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THE CLASS Φ_α OF AUXILIARY FUNCTIONS AND FIXED POINT IN G-METRIC SPACE

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Abstract. Some generalizations of the fixed point theorems of Mohanta [1], Mustafa and Sims [2] and of Vats et al [7] are proved, under a new class Φ_α of auxiliary functions. Also, G-contractive fixed points are obtained for some contraction type conditions.

Keywords: G-complete metric space; the class Φ_α; fixed point; G-contractive fixed point

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1. Introduction

Several fixed point theorems in metric space setting have been proved through contraction type conditions involving different types of auxiliary functions. One such auxiliary function is a mapping $\psi : [0,\infty) \to [0,\infty)$, known as a contractive modulus, with the choice

(1.1) $$\psi(0) = 0 \text{ and } \psi(t) < t \text{ for } t > 0.$$

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The notion of contractive modulus was introduced by Solomon Leader [6]. For instance,

$$\psi_1(t) = \frac{t}{t+1} \text{ and } \psi_2(t) = \frac{t^2}{t+1} \tag{1.2}$$

are contractive moduli. We denote by Ψ, the class of all contractive moduli.

Given a positive integer α, we introduce a generalized class Φ_α as follows:

$$\Phi_\alpha = \{\phi : [0,\infty) \to [0,\infty) | \phi(0) = 0, \phi(\alpha t) < t \text{ for } t > 0\}. \tag{1.3}$$

Remark 1.1. It is obvious that, for $\alpha = 1$, Φ_α reduces to the class Ψ. That is $\Phi_1 = \Psi$. However, in general a contractive modulus need not belong to Φ_α for $\alpha > 1$, as shown in the following example:

Example 1.1. Consider

$$\psi(t) = \begin{cases} \frac{2t}{3}, & t < 1 \\ \frac{t}{2}, & t \geq 1 \end{cases}$$

Obviously, $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$ so that $\psi \in \Psi$. But

$$\psi(2t) = \begin{cases} \frac{4t}{3}, & t < 1/2 \\ t, & t \geq 1/2 \end{cases}$$

so that $\psi(2t) \geq t$ for all $t > 0$. Thus $\psi \notin \Phi_\alpha$.

Definition 1.1. A mapping $\phi \in \Phi_\alpha$ is said to be upper semicontinuous at $t_0 \geq 0$ if $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t_0)$ whenever $\langle t_n \rangle_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} t_n = t_0$, and ϕ is u.s.c if it is u.s.c. at every $t \geq 0$.

Example 1.2. Mappings

$$\frac{t}{t+1}, \frac{t^2}{t+1}, \phi(t) = \begin{cases} qt & (0 \leq t \leq 1) \\ t-q & (t > 1), \end{cases} \tag{1.4}$$

and qt with $0 \leq q < 1$, are continuous contractive modulii, while the contractive modulus

$$\psi(t) = \begin{cases} 0 & (0 \leq t \leq a) \\ t-a & (t > a) \end{cases} \tag{1.5}$$

with $a > 0$, is usc but not continuous.

In this paper, we obtain the fixed points of self-maps satisfying some contraction type conditions in terms of $\phi \in \Phi_\alpha$ for different choices of α in G-metric space. Also, we obtain G-contractive fixed points for some contraction type conditions (See Section 4).

2. **G-metric space**

Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ such that

(G1) $G(x,y,z) = 0$ whenever $x,y,z \in X$ are such that $x = y = z$,

(G2) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,

(G4) $G(x,y,z) = G(\pi(x,y,z))$ for all $x,y,z \in X$, where $\pi(x,y,z)$ is a permutation on the set $\{x,y,z\}$

(G5) $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then G is called a G-metric on X and the pair (X,G), a G-metric space. Axiom (G5) is usually referred to as the rectangle inequality (of the G-metric G). This notion was introduced by Mustafa and Sims [2] in 2006.

In any G-metric space (X,G), we have

$$G(x,y,y) \leq 2G(x,x,y) \text{ for all } x,y \in X. \tag{2.1}$$

A G-metric space (X,G) is said to be symmetric if

$$G(x,y,y) = G(x,x,y) \text{ for all } x,y \in X. \tag{2.2}$$

We use the following notions from of [2] in this paper:

Definition 2.1. Let (X,G) be a G-metric space. A G-ball in X is defined by

$$B_G(x,r) = \left\{ y \in X : G(x,y,y) < r \right\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X.

