{n(k)+1})) \leq$$

$$\psi(\max\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1})\})$$

$$-\phi(\max\{d(x_{m(k)}, x_{n(k)}), d(x_{n(k)}, x_{n(k)+1})\})$$

Letting $k \longrightarrow +\infty$ in the above inequality and using the continuity of ψ and ϕ, we have :

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$$

which is a contradiction by virtue of a property of ϕ.

Consequently, $(x_n)_n$ is a right-Cauchy sequence in (X, d).

Secondly we show $(x_n)_n$ is a left-Cauchy sequence, if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $(m(k))_n$and $(n(k))_n$ such that for all positive integers k, $n(k) > m(k) > k$,

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon$$

and

$$d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$$

we have :

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})$$

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + \varepsilon$$

Taking the limit as $k \to +\infty$, we obtain :

$$d(x_{n(k)}, x_{m(k)}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

Again

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1})$$

So,

$$d(x_{n(k)+1}, x_{m(k)+1}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

Setting $x = x_{n(k)}$ and $y = x_{m(k)}$ in (3.1) we obtain

$$\psi(d(x_{n(k)+1}, x_{m(k)+1})) \leq$$

$$\psi(\max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\})$$

$$-\phi(\max\{d(x_{n(k)}, x_{m(k)}), d(x_{m(k)}, x_{m(k)+1})\})$$

Letting $k \longrightarrow \infty$ in the above inequality and using the continuity of $\psi$and ϕ, we have $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$, which is a contradiction by virtue of a property of ϕ Consequently, $(x_n)_n$ is a left-Cauchy sequence in (X,d). By Remark, we deduce that x_n is a Cauchy sequence in complete quasi-metric space (X,d). It implies that there exists, a $p \in X$ such that $\lim_{n\to\infty} d(x_n, p)$=$\lim_{n\to\infty} d(p, x_n) = 0$.

Third step. Putting $x = x_n$ and $y = p$ in (3.1) we have :

$$\psi(d(x_{n+1}, Tp)) \leq \psi(\max\{d(x_n, p), d(x_n, x_{n+1}), d(p, Tp)\}) - \phi(\max\{d(x_n, p), d(p, Tp)\})$$

Since,

$$d(p,Tp)-d(p,x_{n+1})\leq d(x_{n+1},Tp)\leq d(x_{n+1},p)+d(p,Tp)$$

and

$\lim_{n\to\infty} d(x_n,p)$=$\lim_{n\to\infty} d(p,x_n)=0$, so taking the limit as $n\longrightarrow\infty$ in the above precedent inequality, we obtain :

$$\psi(d(p,Tp))\leq\psi(d(p,Tp))-\phi(d(p,Tp))$$

Imply $d(p,Tp)=0$ i.e. $p=Tp$. Hence p is a fixed point of T.

Uniqueness. Let $q\in X$ such that $Tq=q$.

Putting $x=p$ and $y=q$ in (3.1) we have :

$$\psi(d(p,q))\leq\psi(\max\{d(p,q)\})-\phi(d(p,q))$$

$$\psi(d(p,q))\leq\psi(d(p,q))-\phi(d(p,q))$$

So $\phi(d(p,q)))\leq 0$ i.e. $p=q$.This completes the proof.

Corollary 3.2. *Let (X,d) be a complete quasi-metric space and T a self mapping of X such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq m(x,y)-\phi(\max(d(x,y),d(y,Ty)))$$

where

$$m(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and

$$d^{-1}(x,y)\leq d^{-1}(x,T^2y)$$

Then, T has a unique fixed point.

Example 3.3. *Let $X=\mathbb{R}_+$ and, for all $(x,y)\in X$, $d(x,y)=\max\{y-x,0\}$.*

(X,d) is complete quasi-metric space.

Define $T : X \to X$ *by :* $T(x) = \ln(\frac{x}{2}+1)$, *for all* $x \in X$.

