RESULTS ON CYCLIC $\phi$-WEAK CONTRACTIONS IN FUZZY METRIC SPACES

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Abstract. We develop the fixed point theorems for $\phi$-weak contractions in fuzzy metric spaces. We also define $\psi$-weak contractive condition and establish the fixed point in G-complete fuzzy metric spaces.

Keywords: cyclic $\phi$-weak contraction, fixed point, G-Cauchy, G-complete fuzzy metric space.

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1. Introduction

Grabiec [5] established the Banach contraction theorem and Edelstein fixed point theorem in fuzzy metric spaces. Vasuki [16] generalised Grabiec’s fuzzy Banach contraction. In Vasuki [16] defined a generalization of Grabiecs fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Cho [4] defined the concept of compatible mappings and proved common fixed point theorems in fuzzy metric spaces. Pacurar and Rusin [12] introduced the concept of $\phi$-contraction. They developed some fixed point theorems using cyclic $\phi$- contraction in complete metric space. Based on these ideas Shen et.al [14] came up with notion of cyclic $\phi$- contraction in fuzzy metric spaces. In addition,

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several problems in connection with the fixed point are investigated. In this paper, we generalize
the fixed point theorems of Shen et. al in G-fuzzy metric spaces.

2. Preliminaries

Definition 2.1. [18] A binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous triangular
norm (in short, continuous $t$-norm) if it satisfies the following conditions:

(i) $T$ is commutative and associative;
(ii) $T$ is continuous;
(iii) $T(a, 1) = a, \forall a \in [0, 1]$;
(iv) $T(a, b) \leq T(c, d)$ whenever $a \leq b$ and $c \leq d. a, b, c, d \in [0, 1]$.

Generally $t$-norm $T$ can be expressed (by associativity) in a unique way to an $n$-ary operator
taking for $(x_1, x_2, \cdots, x_n) \in [0, 1]^n, n \in N$, the value $T(x_1, x_2, \cdots, x_n)$ is defined, in [11], by

$$T_{i=1}^{0} = T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, \cdots, x_n).$$

Definition 2.2.[12] Let $X$ be a nonempty set, $m$ a positive integer and $f : X \rightarrow X$ an operator.
$X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of $X$ with respect to $f$ if

(1) $X_i, i = 1, 2, \cdots, m$ are nonempty sets;
(2) $f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 2.3. [3] A fuzzy metric space is an ordered triple $(X, M, T)$ such that $X$ is a nonempty
set, $T$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following
conditions, for all $x, y, z \in X, s, t > 0$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $T \left( M(x, y, t), M(y, z, s) \right) \leq M(x, z, t + s)$;
(v) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Definition 2.4. [5] Let $(X, M, T)$ be a fuzzy metric space. Then
(i) A sequence \( \{x_n\} \) in \( X \) is said to converge to \( x \in X \), denoted by \( x_n \to x \), if and only if 
\[
\lim_{n \to \infty} M(x_n, x, t) = 1 \quad \text{for all } t > 0,
\]
i.e. for each \( r \in (0, 1) \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that 
\[
M(x_n, x, t) > 1 - r \quad \text{for all } n \geq n_0.
\]
(ii) A sequence \( \{x_n\} \) is a \( G \)-Cauchy sequence if and only if 
\[
\lim_{n \to \infty} M(x_n + p, x_n, t) = 1 \quad \text{for any } p > 0 \text{ and } t > 0.
\]
(iii) The fuzzy metric space \( (X, M, T) \) is called \( G \)-complete if every \( G \)-Cauchy sequence is convergent.

**Definition 2.5.** [14] A function \( \phi : [0, 1] \to [0, 1] \) is called a comparison function if it satisfies

1. \( \phi \) is nondecreasing and left continuous;
2. \( \phi(t) > t \) for all \( t \in (0, 1) \).

**Lemma 2.1.** [14] If \( \phi \) be a comparison function, then

(i) \( \phi(1) = 1 \)
(ii) \( \lim_{n \to +\infty} \phi^n(t) = 1 \) for all \( t \in (0, 1) \), where \( \phi^n(t) \) denotes the composition of \( \phi(t) \) with itself \( n \) times.