Also

$$\rho_G(x,y) = G(x,y,y) + G(x,x,y) \text{ for all } x,y \in X. \tag{2.3}$$

induces a metric on X, and the G-metric topology coincides with the metric topology induced by the metric ρ_G. This allows us to readily transform many concepts from metric space into the setting of G-metric space.

Definition 2.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-*convergent* with limit $p \in X$ if it converges to p in the G-metric topology $\tau(G)$.

Definition 2.3. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-Cauchy if $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$.

Definition 2.4. A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in X converges in it.

3. Fixed point theorems involving the class Φ_α

Our first result is

Theorem 3.1. *Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \le \phi\Big(\max\big\{&G(x,y,z),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz),\\ &G(x,fy,fy),G(y,fz,fz),G(z,fx,fx)\big\}\Big) \\ &\textit{for all } x,y,z \in X, \end{aligned} \tag{3.1}$$

where $\phi \in \Phi_2$ is nondecreasing and upper semicontinuous. Then f will have a unique fixed point p.

Proof. Let $x_0 \in X$ be arbitrary. Define $\langle x_n \rangle_{n=1}^{\infty} \subset X$ by

$$x_n = fx_{n-1} \text{ for } n \ge 1. \tag{3.2}$$

Writing with $x = x_{n-1}$ and $y = z = x_n$ in (3.1) and then using (2.1), we get

$$
\begin{aligned}
G(fx_{n-1}, fx_n, fx_n) &= G(x_n, x_{n+1}, x_{n+1}) \\
&\leq \phi\big(\max\big\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, fx_{n-1}, fx_{n-1}), G(x_n, fx_n, fx_n), \\
&\qquad G(x_n, fx_n, fx_n), G(x_{n-1}, fx_n, fx_n), G(x_n, fx_n, fx_n), \\
&\qquad\qquad G(x_n, fx_{n-1}, fx_{n-1})\big\}\big) \\
&\leq \phi\big(\max\big\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
&\qquad G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}), \\
&\qquad\qquad G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_n)\big\}\big) \\
&\leq \phi\big(\max\big\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1})\big\}\big) \\
&\leq \phi\big(\max\big\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
&\qquad G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\big\}\big) \\
&\leq \phi\big(G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\big)
\end{aligned}
$$

Define

$$t_n = G(x_{n-1}, x_n, x_n) \text{ for } n \geq 1. \tag{3.3}$$

Then the above inequality can be written as

$$t_{n+1} \leq \phi\big(t_n + t_{n+1}\big) \tag{3.4}$$

We now prove that

$$t_n \geq t_{n+1} \text{ for } n \geq 1. \tag{3.5}$$

If possible, suppose that $t_m < t_{m+1}$ for some $m \geq 1$. Then $t_{m+1} > 0$. Since ϕ is nondecreasing, from (3.4) it follows that

$$t_{m+1} \leq \phi\big(t_{m+1} + t_m\big) < t_{m+1},$$

which is a contradiction. This proves (3.5). In other words, $\langle t_n \rangle_{n=1}^{\infty}$ is a decreasing sequence of nonnegative real numbers and hence converges to some $t \geq 0$.

Now using (3.5) in (3.4), we get

$$t_{n+1} \leq \phi\left(t_{n+1} + t_n\right) \leq \phi(2t_n) \text{ for } n \geq 1.$$

Taking the limit superior as $n \to \infty$ in this and then using the upper semicontinuity of ϕ, we obtain that

$$t \leq \phi(2t). \tag{3.6}$$

If $t > 0$ in (3.6), then the choice of ϕ implies that $t \leq \phi(2t) < t$, which is a contradiction. Thus

$$t = \lim_{n\to\infty} t_n = \lim_{n\to\infty} G(x_{n-1}, x_n, x_n) = 0. \tag{3.7}$$

We now prove that $\langle x_n \rangle_{n=1}^{\infty}$ is a G-Cauchy sequence in X.