Define ψ *and* ϕ *by :*

$$\psi(t) = t, \text{ for all } t \in [0,+\infty[$$

$$\phi(t) = \frac{t}{4}, \text{ for all } t \in [0,+\infty[.$$

Let $(x,y) \in X^2$,

we have : $d(T^2y,x) = \max\{x - T^2y, 0\}$ *and* $T^2y = \ln(\frac{1}{2}\ln(\frac{y}{2}+1)+1)$, *so*

$$\max\{x-y,0\} \leq \max\{x - \ln(\frac{1}{2}\ln(\frac{y}{2}+1)+1),0\} \text{ i.e. } d(y,x) \leq d(T^2y,x)$$

we have also :

$$d(Tx,Ty) = \max\{\ln(\frac{y}{2}+1) - \ln(\frac{x}{2}+1), 0\}$$

$m(x,y) = \max\{\max\{y-x,0\}, \max\{\ln(\frac{x}{2}+1)-x,0\}, \max\{\ln(\frac{y}{2}+1)-y,0\},\}$

and

$$\max\{d(x,y),d(y,Ty)\} = \max\{\max\{y-x,0\}, \max\{\ln(\frac{y}{2}+1)-y,0\}\}$$

Case 1 : $x \geq y$

we have : $d(Tx,Ty) = 0$, $m(x,y) = 0$ *and* $\max\{d(x,y),d(y,Ty)\} = 0$

So,

$$\psi(d(Tx,Ty)) = \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

Case 2 : $y > x$

we have : $d(Tx,Ty) = \ln(\frac{y}{2}+1) - \ln(\frac{x}{2}+1)$, $\quad m(x,y) = \max\{y-x,0,0\} = y-x$

and

$$\max\{d(x,y),d(y,Ty)\} = y-x$$

So, $\psi(d(Tx,Ty)) = \ln(\frac{y}{2}+1) - \ln(\frac{x}{2}+1)$, $\psi(m(x,y)) = y-x$ *and*
$\phi(\max\{d(x,y),d(y,Ty)\}) = \frac{y-x}{4}$.

Imply

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

Then, 0 *is a unique fixed point.*

If we remove our condition $\forall x,y \in X, d^{-1}(x,y) \leq d^{-1}(x,T^2y)$, it may be that T does not admit a fixed point.

Counter-example 3.4. *Let* $X = \{(\frac{1}{3})^k \times n\,;\,(k,n) \in \mathbb{N}^2\}$, *for all* $(x,y) \in X$

$$d(x,y) = \max\{y-x,0\}$$

(X,d) *is complete quasi-metric space.*

Define $T : X \to X$ *by :*

$$Tx = \frac{1}{3}(x+1)$$

for all $x \in X$.

Define ψ *and* ϕ *by :*

$$\psi(t) = \sqrt{t},\ \textit{for all } t \in [0,+\infty[$$

$$\phi(t) = \frac{\sqrt{t}}{16},\ \textit{for all } t \in [0,+\infty[.$$

Let $(x,y) \in X^2$,

we have : $d(T^2y,x) = \max\{x-T^2y,0\}$ *and* $T^2y = \frac{1}{3}(\frac{1}{3}y+\frac{1}{3})+\frac{1}{3}$.

If $x > y$ *and* $y = 0$

$$\max\{x-y,0\} = x > \max\{x-(\frac{1}{3}(\frac{1}{3}y+\frac{1}{3})+\frac{1}{3}),0\}\ \textit{i.e. } d(y,x) > d(T^2y,x)$$

We have :

$$d(Tx,Ty) = \max\{\frac{1}{3}(y-x),0\}$$

$m(x,y) = \max\{\max\{y-x,0\},\max\{-\frac{2}{3}x+\frac{1}{3},0\},\max\{-\frac{2}{3}y+\frac{1}{3},0\}\}$

and

$$\max\{d(x,y),d(y,Ty)\} = \max\{\max\{y-x,0\},\max\{-\frac{2}{3}y+\frac{1}{3},0\}\}$$

