THE CLASS $\Phi_\alpha$ OF AUXILIARY FUNCTIONS AND FIXED POINT IN $G$-METRIC SPACE

T. PHANEENDRA*, S. SARAVANAN

Department of Mathematics, School of Advanced Sciences, VIT University, Vellore-632014, Tamil Nadu, India

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Abstract. Some generalizations of the fixed point theorems of Mohanta [1], Mustafa and Sims [2] and of Vats et al [7] are proved, under a new class $\Phi_\alpha$ of auxiliary functions. Also, $G$-contractive fixed points are obtained for some contraction type conditions.

Keywords: $G$-complete metric space; the class $\Phi_\alpha$; fixed point; $G$-contractive fixed point

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1. Introduction

Several fixed point theorems in metric space setting have been proved through contraction type conditions involving different types of auxiliary functions. One such auxiliary function is a mapping $\psi : [0, \infty) \to [0, \infty)$, known as a contractive modulus, with the choice

\begin{equation}
\psi(0) = 0 \text{ and } \psi(t) < t \text{ for } t > 0.
\end{equation}

*Corresponding author

E-mail address: drtp.indra@gmail.com

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The notion of contractive modulus was introduced by Solomon Leader [6]. For instance,
\[ \psi_1(t) = \frac{t}{t+1} \text{ and } \psi_2(t) = \frac{t^2}{t+1} \]
are contractive moduli. We denote by \( \Psi \), the class of all contractive moduli.

Given a positive integer \( \alpha \), we introduce a generalized class \( \Phi_\alpha \) as follows:
\[ \Phi_\alpha = \{ \phi : [0, \infty) \to [0, \infty) | \phi(0) = 0, \phi(\alpha t) < t \text{ for } t > 0 \} \]

**Remark 1.1.** It is obvious that, for \( \alpha = 1 \), \( \Phi_\alpha \) reduces to the class \( \Psi \). That is \( \Phi_1 = \Psi \). However, in general a contractive modulus need not belong to \( \Phi_\alpha \) for \( \alpha > 1 \), as shown in the following example:

**Example 1.1.** Consider
\[ \psi(t) = \begin{cases} \frac{2t}{3}, & t < 1 \\ \frac{t}{2}, & t \geq 1 \end{cases} \]
Obviously, \( \psi(0) = 0 \) and \( \psi(t) < t \) for all \( t > 0 \) so that \( \psi \in \Psi \). But
\[ \psi(2t) = \begin{cases} \frac{4t}{3}, & t < 1/2 \\ t, & t \geq 1/2 \end{cases} \]
so that \( \psi(2t) \geq t \) for all \( t > 0 \). Thus \( \psi \notin \Phi_\alpha \).

**Definition 1.1.** A mapping \( \phi \in \Phi_\alpha \) is said to be upper semicontinuous at \( t_0 \geq 0 \) if \( \limsup_{n \to \infty} \phi(t_n) \leq \phi(t_0) \) whenever \( \langle t_n \rangle_{n=1}^\infty \) is such that \( \lim_{n \to \infty} t_n = t_0 \), and \( \phi \) is u.s.c if it is u.s.c. at every \( t \geq 0 \).

**Example 1.2.** Mappings
\[ \phi(t) = \begin{cases} qt & (0 \leq t \leq 1) \\ t - q & (t > 1) \end{cases} \]
and \( qt \) with \( 0 \leq q < 1 \), are continuous contractive modulii, while the contractive modulus
\[ \psi(t) = \begin{cases} 0 & (0 \leq t \leq a) \\ t - a & (t > a) \end{cases} \]
with $a > 0$, is use but not continuous.

In this paper, we obtain the fixed points of self-maps satisfying some contraction type conditions in terms of $\phi \in \Phi_{\alpha}$ for different choices of $\alpha$ in $G$-metric space. Also, we obtain $G$-contractive fixed points for some contraction type conditions (See Section 4).