If possible we suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is not G-Cauchy. Then for some $\varepsilon > 0$, we choose sequences $\langle x_{m_k} \rangle_{k=1}^{\infty}$ and $\langle x_{m_k} \rangle_{k=1}^{\infty}$ of positive integers such that $m_k > n_k > k$ and

$$G(x_{m_k}, x_{n_k}, x_{n_k}) \geq \varepsilon \text{ for } k = 1, 2, 3, \ldots. \tag{3.8}$$

Suppose that m_k is the smallest integer exceeding n_k which satisfies (3.8). That is

$$G(x_{m_k-1}, x_{n_k}, x_{n_k}) < \varepsilon. \tag{3.9}$$

Now by rectangle inequality of G, we see that

$$\begin{aligned} \varepsilon \leq G(x_{m_k}, x_{n_k}, x_{n_k}) &\leq G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{m_k-1}, x_{n_k}, x_{n_k}) \\ &< G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + \varepsilon \end{aligned} \tag{3.10}$$

and from (3.7), we see that

$$\lim_{k\to\infty} G(x_{m_k-1}, x_{m_k}, x_{m_k}) = 0 \tag{3.11}$$

and

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{n_k}, x_{n_k}) = 0 \tag{3.12}$$

Using (3.11) in (3.10), we get

$$\lim_{k\to\infty} G(x_{m_k}, x_{n_k}, x_{n_k}) = \varepsilon. \tag{3.13}$$

Also by rectangle inequality of G and (2.1), we get

$$\begin{aligned} G(x_{n_k-1}, x_{m_k}, x_{m_k}) &\leq G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\leq G(x_{n_k-1}, x_{n_k}, x_{n_k}) + 2G(x_{n_k}, x_{n_k}, x_{m_k}). \end{aligned}$$

As $k \to \infty$ this in view of (3.12) and (3.13), gives

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{m_k}, x_{m_k}) = 2\varepsilon. \tag{3.14}$$

On the other hand, writing $x = x_{m_k-1}$, $y = z = x_{n_k-1}$ in (3.1), we have

$$\begin{aligned} G(fx_{m_k-1}, fx_{n_k-1}, fx_{n_k-1}) &= G(x_{m_k}, x_{n_k}, x_{n_k}) \\ &\leq \phi\big(\max\big\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k-1}, fx_{m_k-1}, fx_{m_k-1}), \\ &\qquad G(x_{n_k-1}, fx_{n_k-1}, fx_{n_k-1}), G(x_{n_k-1}, fx_{n_k-1}, fx_{n_k-1}), \\ &\qquad G(x_{m_k-1}, fx_{n_k-1}, fx_{n_k-1}), G(x_{n_k-1}, fx_{n_k-1}, fx_{n_k-1}), \\ &\qquad G(x_{n_k-1}, fx_{m_k-1}, fx_{m_k-1})\big\}\big), \end{aligned}$$

or

$$\begin{aligned} \varepsilon &\leq G(x_{m_k}, x_{n_k}, x_{n_k}) \\ &\leq \phi\big(\max\big\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k-1}, x_{m_k}, x_{m_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), \\ &\qquad G(x_{n_k-1}, x_{n_k}, x_{n_k}), G(x_{m_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), \\ &\qquad G(x_{n_k-1}, x_{m_k}, x_{m_k})\big\}\big) \\ &= \phi\big(\max\big\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k-1}, x_{m_k}, x_{m_k}), G(x_{n_k-1}, x_{n_k}, x_{n_k}), \\ &\qquad G(x_{m_k-1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{m_k}, x_{m_k})\big\}\big). \end{aligned} \tag{3.15}$$

Proceeding the limit as $n \to \infty$ in (3.15) and then using upper semicontinuity of ϕ, (3.9), (3.11), (3.12),(3.13) and (3.14) we get

$$\varepsilon \leq \phi\big(\max\big\{\varepsilon, 0, 0, \varepsilon, 2\varepsilon\big\}\big) = \phi(2\varepsilon).$$

Since ϕ is nondecreasing, this finally gives

$$\varepsilon \leq \phi(2\varepsilon) < \varepsilon, \tag{3.16}$$

which is a contradiction. Hence $\langle x_n \rangle_{n=1}^{\infty}$ must be a G-Cauchy sequence in X.