Case 1 : $x \geq y$

$d(Tx,Ty) = 0$, $m(x,y) = \max\{0,-\frac{2}{3}x+\frac{1}{3},-\frac{2}{3}y+\frac{1}{3}\} = -\frac{2}{3}y+\frac{1}{3}$

and $\max\{d(x,y),d(y,Ty)\} = -\frac{2}{3}y+\frac{1}{3}$. *Since*

$$0 \leq \frac{15}{16}\sqrt{-\frac{2}{3}y+\frac{1}{3}}$$

so,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

Case 2 : $y > x$

$d(Tx,Ty) = \frac{1}{3}(y-x)$, $m(x,y) = \max\{y-x, -\frac{2}{3}x+\frac{1}{3}, -\frac{2}{3}y+\frac{1}{3}\}$

and $\max\{d(x,y),d(y,Ty)\} = \max\{y-x, -\frac{2}{3}y+\frac{1}{3}\}$

If $m(x,y) = y-x$, *then* $\max\{d(x,y),d(y,Ty)\} = y-x$. *Since*

$$\sqrt{\frac{1}{3}(y-x)} \leq \frac{15}{16}\sqrt{y-x}$$

so,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

If $m(x,y) = -\frac{2}{3}x+\frac{1}{3}$, *then* $\max\{d(x,y),d(y,Ty)\} = y-x$ *or* $-\frac{2}{3}y+\frac{1}{3}$

$$\psi(m(x,y) = \sqrt{-\frac{2}{3}x+\frac{1}{3}} \text{ and } \phi(\max(d(x,y),d(y,Ty))) = \frac{\sqrt{y-x}}{16} \text{ or } \frac{\sqrt{-\frac{2}{3}y+\frac{1}{3}}}{16}$$

We obtain :

$$\begin{cases} \sqrt{\frac{1}{3}(y-x)} \leq \sqrt{-\frac{2}{3}x+\frac{1}{3}} - \frac{\sqrt{y-x}}{16} \\ or \\ \sqrt{\frac{1}{3}(y-x)} \leq \sqrt{-\frac{2}{3}x+\frac{1}{3}} - \frac{\sqrt{-\frac{2}{3}y+\frac{1}{3}}}{16} \end{cases}$$

Hence,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

Then, T *has no fixed point.*

Theorem 3.5. *Let* (X,d) *be a complete quasi-metric space,*T *be a self mapping of* X *such that for all* $x,y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(m(x,y))$$

where

$$m(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and

$$d^{-1}(x,y) \leq d^{-1}(x,T^2y)$$

Then, T has a unique fixed point.

Proof. It's the same proof of theorem 3.1.

Corollary 3.6. *Let (X,d) be a complete quasi-metric space, T be a self mapping of X, it exists a positive Lebesgue integrable function φ on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} \varphi(t)\,\mathrm{d}t > 0$, for each $\varepsilon > 0$,*

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,\mathrm{d}t \leq \int_0^{\psi(m(x,y))} \varphi(t)\,\mathrm{d}t - \int_0^{\phi(\max\{d(x,y),d(y,Ty)\})} \varphi(t)\,\mathrm{d}t$$

and

$$d^{-1}(x,y) \leq d^{-1}(x,T^2y)$$

for all $x,y \in X$. Then, T has a unique fixed point.

Proof. Consider the function Φ define on $[0,+\infty[$ by :

$$\Phi(u) = \int_0^{u} \varphi(t)\,\mathrm{d}t$$

Then, for all $(x,y) \in X^2$,

$$(\Phi \circ \psi)(d(Tx,Ty)) \leq (\Phi \circ \psi)(m(x,y)) - (\Phi \circ \phi)(\max\{d(x,y),d(y,Ty)\})$$

Applying Theorem 3.1, we obtain T has at least one fixed point.

It's easy to verify that :

. $\Phi \circ \phi$ continuous,

. $\Phi \circ \psi$ is monotone nondecreasing and continuous,

. $\Phi \circ \phi(t) = 0$ (resp. $\Phi \circ \psi(t) = 0$) if and only if $t = 0$.