2. $G$-metric space

Let $X$ be a nonempty set and $G : X \times X \times X \to [0, \infty)$ such that

(G1) $G(x, y, z) = 0$ whenever $x, y, z \in X$ are such that $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(\pi(x, y, z))$ for all $x, y, z \in X$, where $\pi(x, y, z)$ is a permutation on the set \{x, y, z\}

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then $G$ is called a $G$-metric on $X$ and the pair $(X, G)$, a $G$-metric space. Axiom (G5) is usually referred to as the rectangle inequality (of the $G$-metric $G$). This notion was introduced by Mustafa and Sims [2] in 2006.

In any $G$-metric space $(X, G)$, we have

(2.1) \quad $G(x, y, y) \leq 2G(x, x, y)$ for all $x, y \in X$.

A $G$-metric space $(X, G)$ is said to be symmetric if

(2.2) \quad $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

We use the following notions from of [2] in this paper:

**Definition 2.1.** Let $(X, G)$ be a $G$-metric space. A $G$-ball in $X$ is defined by

$$B_G(x, r) = \{y \in X : G(x, y, y) < r\}.$$  

It is easy to see that the family of all $G$-balls forms a base topology, called the $G$-metric topology $\tau(G)$ on $X$. 

Also

\[ \rho_G(x, y) = G(x, y, y) + G(x, x, y) \quad \text{for all } x, y \in X. \]

induces a metric on \( X \), and the \( G \)-metric topology coincides with the metric topology induced by the metric \( \rho_G \). This allows us to readily transform many concepts from metric space into the setting of \( G \)-metric space.

**Definition 2.2.** A sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in a \( G \)-metric space \( (X, G) \) is said to be \( G \)-convergent with limit \( p \in X \) if it converges to \( p \) in the \( G \)-metric topology \( \tau(G) \).

**Definition 2.3.** A sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in a \( G \)-metric space \( (X, G) \) is said to be \( G \)-Cauchy if

\[
\lim_{n,m \to \infty} G(x_n, x_m, x_m) = 0.
\]

**Definition 2.4.** A \( G \)-metric space \( (X, G) \) is said to be \( G \)-complete if every \( G \)-Cauchy sequence in \( X \) converges in it.

### 3. Fixed point theorems involving the class \( \Phi_\alpha \)

Our first result is

**Theorem 3.1.** Suppose that \( (X, G) \) is a complete \( G \)-metric space and \( f \), a self-map on \( X \) satisfying the condition

\[
G(fx, fy, fz) \leq \phi \left( \max \left\{ G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), G(x, fy, fy), G(y, fz, fz), G(z, fx, fx) \right\} \right)
\]

\[(3.1)\]

for all \( x, y, z \in X \),

where \( \phi \in \Phi_2 \) is nondecreasing and upper semicontinuous. Then \( f \) will have a unique fixed point \( p \).

**Proof.** Let \( x_0 \in X \) be arbitrary. Define \( \langle x_n \rangle_{n=1}^{\infty} \subset X \) by

\[
x_n = fx_{n-1} \quad \text{for } n \geq 1.
\]

Writing with \( x = x_{n-1} \) and \( y = z = x_n \) in (3.1) and then using (2.1), we get
\[ G(f_{x_{n-1}}, f_{x_n}, f_{x_n}) = G(x_n, x_{n+1}, x_{n+1}) \]
\[ \leq \phi\left( \max \left\{ G(x_{n-1}, x_n, x_n), G(x_{n-1}, f_{x_{n-1}}, f_{x_{n-1}}), G(x_n, f_{x_n}, f_{x_n}), \right. \]
\[ \left. G(x_n, f_{x_n}, f_{x_n}), G(x_{n-1}, f_{x_{n-1}}, f_{x_{n-1}}), G(x_n, f_{x_n}, f_{x_n}), \right. \]
\[ \left. G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), \right. \]
\[ \left. G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n) \right\} \]
\[ \leq \phi\left( \max \left\{ G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), \right. \]
\[ \left. G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \right. \]
\[ \left. G(x_{n-1}, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n) \right\} \]
\[ \leq \phi\left( \max \left\{ G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_{n+1}, x_{n+1}), \right. \]
\[ \left. G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \right\} \]
\[ \leq \phi\left( G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \right) \]