Since (X,G) is G-Complete, there exists a point $p \in X$ such that $\langle x_n \rangle_{n=1}^{\infty}$ is G-convergent to p. That is

$$\lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = p. \tag{3.17}$$

We now establish that p is a fixed point of f. In fact, writing $x = x_{n-1}$ and $y = z = p$ in (3.1)

$$\begin{aligned}
G(fx_{n-1}, fp, fp) = G(x_n, fp, fp)&\\
&\leq \phi\big(\max\big\{G(x_{n-1}, p, p), G(x_{n-1}, fx_{n-1}, fx_{n-1}), G(p, fp, fp),\\
&\qquad G(p, fp, fp), G(x_{n-1}, fp, fp), G(p, fp, fp),\\
&\qquad\qquad G(p, fx_{n-1}, fx_{n-1})\big\}\big)\\
&\leq \phi\big(\max\big\{G(x_{n-1}, p, p), G(x_{n-1}, x_n, x_n), G(p, fp, fp),\\
&\qquad G(x_{n-1}, fp, fp), G(p, x_n, x_n)\big\}\big). \qquad (3.18)
\end{aligned}$$

Proceeding the limit as $n \to \infty$ in (3.18) and then using (3.17), we get

$$\begin{aligned}
G(p, fp, fp) &\leq \phi\big(\max\big\{0, 0, G(p, fp, fp), G(p, fp, fp), 0\big\}\big)\\
&= \phi(G(p, fp, fp)). \qquad (3.19)
\end{aligned}$$

If $p \neq fp$, then $G(p, fp, fp) > 0$. Since ϕ is nondecreasing, (3.19) gives

$$0 < G(p, fp, fp) \leq \phi(G(p, fp, fp)) < G(p, fp, fp),$$

which is a contradiction. Hence $p = fp$.

To establish the uniqueness of the fixed point, we suppose that p and q are fixed points of f with $p \neq q$. Then Writing $x = p$ and $y = z = q$ in (3.1), we get

$$\begin{aligned} G(fp, fq, fq) \leq &\phi\big(\max\{G(p,q,q), G(p,fp,fp), G(q,fq,fq), G(q,fq,fq), \\ &G(p,fq,fq), G(q,fq,fq), G(q,fp,fp)\}\big) \end{aligned}$$

which on using (2.1) implies that

$$\begin{aligned} G(p,q,q) &\leq \phi\big(\max\{G(p,q,q), 0, 0, 0, G(p,q,q), 0, G(q,p,p)\}\big) \\ &\leq \phi\big(\max\{G(p,q,q), 2G(p,q,q)\}\big) \\ &= \phi(2G(p,q,q)). \end{aligned}$$

Since ϕ is nondecreasing, this gives

$$G(p,q,q) \leq \phi(2G(p,q,q)) < G(p,q,q),$$

which is again a contradiction. Therefore, $p = q$. □

Remark 3.1. Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/2$ in Theorem 3.1. Then $\phi(0) = 0$ and $\phi(2t) = 2kt < t$ for all $t > 0$. Therefore, we get

Corollary 3.1 (Theorem 2.1, [3])**.** *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying the condition*

$$\begin{aligned} G(fx, fy, fz) \leq &k \max\{G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ &G(x,fy,fy), G(y,fz,fz), G(z,fx,fx)\} \end{aligned}$$

$$\text{for all } x, y, z \in X, \tag{3.20}$$

where $0 < k < 1/2$. Then f will have a unique fixed point p.

Just similar to Theorem 3.1, we can prove

Theorem 3.2. *Suppose that* (X,G) *is a complete G-metric space and* f*, a self-map on* X *satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq \phi\Big(\max\big\{&G(x,fx,fx),G(x,fy,fy),\\ &G(x,fz,fz),G(y,fy,fy),G(y,fx,fx),\\ &G(y,fz,fz),G(z,fz,fz),G(z,fx,fx),\\ &G(z,fy,fy)\big\}\Big) \text{ for all } x,y,z\in X, \end{aligned} \tag{3.21}$$

where $\phi \in \Phi_2$ *is nondecreasing and upper semicontinuous. Then* f *will have a unique fixed point* p*.*