Corollary 3.7. *Let (X,d) be a complete quasi-metric space,T be a self mapping of X, it exists a positive Lebesgue integrable function φ on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} \varphi(t)\,\mathrm{d}t > 0$for each $\varepsilon > 0$*

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)\,\mathrm{d}t \leq \int_0^{\psi(m(x,y))} \varphi(t)\,\mathrm{d}t - \int_0^{\phi(m(x,y))} \varphi(t)\,\mathrm{d}t$$

and

$$d^{-1}(x,y) \leq d^{-1}(x,T^2y)$$

for all $x,y \in X$. Then, T has a unique fixed point.

Proof. It's the same proof of previous corollary.

Now, we consider $(X,\leq)$ an ordered quasi-metric space.

Theorem 3.8. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a quasi-metric d such that (X,d) is a complete quasi-metric space. Let T a self mapping of X be a non-decreasing map satisfying, for all $x,y \in X$ such that x and y comparable,*

$$\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(\max(d(x,y),d(y,Ty)))$$

where

$$m(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty)\}$$

and for all $x,y \in X$ such that $y \leq x$,

$$d^{-1}(x,y) \leq d^{-1}(x,T^2y)$$

If there exist $x_0 \in X$ satisfying $x_0 \leq Tx_0$ and if, for every increasing sequence $(x_n)_{n\geq 0}$ in X :

$$(x_n)_{n\geq 0} \text{ converge to } z \text{ implies that } x_n \leq z \text{ for all } n \in \mathbb{N}$$

Then, there exists $x \in X$ such that $Tx = x$.

Proof. Let $x_0 \in X$, we define a sequence $(x_n)_n$ in X such that $x_{n+1} = Tx_n$, for all integer $n \in \mathbb{N}$. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T.
Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_0 \leq Tx_0$ and T nondecreasing. we obtain by induction

$$x_0 \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq .... \leq T^nx_0 \leq T^{n+1}x_0 \leq ....$$

We show similarly that of theorem 3.1), that there exists, a $z \in X$ such that $\lim_{n\to\infty} d(x_n,z) = \lim_{n\to\infty} d(z,x_n) = 0$.
And since by hypothesis x_n and z are comparable, for all $n \in \mathbb{N}$, we obtain :

$$\psi(d(z,Tz)) \leq \psi(d(z,Tz)) - \phi(d(z,Tz))$$

Hence, z is a fixed point of T.

Theorem 3.9. *Let (X,d) be a complete quasi-metric space, T be a self mapping of X such that, for all $x,y \in X$,*

$$\phi_1(d(Tx,Ty)) \leq \psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty)) \tag{3.5}$$

and

$$d^{-1}(x,y) \leq d^{-1}(x,T^2y)$$

where $\psi_1,\psi_2 : [0,+\infty)^3 \to [0,+\infty)$ are a continuous functions with ψ_1 is monotone increasing in all the three variables and ($\psi_1(x,y,z) = \psi_2(x,y,z) = 0$ if and only if $x = y = z = 0$). and $\phi_1 : x \mapsto \psi_1(x,x,x)$.
Then, T has a unique fixed point.

Proof. First step. For any $x_0 \in X$, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in X by taking $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.
If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T.
Hence we shall assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Putting $x = x_n$ and $y = x_{n+1}$ in (3.5), we have :

$$\phi_1(d(x_{n+1},x_{n+2})) \leq$$

$$\psi_1(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})) - \psi_2(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))$$

Suppose that $d(x_{n+1},x_{n+2}) \geq d(x_n,x_{n+1})$ for some positive integer n, so :

$$\phi_1(d(x_{n+1},x_{n+2})) \leq \phi_1(d(x_{n+1},x_{n+2})) - \psi_2(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))$$

Which is a contradiction that :

$$\psi_2(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})) \neq 0, \text{ whenever } d(x_{n+1},x_{n+2}) \neq 0.$$

Hence, $(d(x_n,x_{n+1}))_n$ is monotone decreasing sequence of non negative real numbers.