With the inspiration from cyclic \( \phi \)-contraction in [14] we present a contraction in fuzzy metric space, with \( P_{cl} \), the collection of closed subsets of \( X \).

**Definition 2.6.** Let \( (X, M, T) \) be a fuzzy metric space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^{m} A_i \) and \( f : Y \to Y \) an operator. If

(i) \( \bigcup_{i=1}^{m} A_i \) is cyclic representation of \( Y \) with respect to \( f \);
(ii) there exists a comparison function \( \phi : [0, 1] \to [0, 1] \) such that

\[
M(fx, fy, t) \geq \phi(\min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\})
\]

for any \( x \in A_i, y \in A_{i+1} \) and \( t > 0 \), where \( A_{m+1} = A_1 \), then \( f \) is called cyclic \( \phi \)-weak contraction in the fuzzy metric space \( (X, M, T) \).

**Definition 2.7.** [14] Let \( (X, M, T) \) be a fuzzy metric space and let \( \{f_n\} \) be a sequence of self-mappings on \( X \). \( f_0 : X \to X \) is a given mapping. The sequence \( \{f_n\} \) is said to converge uniformly to \( f_0 \) if for each \( \varepsilon \in (0, 1) \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that

\[
M(f_n(x), f_0(x), t) > 1 - \varepsilon
\]
for all \( n \geq n_0 \) and \( x \in X \).

3. Main results

**Theorem 3.1.** Let \((X, M, T)\) be a \(G\)-complete fuzzy metric space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \in P_c l(X) \), \( Y = \bigcup_{i=1}^{m} A_i \), \( \phi : [0, 1] \rightarrow [0, 1] \) a comparison function and \( f : Y \rightarrow Y \) an operator. Assume that

(i) \( \bigcup_{i=1}^{m} A_i \) is cyclic representation of \( Y \) with respect to \( f \);

(ii) \( f \) is a cyclic \( \phi \)-weak contraction.

Then \( f \) has a unique fixed point \( x' \in \bigcap_{i=1}^{m} A_i \) and the iterative sequence \( \{x_n\}_{n \geq 0}, (x_n = f(x_{n-1}) n \in N) \) converges to \( x' \) for any starting point \( x_0 \in Y \).

**Proof.** Let \( x_0 \in Y = \bigcap_{i=1}^{m} A_i \) be starting point, since \( x_n = f(x_1)(n \geq 1) \), we have

\[ M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \text{ for any } t > 0. \]

For any \( n \geq 0 \), there exists \( i_n \in 1, 2, \ldots, m \) such that \( x_n \in A_{i_n} \) and \( x_{n+1} \in A_{i_{n+1}} \). Therefore, we can get

\[ M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \]

\[ \geq \phi\left(\min\left\{ (M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}, t), M(x_n, f(x_n, t)) \right) \right) \]

\[ = \phi\left(\min\left\{ M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \right) \right) \]

\[ = \phi\left(\min\left\{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \right) \right). \]

If \( \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_n, x_{n+1}, t) \) it leads a contradiction that \( M(x_n, x_{n+1}, t) > M(x_n, x_{n+1}, t) \). Hence \( \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\} = M(x_{n-1}, x_n, t) \). Thus we get

\[ M(x_n, x_{n+1}, t) \geq \phi(M(x_{n-1}, x_n, t)). \]

Using the definition of \( \phi \), we get by induction that

\[ M(x_n, x_{n+1}, t) \geq \phi^n(M(x_0, x_1, t)). \]
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Therefore, for any $p > 0$, we have

$$M(x_n, x_{n+p}, t) \geq T(M(x_n, x_{n+1}, t/p), M(x_{n+1}, x_{n+2}, t/p), \cdots, M(x_{n+p-1}, x_{n+p}, t/p))$$

$$\geq T(\phi^n(M(x_0, x_1, t/p)), \phi^{n+1}(M(x_0, x_1, t/p)), \cdots, \phi^{n+p-1}(M(x_0, x_1, t/p)))$$

$$= T_{i=0}^{p-1} \phi^{n+i}(M(x_0, x_1, t/p)).$$

According to Lemma [2.1], for every $i \in 0, 1, \cdots, p - 1$, we obtain that $\lim_{n \to \infty} \phi^{n+i}(M(x_0, x_1, t/p)) = 1$. As $T$ is continuous $t$-norm, $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n \geq 0}$ is a $G$-Cauchy sequence in the $G$-complete subspace $Y$. Hence there is $x' \in Y$ such that $\lim_{n \to \infty} x_n = x'$.