Define

(3.3) \[ t_n = G(x_{n-1}, x_n, x_n) \text{ for } n \geq 1. \]

Then the above inequality can be written as

(3.4) \[ t_{n+1} \leq \phi\left( t_n + t_{n+1} \right) \]

We now prove that

(3.5) \[ t_n \geq t_{n+1} \text{ for } n \geq 1. \]

If possible, suppose that \( t_m < t_{m+1} \) for some \( m \geq 1 \). Then \( t_{m+1} > 0 \). Since \( \phi \) is nondecreasing, from (3.4) it follows that

\[ t_{m+1} \leq \phi\left( t_{m+1} + t_m \right) < t_{m+1}, \]
which is a contradiction. This proves (3.5). In other words, \((t_n)_{n=1}^\infty\) is a decreasing sequence of nonnegative real numbers and hence converges to some \(t \geq 0\).

Now using (3.5) in (3.4), we get

\[
t_{n+1} \leq \phi(t_{n+1} + t_n) \leq \phi(2t_n) \quad \text{for } n \geq 1.
\]

Taking the limit superior as \(n \to \infty\) in this and then using the upper semicontinuity of \(\phi\), we obtain that

(3.6) \hspace{1cm} t \leq \phi(2t).

If \(t > 0\) in (3.6), then the choice of \(\phi\) implies that \(t \leq \phi(2t) < t\), which is a contradiction. Thus

(3.7) \hspace{1cm} t = \lim_{n \to \infty} t_n = \lim_{n \to \infty} G(x_{n-1}, x_n, x_n) = 0.

We now prove that \(\langle x_n \rangle_{n=1}^\infty\) is a \(G\)-Cauchy sequence in \(X\).

If possible we suppose that \(\langle x_n \rangle_{n=1}^\infty\) is not \(G\)-Cauchy. Then for some \(\varepsilon > 0\), we choose sequences \(\langle x_{m_k} \rangle_{k=1}^\infty\) and \(\langle x_{n_k} \rangle_{k=1}^\infty\) of positive integers such that \(m_k > n_k > k\) and

(3.8) \hspace{1cm} G(x_{m_k}, x_{n_k}, x_{n_k}) \geq \varepsilon \quad \text{for } k = 1, 2, 3, \ldots.

Suppose that \(m_k\) is the smallest integer exceeding \(n_k\) which satisfies (3.8). That is

(3.9) \hspace{1cm} G(x_{m_k-1}, x_{n_k}, x_{n_k}) < \varepsilon.

Now by rectangle inequality of \(G\), we see that

\[
\varepsilon \leq G(x_{m_k}, x_{n_k}, x_{n_k}) \leq G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{m_k-1}, x_{n_k}, x_{n_k}) < G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + \varepsilon
\]

(3.10) \hspace{1cm} \text{and from (3.7), we see that}

(3.11) \hspace{1cm} \lim_{k \to \infty} G(x_{m_k-1}, x_{m_k}, x_{m_k}) = 0.
As \( \phi = \max \{ G(x_{n_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k}, x_{m_k}, x_{m_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}) \} \),

or

\[
\varepsilon \leq G(x_{m_k}, x_{n_k}, x_{n_k}) \]

\[
\leq \phi \left( \max \{ G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k-1}, x_{m_k}, x_{m_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}) \} \right)
\]

\[
= \phi \left( \max \{ G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k-1}, x_{m_k}, x_{m_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}) \} \right)
\]

(3.15) 

\[
G(x_{m_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{m_k}, x_{m_k}) \}
\]
Proceeding the limit as \( n \to \infty \) in (3.15) and then using upper semicontinuity of \( \phi \), (3.9), (3.11), (3.12), (3.13) and (3.14) we get

\[
\varepsilon \leq \phi \left( \max \{ \varepsilon, 0, 0, \varepsilon, 2\varepsilon \} \right) = \phi (2\varepsilon).
\]

Since \( \phi \) is nondecreasing, this finally gives

\[
(3.16) \quad \varepsilon \leq \phi (2\varepsilon) < \varepsilon,
\]

which is a contradiction. Hence \( \langle x_n \rangle_{n=1}^{\infty} \) must be a \( G \)-Cauchy sequence in \( X \).