Remark 3.2. Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/2$ in Theorem 3.2. Then $\phi(0) = 0$ and $\phi(2t) = 2kt < t$ for all $t > 0$. Therefore, we get

Corollary 3.2 (Theorem 1, [7])**.** *Suppose that* (X,G) *is a complete G-metric space and* f*, a self-map on* X *satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq k\max\big\{&G(x,fx,fx),G(x,fy,fy),\\ &G(x,fz,fz),G(y,fy,fy),G(y,fx,fx),\\ &G(y,fz,fz),G(z,fz,fz),G(z,fx,fx),\\ &G(z,fy,fy)\big\} \text{ for all } x,y,z\in X, \end{aligned} \tag{3.22}$$

where $0 < k < 1/2$*. Then* f *will have a unique fixed point* p*.*

With an argument, similar to that of Theorem 3.1, we can prove the following:

Theorem 3.3. *Suppose that* (X,G) *is a complete G-metric space and* f*, a self-map on* X *satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq \phi\Big(\max\big\{&G(x,fy,fy)+G(y,fx,fx)+G(z,fz,fz),\\ &G(y,fz,fz)+G(z,fy,fy)+G(x,fx,fx),\\ &G(z,fx,fx)+G(x,fz,fz)+G(y,fy,fy)\big\}\Big)\\ &\text{for all } x,y,z\in X, \end{aligned} \tag{3.23}$$

where $\phi \in \Phi_3$ is nondecreasing and upper semicontinuous. Then f will have a unique fixed point p.

Remark 3.3. Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/3$ in Theorem 3.3. Then $\phi(0) = 0$ and $\phi(3t) = 3kt < t$ for $t > 0$. Therefore, we have

Corollary 3.3 (Theorem 3.9, [1])**.** *Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq & k \max\big\{G(x,fy,fy)+G(y,fx,fx)+G(z,fz,fz), \\ & G(y,fz,fz)+G(z,fy,fy)+G(x,fx,fx), \\ & G(z,fx,fx)+G(x,fz,fz)+G(y,fy,fy)\big\} \\ & \text{for all } x,y,z \in X, \end{aligned} \tag{3.24}$$

where $0 < k < 1/3$. Then f will have a unique fixed point p.

The fourth main result is given below without proof:

Theorem 3.4. *Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq & \phi\big(\max\big\{G(x,fx,fx)+G(y,fy,fy)+G(z,fz,fz), \\ & G(x,fy,fy)+G(y,fx,fx)+G(z,fy,fy), \\ & G(x,fz,fz)+G(y,fz,fz)+G(z,fx,fx)\big\}\big) \\ & \text{for all } x,y,z \in X, \end{aligned} \tag{3.25}$$

where $\phi \in \Phi_4$ is nondecreasing upper semicontinuous. Then f will have a unique fixed point p.

Remark 3.4. Set $\phi(t) = kt$ for all $t \geq 0$, where $0 < k < 1/4$ in Theorem 3.4. Then $\phi(0) = 0$ and $\phi(4t) = 4kt < t$ for $t > 0$. Therefore, we have

Corollary 3.4 (Vats et al, [7]). *Suppose that (X,G) is a complete G-metric space and f, a self-map on X satisfying the condition*

$$\begin{aligned} G(fx,fy,fz) \leq k\max\big\{&G(x,fx,fx)+G(y,fy,fy)+G(z,fz,fz),\\ &G(x,fy,fy)+G(y,fx,fx)+G(z,fy,fy),\\ &G(x,fz,fz)+G(y,fz,fz)+G(z,fx,fx)\big\}\\ &\text{for all } x,y,z\in X, \end{aligned} \tag{3.26}$$

where $0<k<1/4$. Then f will have a unique fixed point p.

The final main result of this paper is

Theorem 3.5. *Let (X,G) be a complete G-metric space and f be a self-map on X such that*

$$\begin{aligned} G(fx,fy,fz) \leq \phi\big(\max\big\{&G(x,fx,fx)+G(x,fy,fy)+G(x,fz,fz),\\ &G(y,fy,fy)+G(y,fx,fx)+G(y,fz,fz),\\ &G(z,fz,fz)+G(z,fx,fx)+G(z,fy,fy)\big\}\big)\\ &\text{for all } x,y,z\in X, \end{aligned} \tag{3.27}$$

where $\phi\in\Phi_5$ is nondecreasing upper semicontinuous. Then f will have a unique fixed point p.