Using the condition (i) in this theorem, it follows that the iterative sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each $A_i, i = 1, 2, \cdots, m$. Since $Y$ is $G$-complete, from each $A_i, i = 1, 2, \cdots, m$, we can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to $x'$ as well. Because each $A_i, i = 1, 2, \cdots, m$ is closed, we conclude that $x' \in \bigcap_{i=1}^{m} A_i$ and thus $\bigcap_{i=1}^{m} A_i$ is non empty.

Set $Z = \bigcap_{i=1}^{m} A_i$ Obviously, $Z$ is also closed and $G$-complete. Consider the restriction of $f$ to $Z$, that is, $f|_Z : Z \to Z$. Next, we will prove that $f|_Z$ has a unique fixed point in $Z \subset Y$. Now $x' \in Z$, since $f|_Z(x') \in Z$ and $x_n \in A_{i_n}$, we can choose $A_{i_n+1}$ such that $f|_Z(x') \in A_{i_n+1}$. Hence, for any $t > 0$, we have

$$M(f|_Z(x'), x', t) = M(f(x'), x', t)$$

$$\geq T(M(f(x'), f(x_n), t/2), M(x_{n+1}, x', t/2))$$

$$\geq T(\phi(x', x_n, t/2), M(x_{n+1}, x', t/2)) \to T(1, 1) = 1(n \to \infty).$$

Clearly, we get $f|_Z(x') = x'$, namely, $x'$ a fixed point, which is obtained by iteration from starting point $x_0$. To show uniqueness, we assume that $z \in \bigcap_{i=1}^{m} A_i$ is another fixed point of $f|_Z$. Since $x', z \in A_i$ for all $i \in N$, we can obtain

$$M(x', z, t) = M(f|_Z(x'), f|_Z(z), t)$$

$$= M(f(x'), f(z), t)$$

$$\geq \phi(\min M(x', z, t), M(x', f(x'), t), M(z, f(z), t))$$

$$> M(x', z, t).$$
This leads to a contradiction. Thus, $x'$ is the unique fixed point of $f|_{Z}$ for any starting point $x_0 \in Z \subseteq Y$. Now, we still have to prove that the iterative sequence $x_n$, $n \geq 0$ converges to $x'$ for any initial point $x_0 \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$, there exists $i_0 \in 1, 2, \ldots, m$ such that $x \in A_{i_0}$. As $x' \in \bigcap_{i=1}^{m} A_i$, it follows that $x' \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$M(f(x), f(x'), t) \geq \phi(M(x, x', t)).$$

By induction and Definition [2.6], we can obtain

$$M(x_n, x', t) = M(f_n(x_0), x', t)$$
$$= M(f_n(x_0), f(x'), t)$$
$$= M(f(f_{n-1}(x_0)), f(x'), t)$$
$$\geq \phi(\min\{M(f_{n-1}(x_0), x', t), M(f_{n-1}(x_0), f(f_{n-1}(x_0)), t), M(x', f(x'), t)\})$$
$$\geq \phi(M(f_{n-1}(x_0), x', t))$$
$$\geq \phi^n(M(x_0, x', t)).$$

Supposing $x_0 \neq x'$, it follows immediately that $x_n \to x'$ as $n \to \infty$. So the iterative sequence $\{x_n\}, n \geq 0$ converges to the unique fixed point $x'$ of $f$ for any starting point $x_0 \in Y$.

**Definition 3.1.** Let $(X, M, T)$ be a fuzzy metric space, $m$ a positive integer, $A_1, A_2, \ldots, A_m \in P_c(X), Y = \bigcup_{i=1}^{m} A_i$ and $f : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of $Y$ with respect to $f$;
(ii) there exists a comparison function $\psi : [0, 1] \to [0, 1]$ such that

$$M(fx, fy, t) \geq \psi(\min\{M(x, y, t), M(x, fx, t), M(y, fy, t)\}),$$

for any $x \in A_i, y \in A_{i+1}$ and $t > 0$, where $A_{m+1} = A_1$, then $f$ is called cyclic $\psi$-contraction in the fuzzy metric space $(X, M, T)$.