Since \( (X, G) \) is \( G \)-Complete, there exists a point \( p \in X \) such that \( \langle x_n \rangle_{n=1}^{\infty} \) is \( G \)-convergent to \( p \). That is

\[
(3.17) \quad \lim_{n \to \infty} x_{n-1} = \lim_{n \to \infty} x_n = p.
\]

We now establish that \( p \) is a fixed point of \( f \). In fact, writing \( x = x_{n-1} \) and \( y = z = p \) in (3.1)

\[
G(fx_{n-1}, fp, fp) = G(x_n, fp, fp)
\]

\[
\leq \phi \left( \max \{ G(x_{n-1}, p, p), G(x_{n-1}, fx_{n-1}, fx_{n-1}), G(p, fp, fp), G(p, fp, fp), G(x_{n-1}, fp, fp), G(0, 0, 0) \} \right)
\]

\[
\leq \phi \left( \max \{ G(x_{n-1}, p, p), G(x_{n-1}, x_n), G(p, fp, fp), G(p, x_n, x_n) \} \right).
\]

(3.18)

Proceeding the limit as \( n \to \infty \) in (3.18) and then using (3.17), we get

\[
G(p, fp, fp) \leq \phi \left( \max \{ 0, 0, G(p, fp, fp), G(p, fp, fp) \} \right)
\]

(3.19)

\[
= \phi (G(p, fp, fp)).
\]

If \( p \neq fp \), then \( G(p, fp, fp) > 0 \). Since \( \phi \) is nondecreasing, (3.19) gives

\[
0 < G(p, fp, fp) \leq \phi (G(p, fp, fp)) < G(p, fp, fp),
\]

which is a contradiction. Hence \( p = fp \).
To establish the uniqueness of the fixed point, we suppose that \( p \) and \( q \) are fixed points of \( f \) with \( p \neq q \). Then Writing \( x = p \) and \( y = z = q \) in (3.1), we get

\[
G(fp, fq, fq) \leq \phi \left( \max \left\{ G(p, q, q), G(p, fp, fp), G(q, fq, fq), G(q, fq, fq), \right. \right.
\]
\[
G(p, fp, fp), G(q, fq, fq), G(q, fp, fp) \right\} \right)
\]

which on using (2.1) implies that

\[
G(p, q, q) \leq \phi \left( \max \left\{ G(p, q, q), 0, 0, 0, G(p, q, q), 0, G(q, p, p) \right\} \right)
\]
\[
\leq \phi \left( \max \left\{ G(p, q, q), 2G(p, q, q) \right\} \right)
\]
\[
= \phi(2G(p, q, q)).
\]

Since \( \phi \) is nondecreasing, this gives

\[
G(p, q, q) \leq \phi(2G(p, q, q)) < G(p, q, q),
\]

which is again a contradiction. Therefore, \( p = q \). □

**Remark 3.1.** Set \( \phi(t) = kt \) for all \( t \geq 0 \), where \( 0 < k < 1/2 \) in Theorem 3.1. Then \( \phi(0) = 0 \) and \( \phi(2t) = 2kt < t \) for all \( t > 0 \). Therefore, we get

**Corollary 3.1** (Theorem 2.1, [3]). Suppose that \((X, G)\) is a complete G-metric space and \( f \), a self-map on \( X \) satisfying the condition

\[
G(fx, fy, fz) \leq k \max \left\{ G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \right. \right.
\]
\[
G(x, fy, fy), G(y, fz, fz), G(z, fx, fx) \right\}
\]
\[
(3.20)
\]

for all \( x, y, z \in X \),

where \( 0 < k < 1/2 \). Then \( f \) will have a unique fixed point \( p \).