Putting $x = x_{n+1}$ and $y = x_n$ in (3.5), we obtain :

$$\phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\psi_1(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}))$$

Since $(d(x_n,x_{n+1}))_n$ is monotone decreasing, so $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$, and then

$$\phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\psi_1(d(x_{n+1},x_n),d(x_n,x_{n+1}),d(x_n,x_{n+1})) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2}))$$

Suppose that $d(x_{n+2},x_{n+1}) \geq d(x_{n+1},x_n)$ for some positive integer n.

Case 1 : $d(x_{n+1},x_n) \geq d(x_n,x_{n+1})$

$$\phi_1(d(x_{n+1},x_n)) \leq \phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\phi_1(d(x_{n+1},x_n)) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2}))$$

Which is a contradiction that :

$$\psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2})) \neq 0, \text{ whenever } d(x_{n+1},x_n) \neq 0.$$

Case 2 : $d(x_n,x_{n+1}) \geq d(x_{n+1},x_n)$

$$\phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\phi_1(d(x_n,x_{n+1})) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2}))$$

Since, $d^{-1}(x,y) \leq d^{-1}(x,T^2y)$, for all $x,y \in X$, then $d(x_n,x_{n+1}) \leq d(x_{n+2},x_{n+1})$.

$$\phi_1(d(x_n,x_{n+1})) \leq \phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\phi_1(d(x_n,x_{n+1})) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2}))$$

Which is a contradiction that :

$$\psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_{n+1},x_{n+2})) \neq 0, \text{ whenever } d(x_{n+1},x_n) \neq 0.$$

Hence $(d(x_{n+1},x_n))_n$ is monotone decreasing.

Consequently, there exists $r > 0$, such that :

$$d(x_n,x_{n+1}) \longrightarrow r \;\; as \quad n \longrightarrow \;\; \infty$$

Since,

$$\phi_1(d(x_{n+1},x_{n+2})) \leq$$

$$\psi_1(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})) - \psi_2(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))$$

Letting $n \longrightarrow \infty$ in this inequality, we obtain :

$$\phi_1(r) \leq \psi_1(r,r,r) - \psi_2(r,r,r)$$

So,

$$\psi_2(r,r,r) \leq 0 \text{ i.e. } r = 0$$

$$d(x_n,x_{n+1}) \longrightarrow 0 \;\; as \quad n \longrightarrow \;\; \infty$$

Also, there exists $r' > 0$, such that :

$$d(x_{n+1},x_n) \longrightarrow r' \;\; as \quad n \longrightarrow \;\; \infty$$

Since,

$$\phi_1(d(x_{n+2},x_{n+1})) \leq$$

$$\psi_1(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})) - \psi_2(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1}))$$

Letting $n \longrightarrow \infty$ in this inequality, we obtain :

$$\phi_1(r') \leq \psi_1(r',0,0) - \psi_2(r',0,0)$$

and then,

$$\phi_1(r') \leq \psi_1(r',r',r') - \psi_2(r',0,0)$$

$$\psi_2(r',0,0) \leq 0 \text{ i.e. } r' = 0$$

$$d(x_{n+1},x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

Second step. Next we show that $(x_n)_n$is a Cauchy sequences.