**Theorem 3.2.** Let $(X, M, T)$ be a G-complete fuzzy metric space, $m$ a positive integer, $A_1, A_2, \ldots, A_m \in P_c(X), Y = \bigcup_{i=1}^{m} A_i, \phi : [0, 1] \to [0, 1]$ a comparison function and $f : Y \to Y$ an operator. Assume that

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of $Y$ with respect to $f$;

(ii) $f$ is a cyclic $\psi$-contraction.

Then $f$ has a unique fixed point $x' \in \bigcap_{i=1}^{m} A_i$ and the iterative sequence $\{x_n\}_{n \geq 0}, (x_n = f(x_{n-1})) n \in N$ converges to $x'$ for any starting point $x_0 \in Y$.

**Proof.** Let the point $x_0 \in Y = \bigcap_{i=1}^{m} A_i$ be a starting point. Since $x_n = f(x_{n-1})(n \geq 1)$, we have $M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t)$ for any $t > 0$. Besides, for any $n \geq 0$, there exists $i_n \in 1, 2, \ldots, m$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Therefore, we can get

$$M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t)$$

$$\geq \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_{n-1}), t)\})$$

$$= \psi(\min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_n, t)\})$$

$$= \psi(\min\{M(x_{n-1}, x_n, t), 1\})$$

$$= \psi(M(x(n-1), x_n, t)).$$

Consider the definition of $\psi$, we get by induction that

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t)).$$

Thus, for any $p > 0$, we have

$$M(x_n, x_{n+p}, t) \geq T(M(x_n, x_{n+1}, t/p), M(x_{n+1}, x_{n+2}, t/p), \ldots, M(x_{n+p-1}, x_{n+p}, t/p))$$

$$\geq T(\psi^n(M(x_0, x_1, t/p)), \psi^{n+1}(M(x_0, x_1, t/p)), \ldots, \psi^{n+p-1}(M(x_0, x_1, t/p)))$$

$$= T_{i=0}^{p-1} \psi^{n+i}(M(x_0, x_1, t/p)).$$

Using Lemma [2.1], for every $i \in 0, 1, \ldots, p - 1$, we obtain that $\lim_{n \to \infty} \psi^{n+i}(M(x_0, x_1, t/p)) = 1$. As $T$ is continuous $t$-norm, $M(x_n, x_{n+p}, t) \to 1$ as $n \to \infty$. It shows that $\{x_n\}_{n \geq 0}$ is a $G$-Cauchy sequence in the $G$-complete subspace $Y$. So there exists $x' \in Y$ such that $\lim_{n \to \infty} x_n = x'$.

Now using the condition (i) in this theorem, it follows that the iterative sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each $A_i, i = 1, 2, \ldots, m$. Since $Y$ is $G$-complete, from each $A_i, i = 1, 2, \ldots, m$, one can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to $x'$ as well. Because each $A_i, i = 1, 2, \ldots, m$ is closed, we conclude that $x' \in \bigcap_{i=1}^{m} A_i$ and thus $\bigcap_{i=1}^{m} A_i$ is non empty. Set $Z = \bigcap_{i=1}^{m} A_i$. Obviously, $Z$ is also closed and $G$-complete. Consider the restriction of
Next, we will prove that \( f|_Z \) has a unique fixed point in \( Z \subset Y \). For the foregoing \( x' \in Z \), since \( f|_Z(x') \in Z \) and \( x_n \in A_{i_n} \), we can choose \( A_{i_{n+1}} \) such that \( f|_Z(x') \in A_{i_{n+1}} \).