Just similar to Theorem 3.1, we can prove
Theorem 3.2. Suppose that \((X, G)\) is a complete \(G\)-metric space and \(f\), a self-map on \(X\) satisfying the condition

\[
G(fx, fy, fz) \leq \phi \left( \max \left\{ G(x, fx, fx), G(y, fy, fy), G(y, fx, fx), \right. \\
G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), \\
G(z, fz, fz), G(z, fx, fz), G(z, fx, fx), \\
G(z, fy, fy) \right\} \right) \quad \text{for all } x, y, z \in X,
\]

(3.21)

where \(\phi \in \Phi_2\) is nondecreasing and upper semicontinuous. Then \(f\) will have a unique fixed point \(p\).

Remark 3.2. Set \(\phi(t) = kt\) for all \(t \geq 0\), where \(0 < k < 1/2\) in Theorem 3.2. Then \(\phi(0) = 0\) and \(\phi(2t) = 2kt < t\) for all \(t > 0\). Therefore, we get

Corollary 3.2 (Theorem 1, [7]). Suppose that \((X, G)\) is a complete \(G\)-metric space and \(f\), a self-map on \(X\) satisfying the condition

\[
G(fx, fy, fz) \leq k \max \left\{ G(x, fx, fx), G(x, fy, fy), \right. \\
G(x, fz, fz), G(y, fy, fy), G(y, fx, fx), \\
G(y, fz, fz), G(z, fz, fz), G(z, fx, fx), \\
G(z, fz, fz), G(z, fy, fy) \right\} \quad \text{for all } x, y, z \in X,
\]

(3.22)

where \(0 < k < 1/2\). Then \(f\) will have a unique fixed point \(p\).

With an argument, similar to that of Theorem 3.1, we can prove the following:

Theorem 3.3. Suppose that \((X, G)\) is a complete \(G\)-metric space and \(f\), a self-map on \(X\) satisfying the condition

\[
G(fx, fy, fz) \leq \phi \left( \max \left\{ G(x, fy, fy) + G(y, fx, fx) + G(z, fz, fz), \\
G(y, fz, fz) + G(z, fy, fy) + G(x, fx, fx), \\
G(z, fx, fx) + G(x, fz, fz) + G(y, fy, fy) \right\} \right) \\
\text{for all } x, y, z \in X,
\]

(3.23)
where $\phi \in \Phi_3$ is nondecreasing and upper semicontinuous. Then $f$ will have a unique fixed point $p$.

**Remark 3.3.** Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/3$ in Theorem 3.3. Then $\phi(0) = 0$ and $\phi(3t) = 3kt < t$ for $t > 0$. Therefore, we have

**Corollary 3.3** (Theorem 3.9, [1]). Suppose that $(X, G)$ is a complete $G$-metric space and $f$, a self-map on $X$ satisfying the condition

$$G(fx, fy, fz) \leq k \max \{G(x, fy, y) + G(y, fx, x) + G(z, fz, z),
G(y, fz, z) + G(z, fy, y) + G(x, fx, x),
G(z, fx, x) + G(x, fz, z) + G(y, fy, y)\}$$

(3.24) for all $x, y, z \in X$,

where $0 < k < 1/3$. Then $f$ will have a unique fixed point $p$.

The fourth main result is given below without proof:

**Theorem 3.4.** Suppose that $(X, G)$ is a complete $G$-metric space and $f$, a self-map on $X$ satisfying the condition

$$G(fx, fy, fz) \leq \phi(\max \{G(x, fx, x) + G(y, fy, y) + G(z, fz, z),
G(x, fy, y) + G(y, fx, x) + G(z, fy, y),
G(x, fz, z) + G(y, fz, z) + G(z, fx, x)\})$$

(3.25) for all $x, y, z \in X$,

where $\phi \in \Phi_4$ is nondecreasing upper semicontinuous. Then $f$ will have a unique fixed point $p$.

**Remark 3.4.** Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/4$ in Theorem 3.4. Then $\phi(0) = 0$ and $\phi(4t) = 4kt < t$ for $t > 0$. Therefore, we have
Corollary 3.4 (Vats et al, [7]). Suppose that \((X, G)\) is a complete G-metric space and \(f\), a self-map on \(X\) satisfying the condition

\[
G(fx, fy, fz) \leq k \max \left\{ G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz), \\
G(x, fx, fy) + G(y, fy, fz) + G(z, fz, fy), \\
G(x, fz, fz) + G(z, fz, fz) + G(z, fx, fx) \right\}
\]

(3.26) \hspace{1cm} \text{for all } x, y, z \in X,

where \(0 < k < 1/4\). Then \(f\) will have a unique fixed point \(p\).