Firstly we show $(x_n)_n$ is a right-Cauchy sequence, if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $(m(k))_k$ and $(n(k))_k$ such that for all positive integers k,

$n(k) > m(k) > k$,

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon \text{ and } d(x_{m(k)},x_{n(k)-1}) < \varepsilon$$

We follow the same steps as in the proof of previous theorem 3.1) to justify the :

$$d(x_{m(k)},x_{n(k)}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

$$d(x_{m(k)+1},x_{n(k)+1}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

For $x = x_{m(k)}$ and $y = x_{n(k)}$, we have :

$$\phi_1(d(x_{m(k)+1},x_{n(k)+1})) \leq$$

$$\psi_1(d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{m(k)+1}),d(x_{n(k)},x_{n(k)+1}))$$

$$-\psi_2(d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{m(k)+1}),d(x_{n(k)},x_{n(k)+1}))$$

Letting $k \longrightarrow \infty$ in the above inequality, we obtain :

$$\phi_1(\varepsilon) \leq \psi_1(\varepsilon,0,0) - \psi_2(\varepsilon,0,0) \leq \phi_1(\varepsilon) - \psi_2(\varepsilon,0,0)$$

So, $\psi_2(\varepsilon,0,0) \leq 0$ i.e. $\varepsilon = 0$. Which is a contradiction by virtue of a property of ϕ. Consequently, $(x_n)_n$ is a right-Cauchy sequence in (X,d).

Secondly we show $(x_n)_n$ is a left-Cauchy sequence, if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $(m(k))_k$ and $(n(k))_k$ such that for all positive integers

k

$n(k) > m(k) > k,$

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \text{ and } d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$$

We follow the same steps as in the proof of previous theorem 3.1) to justify the :

$$d(x_{n(k)}, x_{m(k)}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

$$d(x_{n(k)+1}, x_{m(k)+1}) \longrightarrow \varepsilon \quad as \quad k \longrightarrow \infty$$

For $x = x_{n(k)}$ and $y = x_{m(k)}$, we have :

$$\phi_1(d(x_{n(k)+1}, x_{m(k)+1})) \leq$$

$$\psi_1(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}))$$

$$-\psi_2(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}))$$

Letting $k \longrightarrow \infty$ in the above inequality, we obtain :

$$\phi_1(\varepsilon) \leq \psi_1(\varepsilon, 0, 0) - \psi_2(\varepsilon, 0, 0) \leq \phi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0)$$

So, $\psi_2(\varepsilon, 0, 0) \leq 0$ i.e. $\varepsilon = 0$. Which is a contradiction by virtue of a property of ϕ.

Consequently, $(x_n)_n$ is a left-Cauchy sequence in (X, d).

By Remark, we deduce that $(x_n)_n$ is a Cauchy sequence in complete quasi-metric space (X, d).

It implies that there exists, a $p \in X$ such that :

$$\lim_{n \to \infty} d(x_n, p) = \lim_{n \to \infty} d(p, x_n) = 0$$

Third Step. Putting $x = x_n$ and $y = p$ in (3.5), we have :

$$\phi_1(d(x_{n+1}, Tp)) \leq$$

$$\psi_1(d(x_n, p), d(x_n, x_{n+1}), d(p, Tp)) - \psi_2(d(x_n, p), d(x_n, x_{n+1}), d(p, Tp))$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, we obtain :

$$\phi_1(d(p, Tp)) \leq \psi_1(0, 0, d(p, Tp)) - \psi_2(0, 0, d(p, Tp)) \leq \phi_1(d(p, Tp)) - \psi_2(0, 0, d(p, Tp))$$

So, $\psi_2(0,0,d(p,Tp)) \leq 0$ i.e. $d(p,Tp) = 0$. Hence p is a fixed point of T.

Uniqueness of the fixed point : let $u \in X$ such that $u = Tu$.

Putting $x = u$ and $y = p$ in (3.5), we obtain :

$$\phi_1(d(Tu,Tp)) \leq \psi_1(d(u,p),d(u,Tu),d(p,Tp)) - \psi_2(d(u,p),d(u,Tu),d(p,Tp))$$

Hence,

$$\phi_1(d(Tu,Tp)) \leq \psi_1(d(u,p),0,0) - \psi_2(d(u,p),0,0)$$

$$\phi_1(d(u,p)) \leq \phi_1(d(u,p)) - \psi_2(d(u,p),0,0)$$

Imply that $d(u,p) = 0$ i.e. $u = p$.