Hence, for any \( t > 0 \), we have

\[
M(f|_Z(x'), x', t) = M(f(x'), x', t) \\
\geq T(M(f(x'), f(x_n), t/2), M(x_n+1, x', t/2)) \\
\geq T(\psi(x', x_n, t/2), M(x_n+1, x', t/2)) \to T(1, 1) = 1(n \to \infty).
\]

Clearly, we get \( f|_Z(x') = x' \) namely, \( x' \) a fixed point, which is obtained by iteration from starting point \( x_0 \). To show uniqueness, we assume that \( z \in \bigcap_{i=1}^m A_i \) is another fixed point of \( f|_Z \). Since \( x', z \in A_i \) for all \( i \in N \), we can obtain

\[
M(x', z, t) = M(f|_Z(x'), f|_Z(z), t) \\
= M(f(x'), f(z), t) \\
\geq \psi(\min M(x', z, t), M(x', f(x'), t), M(z, f(x'), t)) \\
> M(x', z, t).
\]

This leads to a contradiction. Thus, \( x' \) is the unique fixed point of \( f|_Z \) for any starting point \( x_0 \in Z \subset Y \). Now, we still have to prove that the iterative sequence \( x_n \geq 0 \) converges to \( x' \) for any initial point \( x_0 \in Y \). Let \( x \in Y = \bigcup_{i=1}^m A_i \), there exists \( i_0 \in 1, 2, \ldots, m \) such that \( x \in A_{i_0} \). As \( x' \in \bigcap_{i=1}^m A_i \), it follows that \( x' \in A_{i_{0+1}} \) as well. Then, for any \( t > 0 \), we have

\[
M(f(x), f(x'), t) \geq \psi(M(x, x', t)).
\]

By induction and Definition [2.6], we can obtain

\[
M(x_n, x', t) = M(f_n(x_0), x', t) \\
= M(f_n(x_0), f(x'), t) \\
= M(f(f_n(x_0)), f(x'), t)
\]
\[ \geq \psi(\min\{(M(f_{n-1}(x_0),x',t),M(f_{n-1}(x_0),f(f_{n-1}(x_0)),t),M(x',f(f_{n-1}(x_0)),t))\}) \]
\[ \geq \psi(M(f_{n-1}(x_0),x',t)) \]
\[ \geq \psi^{n}(M(x_0,x',t)). \]

Supposing \( x_0 \neq x' \), it follows immediately that \( x_n \to x' \) as \( n \to \infty \). So the iterative sequence \( \{x_n\}, n \geq 0 \) converges to the unique fixed point \( x' \) of \( f \) for any starting point \( x_0 \in Y \).

**Theorem 3.3.** Let \( f : Y \to Y \) be a self-mapping as in Theorem [3.1]. If there exists an iterative sequence \( \{y_n\} n \in N \) in \( Y \) such that \( M(y_n,f(y_n),t) \to 1 \) as \( n \to \infty \) for any \( t > 0 \), then \( y_n \to x' \) as \( n \to \infty \).

**Proof.** In view the proof of Theorem [3.1], we can find \( x' \) as unique fixed point of \( f \) for any starting point \( x_0 \in Y \). Therefore, for any \( t > 0 \), we have

\[ 1 \geq M(y_n,x',t) \geq T(M(y_n,f(y_n),t/2),M(f(y_n),f(x'),t/2)) \]
\[ \geq T(M(y_n,f(y_n),t/2),\phi(\min\{M(y_n,x',t/2),M(y_n,f(y_n),t/2),M(x',f(x'),t/2)\})) \]
\[ T(M(y_n,f(y_n),t/2),\phi^{n}(M(x_0,x',t/2))). \]

Since \( M(y_n,f(y_n),t/2) \to 1 \) and \( \phi^{n}(M(x_0,x',t/2)) \to 1 \) as \( n \to \infty \), it shows that \( M(y_n,x',t) \to 1 \) which is equivalent to \( y_n \to x' \) as \( n \to \infty \).

**Theorem 3.4.** Let \( f : Y \to Y \) be a self-mapping as in Theorem [3.1]. If there exists a convergent sequence \( \{y_n\} n \in N \) in \( Y \) such that \( M(y_{n+1},f(y_n),t) \to 1 \) as \( n \to \infty \) for any \( t > 0 \), then there exists \( x_0 \in Y \) such that \( M(y_n,f^n(x_0),t) \to 1 \) as \( n \to \infty \).