The final main result of this paper is

Theorem 3.5. Let \((X, G)\) be a complete G-metric space and \(f\) be a self-map on \(X\) such that

\[
G(fx, fy, fz) \leq \phi \left( \max \left\{ G(x, fx, fx) + G(x, fy, fy) + G(x, fz, fz), \\
G(y, fy, fy) + G(y, fx, fx) + G(y, fz, fz), \\
G(z, fz, fz) + G(z, fx, fz) + G(z, fx, fx) \right\} \right)
\]

(3.27) \hspace{1cm} \text{for all } x, y, z \in X,

where \(\phi \in \Phi_5\) is nondecreasing upper semicontinuous. Then \(f\) will have a unique fixed point \(p\).

Remark 3.5. Set \(\phi(t) = kt\) for all \(t \geq 0\), where \(0 < k < 1/5\) in Theorem 3.5. Then \(\phi(0) = 0\) and \(\phi(5t) = 5kt < t\) for \(t > 0\). Therefore, we have

Corollary 3.5. Let \((X, G)\) be a complete G-metric space and \(f\) be a self-map on \(X\) such that

\[
G(fx, fy, fz) \leq k \max \left\{ G(x, fx, fx) + G(x, fy, fy) + G(x, fz, fz), \\
G(y, fy, fy) + G(y, fx, fx) + G(y, fz, fz), \\
G(z, fz, fz) + G(z, fx, fx) + G(z, fy, fy) \right\}
\]

(3.28) \hspace{1cm} \text{for all } x, y, z \in X,

where \(0 \leq k < 1/5\). Then \(f\) will have a unique fixed point \(p\).
4. **G-contractive fixed points**

We begin this section with

**Definition 4.1** (Phaneendra and Kumara Swamy, [4]). A fixed point \( p \) of \( f \) on a \( G \)-metric space \((X, G)\) is a \( G \)-contractive fixed point, if for each \( x_0 \in X \), the orbit \( \mathcal{O}_f(x_0) = \langle x_0, f x_0, ..., f^n x_0, ... \rangle \) is \( G \)-convergent, with limit \( p \).

It was shown in [4] that the unique fixed point of the self-map \( f \) with the following choices is a \( G \)-contractive fixed point.

(a) \( G(f x, f y, f z) \leq qG(x, y, z) \) for all \( x, y, z \in X \), where \( 0 \leq q < 1 \),

(b) \( G(f x, f y, f z) \leq aG(x, f x, f x) + bG(y, f y, f y) + cG(z, f z, f z) + eG(x, y, z) \) for all \( x, y, z \in X \), where \( a, b, c \) and \( e \) are nonnegative real numbers with \( a + b + c + e < 1 \).

In [5], the authors have proved that the unique fixed points of the self-maps are \( G \)-contractive fixed points, under (3.24) and (3.28).

Now, we obtain \( G \)-contractive fixed points for the maps of Corollaries, obtained in the previous sections.

**Theorem 4.1.** Let \( p \) be a unique fixed point of a self-map \( f \) on a complete \( G \)-metric space satisfying (3.22). Then \( p \) will be a \( G \)-contractive fixed point.

**Proof.** Let \( x_0 \in X \) be arbitrary. Writing \( x = f^{n-1}x_0 \) and \( y = z = p \) in (3.22), we get

\[
G(f^n x_0, p, p) = G(f^n x_0, f p, f p) \\
\leq k \max \{ G(f^{n-1} x_0, f^n x_0, f^n x_0), G(f^{n-1} x_0, f p, f p), G(f^{n-1} x_0, f p, f p), \\
G(p, f p, f p), d(p, f^{n-1} x_0, f^{n-1} x_0), G(p, f p, f p), \\
G(p, f p, f p), G(p, f^{n-1} x_0, f^{n-1} x_0), G(p, f p, f p) \} \\
= kM,
\]

where

\[
\max \{ G(f^{n-1} x_0, f^n x_0, f^n x_0), G(f^{n-1} x_0, p, p), G(p, f^n x_0, f^n x_0) \}.
\]
Now, three cases arise:

**Case (a).** Suppose that \( M = G(p, f^n x_0, f^n x_0) \). Then, it can be shown that \( p \) is a \( G \)-contractive fixed point, as in case (a) of the previous proof.