Remark 3.5. Set $\phi(t)=kt$ for all $t\geq 0$, where $0<k<1/5$ in Theorem 3.5. Then $\phi(0)=0$ and $\phi(5t)=5kt<t$ for $t>0$. Therefore, we have

Corollary 3.5. *Let (X,G) be a complete G-metric space and f be a self-map on X such that*

$$\begin{aligned} G(fx,fy,fz) \leq k\max\big\{&G(x,fx,fx)+G(x,fy,fy)+G(x,fz,fz),\\ &G(y,fy,fy)+G(y,fx,fx)+G(y,fz,fz),\\ &G(z,fz,fz)+G(z,fx,fx)+G(z,fy,fy)\big\}\\ &\text{for all } x,y,z\in X, \end{aligned} \tag{3.28}$$

where $0\leq k<1/5$. Then f will have a unique fixed point p.

4. *G*-contractive fixed points

We begin this section with

Definition 4.1 (Phaneendra and Kumara Swamy, [4]). A fixed point p of f on a G-metric space (X,G) is a G-contractive fixed point, if for each $x_0 \in X$, the orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ is G-convergent, with limit p.

It was shown in [4] that the unique fixed point of the self-map f with the following choices is a G-contractive fixed point.

(a) $G(fx, fy, fz) \leq qG(x,y,z)$ for all $x,y,z \in X$, where $0 \leq q < 1$,

(b) $G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) + eG(x,y,z)$ for all $x,y,z \in X$, where a,b,c and e are nonnegative real numbers with $a+b+c+e < 1$.

In [5], the authors have proved that the unique fixed points of the self-maps are G-contractive fixed points, under (3.24) and (3.28).

Now, we obtain G-contractive fixed points for the maps of Corollaries, obtained in the previous sections.

Theorem 4.1. *Let p be a unique fixed point of a self-map f on a complete G-metric space satisfying* (3.22). *Then p will be a G-contractive fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = z = p$ in (3.22), we get

$$\begin{aligned} G(f^n x_0, p, p) &= G(f^n x_0, fp, fp) \\ &\leq k \max\{G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-1}x_0, fp, fp), G(f^{n-1}x_0, fp, fp), \\ &\quad G(p, fp, fp), d(p, f^{n-1}x_0, f^{n-1}x_0), G(p, fp, fp), \\ &\quad G(p, fp, fp), G(p, f^{n-1}x_0, f^{n-1}x_0), G(p, fp, fp)\} \\ &= kM, \end{aligned} \tag{4.1}$$

where

$$\max\{G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-1}x_0, p, p), G(p, f^n x_0, f^n x_0)\}. \tag{4.2}$$

Now, three cases arise:

Case (a). Suppose that $M = G(p, f^n x_0, f^n x_0)$. Then, it can be shown that p is a G-contractive fixed point, as in case (a) of the previous proof.

Case (b). The case of $M = G(f^{n-1}x_0, p, p)$ is obvious, since $k < 1$.

Case (c). Let $M = G(f^{n-1}x_0, f^n x_0, f^n x_0)$. Then, (4.1) can be written as

$$G(f^n x_0, p, p) \leq kG(f^{n-1}x_0, f^n x_0, f^n x_0) \text{ for } n \geq 1. \tag{4.3}$$

But, (3.22) with $x = f^{n-2}x_0$ and $y = z = f^{n-1}x_0$, gives

$$\begin{aligned} G(f^{n-1}x_0, f^n x_0, f^n x_0) &= G(ff^{n-2}x_0, ff^{n-1}x_0, ff^{n-1}x_0) \\ &\leq k \max\big\{G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0), G(f^{n-2}x_0, f^n x_0, f^n x_0), \\ &\qquad G(f^{n-2}x_0, f^n x_0, f^n x_0), G(f^{n-1}x_0, f^n x_0, f^n x_0), 0, \\ &\qquad G(f^{n-1}x_0, f^n x_0, f^n x_0), G(f^{n-1}x_0, f^n x_0, f^n x_0), 0 \\ &\qquad G(f^{n-1}x_0, f^n x_0, f^n x_0)\big\} \\ &\leq k\big[G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + G(f^{n-1}x_0, f^n x_0, f^n x_0)\big] \\ &\leq \left(\tfrac{k}{1-k}\right) G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0), \end{aligned}$$

from which, by induction, it follows that

$$G(f^{n-1}x_0, f^n x_0, f^n x_0) \leq \left(\tfrac{k}{1-k}\right)^{n-1} G(x_0, fx_0, fx_0), n \geq 1.$$