**Proof.** For any \( t > 0 \), let \( y_n \in Y, n \in N \) such that \( M(y_{n+1},f(y_n),t) \to 1, n \to \infty \). Set \( y \) as a limit of \( \{y_n\} n \in N \). By the proof of previous Theorem we note that \( x' \in \cap_{i=1}^{n} A_i \) is the unique fixed point of \( f \) for any starting point \( x_0 \in Y \) and \( t > 0 \). Therefore, for any \( t = t_1 + t_2 \) with \( t_1, t_2 > 0 \) and \( n \geq 0 \), we have

\[ M(y_{n+1},x',t) \geq T(M(y_{n+1},f(y_n),t_1),M(f(y_n),f(x'),t_2)). \]

Now, Suppose that \( M(y_{n+1},x',t) \neq 1, n \to \infty \), there exists \( 0 < \varepsilon < 1 \) and \( t > 0 \) such that

\[ \lim_{n \to \infty} M(y_{n+1},x',t) = M(y,x',t) = 1 - \varepsilon. \]
Then there exists $0 < t_0 < t$ such that

$$M(y, x', t_0) \leq 1 - \varepsilon$$

and

$$\limsup_{n \to \infty} M(y_n, x', t_0) = 1 - \varepsilon.$$

Since $y_n \in Y = \bigcup_{i=1}^{m} A_i$ for each $n \geq 0$, there is $i_n \in 1, 2, \cdots, m$ such that $y_n \in A_{i_n}$. But $x' \in \bigcap_{i=1}^{m} A_i$, so we can select one $A_{i_{n+1}}$ such that $x' \in A_{i_{n+1}}$. Therefore, we can obtain

$$M(y_{n+1}, x', t) \geq T(M(y_{n+1}, f(y_n), t - t_0), \phi(M(y_n, x', t_0))), n \geq 0.$$

As $T$ is continuous $t$-norm, we have

$$1 - \varepsilon = \lim_{n \to \infty} M(y_{n+1}, x', t) = M(y, x', t)$$

$$\geq \limsup_{n \to \infty} T(M(y_{n+1}, f(y_n), t - t_0), \phi(M(y_n, x', t_0)))$$

$$= T(\limsup_{n \to \infty} M(y_{n+1}, f(y_n), t - t_0), \limsup_{n \to \infty} \phi(M(y_n, x', t_0)))$$

$$= T(1, \limsup_{n \to \infty} \phi(M(y_n, x', t_0)))$$

$$= T(1, \limsup_{n \to \infty} \phi(M(y_n, x', t_0)))$$

$$= \phi(1 - \varepsilon) > 1 - \varepsilon,$$

which is a contradiction. Hence, $M(y, x', t) = 1$, namely, $y = x'$. Thus, for any $t > 0$, we have

$$M(y_n, f^n(x_0), t) \to M(y, x', t) \text{ as } n \to \infty.$$

**Theorem 3.5.** Let $f : Y \to Y$ be a self-mapping as in Theorem [3.1] and $f_n : Y \to Y, n \in N$. Moreover if the following three conditions hold:

(i) there exists a fixed point $x'_n$ for each $f_n$;

(ii) $\{f_n\}n \in N$ converges uniformly to $f$;

(iii) the sequence $x'_n, n \in N$ is convergent.

Then, $x'_n \to x'$ as $n \to \infty$.

**Proof.** Suppose that $x'_n n \in N$ converges to $x''$. Since $\{f_n\}, n \in N$ converges uniformly to $f$, for any $\varepsilon \in (0, 1)$ and $t > 0$, there exists an $n_0 \in N$ such that $M(f_n(x), f(x), t) > 1 - \varepsilon$ for all
$n \geq n_0$ and $x \in Y$. That is, for every $x \in Y, M(f_n(x), f(x), t) \to 1$ as $n \to \infty$. By induction, for any $t = t_1 + t_2$ with $t_1, t_2 > 0$, we can easily get $M(x'_n, x', t) = M(f_n(x'_n), f(x'), t_1 + t_2)$

\[
\begin{align*}
&\geq T(M(f_n(x'_n), f(x'_n), t_1), M(f(x'_n), f(x'), t_2)) \\
&\geq T(M(f_n(x'_n), f(x'_n), t_1), \phi(\min\{M(x'_n, x', t_2), M(x'_n, f(x'_n), t_2), M(x', f(x'), t_2)\})) \\
&= T(M(f_n(x'_n), f(x'_n), t_1), \phi(M(x'_n, x', t_2))).
\end{align*}
\]