**Case (b).** The case of \( M = G(f^{n-1} x_0, p, p) \) is obvious, since \( k < 1 \).

**Case (c).** Let \( M = G(f^{n-1} x_0, f^n x_0, f^n x_0) \). Then, (4.1) can be written as

\[
(4.3) \quad G(f^n x_0, p, p) \leq kG(f^{n-1} x_0, f^n x_0, f^n x_0) \quad \text{for } n \geq 1.
\]

But, (3.22) with \( x = f^{n-2} x_0 \) and \( y = z = f^{n-1} x_0 \), gives

\[
G(f^{n-1} x_0, f^n x_0, f^n x_0) = G(f f^{n-2} x_0, f f^{n-1} x_0, f f^{n-1} x_0)
\]

\[
\leq k \max \{ G(f^{n-2} x_0, f^{n-1} x_0, f^{n-1} x_0), G(f^{n-2} x_0, f^n x_0, f^n x_0), G(f^{n-1} x_0, f^n x_0, f^n x_0), 0, G(f^{n-1} x_0, f^n x_0, f^n x_0), G(f^{n-1} x_0, f^n x_0, f^n x_0), 0, G(f^{n-1} x_0, f^n x_0, f^n x_0) \}
\]

\[
\leq k \left[ G(f^{n-2} x_0, f^{n-1} x_0, f^{n-1} x_0) + G(f^{n-1} x_0, f^n x_0, f^n x_0) \right]
\]

\[
\leq \left( \frac{k}{1-k} \right) G(f^{n-2} x_0, f^{n-1} x_0, f^{n-1} x_0),
\]

from which, by induction, it follows that

\[
G(f^{n-1} x_0, f^n x_0, f^n x_0) \leq \left( \frac{k}{1-k} \right)^{n-1} G(x_0, f x_0, f x_0), \quad n \geq 1.
\]

Substituting this in (4.3), we get

\[
(4.4) \quad G(f^n x_0, p, p) \leq k \left( \frac{k}{1-k} \right)^{n-1} G(x_0, f x_0, f x_0) \quad \text{for } n \geq 1.
\]

Applying the limit as \( n \to \infty \) in (4.4), we see that \( G(f^n x_0, p, p) \to 0 \) or \( f^n x_0 \to p \) as \( n \to \infty \). Since \( x_0 \) is arbitrary, we conclude that \( p \) is a \( G \)-contractive fixed point. \( \square \)

Similarly, we have
**Theorem 4.2.** Let \( p \) be a unique fixed point of a self-map \( f \) on a complete \( G \)-metric space satisfying (3.20). Then \( p \) will be a \( G \)-contractive fixed point.

**Theorem 4.3.** Let \( p \) be a unique fixed point of a self-map \( f \) on a complete \( G \)-metric space satisfying (3.26). Then \( p \) will be a \( G \)-contractive fixed point.

**Proof.** Let \( x_0 \in X \) be arbitrary. Writing \( x = f^{n-1}x_0 \) and \( y = z = p \) in (3.26) and using (G5), we get

\[
G(f^n x_0, p, p) = G(f^n x_0, f p, f p)
\]

\[
\leq k \max \left\{ G(f^{n-1} x_0, f^n x_0, f^n x_0) + G(f^{n-1} x_0, f p, f p) + G(f^{n-1} x_0, f p, f p),
G(p, f p, f p) + G(p, f^n x_0, f^n x_0) + G(p, f p, f p),
G(p, f p, f p) + G(p, f^n x_0, f^n x_0) + G(p, f p, f p) \right\}
\]

\[
= k \max \left\{ G(f^{n-1} x_0, f^n x_0, f^n x_0) + 2G(f^{n-1} x_0, p, p),
0 + G(p, f^n x_0, f^n x_0) + 0 + G(p, f^n x_0, f^n x_0) + 0 \right\}
\]