Thus, p is a unique fixed point of T.This completes the proof.

Example 3.10. *Let $X = \mathbb{R}$ and, for all $(x,y) \in X^2$, $d(x,y) = \begin{cases} 0 & if \quad x=y \\ |y| & otherwise \end{cases}$ (X,d) is complete quasi-metric space.*

Define $T : X \to X$ by :

$$Tx = \begin{cases} 0 & if \quad -1<x<1 \\ \frac{5}{11x} & otherwise \end{cases}$$

Define $\psi_1, \psi_2 : [0,+\infty[^3 \to [0,+\infty[$ by for all $(t,y,z) \in [0,+\infty[^3$,

$$\psi_1(t,y,z) = \frac{1}{2}t + \frac{1}{40}y + \frac{1}{40}z$$

$$\psi_2(t,y,z) = \frac{1}{4}t + \frac{1}{40}y + \frac{1}{40}z$$

and

$$\phi_1 : x \mapsto \psi_1(x,x,x) = \frac{11}{20}x, \text{ for all } x \in [0,+\infty[$$

we have : $T^2x = \begin{cases} 0 & if \quad -1<x<1 \\ x & otherwise \end{cases}$, so for all $(x,y) \in X^2$,

$$d(y,x) \leq d(T^2y,x)$$

Let $(x,y) \in X^2$ *such that* $x \neq y$.

Case 1 *:* $-1 < y < 1$, *we have :* $d(Tx,Ty) = 0$ *and*

$$\psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty)) = \frac{1}{4}d(x,y) = \frac{1}{4} \mid y \mid$$

Then,

$$\phi_1(d(Tx,Ty)) \leq \psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty))$$

Case 2 *:* $y \leq -1$ *or* $y \geq 1$, *we have :* $Ty = \frac{5}{11y}$ *and* $\phi_1(d(Tx,Ty)) = \mid \frac{1}{4y} \mid$.
Since $\mid \frac{1}{4y} \mid \leq \mid \frac{1}{4}y \mid$, *then*

$$\phi_1(d(Tx,Ty)) \leq \psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty))$$

"0" *is unique fixed point of* T.

If we remove our condition $\forall x,y \in X, d^{-1}(x,y) \leq d^{-1}(x,T^2y)$, it may be that T does not admit a fixed point.

Counter-example 3.11. *We take* (X,d) *a complete quasi-metric space and* $T : X \to X$ *of our counter-example* 3.4

Define $\psi_1, \psi_2 : [0,+\infty[^3 \to [0,+\infty[$ *by for all* $(t,y,z) \in [0,+\infty[^3$,

$$\psi_1(t,y,z) = \frac{1}{2}t + \frac{1}{3}y + \frac{1}{6}z$$

$$\psi_2(t,y,z) = \frac{1}{6}t + \frac{1}{3}y + \frac{1}{6}z$$

and

$$\phi_1 : x \mapsto \psi_1(x,x,x) = x, \text{ for all } x \in [0,+\infty[$$

We already know that for each $(x,y) \in X^2$, *if* $x > y$ *and* $y = 0$ *we have* $d^{-1}(x,y) > d^{-1}(x,T^2y)$
Let $(x,y) \in X^2$ *such that*
Case 1 *:* $y > x$, *we have :* $d(Tx,Ty) = \frac{1}{3}(y-x)$ *and*

$$\psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty)) = \frac{1}{3}d(x,y) = \frac{1}{3}(y-x)$$

Then,

$$\phi_1(d(Tx,Ty)) = \psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty))$$

Case 2 *:* $y \leq x$*, we have :* $d(Tx,Ty) = 0$ *and*

$$\psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty)) = \frac{1}{3}d(x,y) = 0$$

Then,

$$\phi_1(d(Tx,Ty)) = \psi_1(d(x,y),d(x,Tx),d(y,Ty)) - \psi_2(d(x,y),d(x,Tx),d(y,Ty))$$

Then, T *has no fixed point.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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