Now, let us assume that $x'_n \neq x'$ as $n \to \infty$, i.e., there exist $\eta \in (0, 1)$ and $t > 0$ such that

\[
\lim_{n \to \infty} M(x'_n, x', t) = M(x''_n, x', t) = 1 - \eta.
\]

Then there exists $0 < t_0 < t$ such that $M(x'', x', t_0) \leq 1 - \eta$

and

\[
\limsup_{n \to \infty} M(x'_n, x', t_0) = 1 - \eta.
\]

Thus, we can have

\[
1 - \eta = \lim_{n \to \infty} M(x'_n, x', t) = M(x'', x', t)
\]

\[
\geq \limsup_{n \to \infty} T(M(f_n(x'_n), f(x_n), t - t_0), \phi(\min\{M(x'_n, x', t_0), M(x'_n, f(x'_n), t_0), M(x', f(x'), t_0)\}))
\]

\[
= \limsup_{n \to \infty} T(M(f_n(x'_n), f(x_n), t - t_0), \phi(M(x'_n, x', t_0)))
\]

\[
= T(1, \limsup_{n \to \infty} \phi(M(x'_n, x', t_0)))
\]

\[
= \limsup_{n \to \infty} \phi(M(x'_n, x', t_0))
\]

\[
= \phi(1 - \eta) > 1 - \eta,
\]

which is not true. Hence, $M(x'_n, x', t) \to 1$ as $n \to \infty$, i.e., $x'_n \to x'$ as $n \to \infty$.

**Theorem 3.6.** Let $(X, M, T)$ be a $G$-complete fuzzy metric space, $m$ a positive integer, $A_1, A_2, \cdots, A_m \in P_{cl}(X), Y = \bigcup_{i=1}^m A_i, \phi : [0, 1] \to [0, 1]$ a comparison function and $f : Y \to Y$ an operator. Assume that
(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of $Y$ with respect to $f$;
(ii) $f$ is a cyclic $\psi$-contraction.

If there exists an iterative sequence $\{y_n\}_{n \in \mathbb{N}}$ in $Y$ such that $M(y_n, f(y_n), t) \to 1$ as $n \to \infty$ for any $t > 0$, then $y_n \to x'$ as $n \to \infty$.

**Theorem 3.7.** Let $(X, M, T)$ be a $G$-complete fuzzy metric space, $m$ a positive integer, $A_1, A_2, \cdots, A_m \in \mathcal{P}_{cl}(X)$, $Y = \bigcup_{i=1}^{m} A_i$, $\phi : [0, 1] \to [0, 1]$ a comparison function and $f : Y \to Y$ an operator. Assume that

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of $Y$ with respect to $f$;
(ii) $f$ is a cyclic $\psi$-contraction.

and $f_n : Y \to Y, n \in \mathbb{N}$. Moreover if the following three conditions hold:

(iii) there exists a fixed point $x'_n$ for each $f_n$;
(iv) $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to $f$;
(v) the sequence $x'_n, n \in \mathbb{N}$ is convergent.

Then, $x'_n \to x'$ as $n \to \infty$.

**Theorem 3.8.** Let $(X, M, T)$ be a $G$-complete fuzzy metric space, $m$ a positive integer, $A_1, A_2, \cdots, A_m \in \mathcal{P}_{cl}(X)$, $Y = \bigcup_{i=1}^{m} A_i$, $\phi : [0, 1] \to [0, 1]$ a comparison function and $f : Y \to Y$ an operator. Assume that

(i) $\bigcup_{i=1}^{m} A_i$ is cyclic representation of $Y$ with respect to $f$;
(ii) $f$ is a cyclic $\psi$-contraction.

If there exists a convergent sequence $\{y_n\}_{n \in \mathbb{N}}$ in $Y$ such that $M(y_{n+1}, f(y_n), t) \to 1$ as $n \to \infty$ for any $t > 0$, then there exists $x_0 \in Y$ such that $M(y_n, f^n(x_0), t) \to 1$ as $n \to \infty$.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


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