(4.5) \[ = kM, \]

where

(4.6) \[ M = \max \{ G(f^{n-1} x_0, f^n x_0, f^n x_0) + 2G(f^{n-1} x_0, p, p), G(p, f^n x_0, f^n x_0) \}. \]

We have two cases:

**Case (a).** Suppose that \( M = G(p, f^n x_0, f^n x_0) \). Then, (4.5), in view of (2.1), can be written as

(4.7) \[ G(f^n x_0, p, p) \leq kG(p, f^n x_0, f^n x_0) \leq 2kG(p, p, f^n x_0) \text{ for all } n \geq 1. \]

If \( f^n x_0 \neq p \) for some \( m \), then (4.7) would imply a contradiction that

\[ 0 < G(p, p, f^m x_0) < G(p, p, f^m x_0), \]

since \( 2k < 1 \). Therefore, \( f^n x_0 = p \) for all \( n \), so that \( f^n x_0 \to p \) as \( n \to \infty \). Since \( x_0 \) is arbitrary, we conclude that \( p \) is a \( G \)-contractive fixed point.
Case (b). Let $M = G(f^{n-1}x_0, f^nx_0, f^n x_0) + 2G(f^{n-1}x_0, p, p)$. Then, (4.5) can be written as

\[(4.8) \quad G(f^n x_0, p, p) \leq k[G(f^{n-1}x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p)], n \geq 1.\]

Now, (3.26) with $x = f^{n-2}x_0$ and $y = z = f^{n-1}x_0$, gives

\[
G(f^{n-1}x_0, f^n x_0, f^n x_0) = G(ff^{n-2}x_0, ff^{n-1}x_0, ff^{n-1}x_0) \\
\leq k \max \left\{ G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-2}x_0, f^n x_0, f^n x_0) + 0 + G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + G(f^{n-1}x_0, f^n x_0, f^n x_0) + 0 \right\} \\
\leq k \max \left\{ G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0) \right\} \\
= k[G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0)] \\
\leq \left( \frac{k}{1-2k} \right) G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0),
\]

from which, by induction, it follows that

\[G(f^{n-1}x_0, f^n x_0, f^n x_0) \leq \left( \frac{k}{1-2k} \right)^{n-1} G(x_0, f^n x_0, f^n x_0), n \geq 1.\]

Substituting this in (4.8), we get

\[G(f^n x_0, p, p) \leq k \left[ \left( \frac{k}{1-2k} \right)^{n-1} G(x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p) \right], n \geq 1,\]

which, again by induction, gives

\[G(f^n x_0, p, p) \leq k \left[ 1 + (2k)^2 + \cdots + (2k)^{n-1} \right] \left( \frac{k}{1-2k} \right)^{n-1} G(x_0, f^n x_0, f^n x_0) + (2k)^n G(x_0, p, p) \]

\[= k \left[ \frac{1-(2k)^n}{1-2k} \right] \left( \frac{k}{1-2k} \right)^{n-1} G(x_0, f^n x_0, f^n x_0) + (2k)^n G(x_0, p, p) \text{ for all } n \geq 1.\]

Note that $2k < 1$. Therefore, applying the limit as $n \to \infty$ in (4.9), we see that $G(f^n x_0, p, p) \to 0$ or $f^n x_0 \to p$ as $n \to \infty$. Since $x_0$ is arbitrary, we conclude that $p$ is a $G$-contractive fixed point. \qed
Conclusion: A new class $\Phi_\alpha$ of auxiliary functions has been introduced and then the generalizations of the fixed point theorems of Mustafa and Sims [2], Mohanta [1] and of Vats et al [7] have been proved. Also, $G$-contractive fixed points are obtained for self-maps satisfying the contractive type conditions (3.20), (3.22) and (3.26).

Conflict of Interests
The authors declare that there is no conflict of interests.

References