Substituting this in (4.3), we get

$$G(f^n x_0, p, p) \leq k \left(\tfrac{k}{1-k}\right)^{n-1} G(x_0, fx_0, fx_0) \text{ for } n \geq 1. \tag{4.4}$$

Applying the limit as $n \to \infty$ in (4.4), we see that $G(f^n x_0, p, p) \to 0$ or $f^n x_0 \to p$ as $n \to \infty$. Since x_0 is arbitrary, we conclude that p is a G-contractive fixed point. □

Similarly, we have

Theorem 4.2. *Let p be a unique fixed point of a self-map f on a complete G-metric space satisfying* (3.20)*. Then p will be a G-contractive fixed point.*

Theorem 4.3. *Let p be a unique fixed point of a self-map f on a complete G-metric space satisfying* (3.26)*. Then p will be a G-contractive fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = z = p$ in (3.26) and using (G5), we get

$$
\begin{aligned}
G(f^n x_0, p, p) &= G(f^n x_0, fp, fp) \\
&\leq k \max\big\{G(f^{n-1}x_0, f^n x_0, f^n x_0) + G(f^{n-1}x_0, fp, fp) + G(f^{n-1}x_0, fp, fp), \\
&\qquad G(p, fp, fp) + G(p, f^n x_0, f^n x_0) + G(p, fp, fp), \\
&\qquad G(p, fp, fp) + G(p, f^n x_0, f^n x_0) + G(p, fp, fp)\big\} \\
&= k \max\{G(f^{n-1}x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p), \\
&\qquad 0 + G(p, f^n x_0, f^n x_0) + 0, 0 + G(p, f^n x_0, f^n x_0) + 0\} \\
&= kM, \qquad (4.5)
\end{aligned}
$$

where

$$M = \max\{G(f^{n-1}x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p), G(p, f^n x_0, f^n x_0)\}. \qquad (4.6)$$

We have two cases:

Case (a). Suppose that $M = G(p, f^n x_0, f^n x_0)$. Then, (4.5), in view of (2.1), can be written as

$$G(f^n x_0, p, p) \leq kG(p, f^n x_0, f^n x_0) \leq 2kG(p, p, f^n x_0) \text{ for all } n \geq 1. \qquad (4.7)$$

If $f^n x_0 \neq p$ for some m, then (4.7) would imply a contradiction that

$$0 < G(p, p, f^m x_0) < G(p, p, f^m x_0),$$

since $2k < 1$. Therefore, $f^n x_0 = p$ for all n, so that $f^n x_0 \to p$ as $n \to \infty$. Since x_0 is arbitrary, we conclude that p is a G-contractive fixed point.

Case (b). Let $M = G(f^{n-1}x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p)$. Then, (4.5) can be written as

$$G(f^n x_0, p, p) \le k[G(f^{n-1}x_0, f^n x_0, f^n x_0) + 2G(f^{n-1}x_0, p, p)], n \ge 1. \tag{4.8}$$

Now, (3.26) with $x = f^{n-2}x_0$ and $y = z = f^{n-1}x_0$, gives

$$\begin{aligned}
G(f^{n-1}x_0, f^n x_0, f^n x_0) &= G(ff^{n-2}x_0, ff^{n-1}x_0, ff^{n-1}x_0)\\
&\le k \max \big\{ G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0),\\
&\qquad G(f^{n-2}x_0, f^n x_0, f^n x_0) + 0 + G(f^{n-1}x_0, f^n x_0, f^n x_0),\\
&\qquad G(f^{n-2}x_0, f^n x_0, f^n x_0) + G(f^{n-1}x_0, f^n x_0, f^n x_0) + 0 \big\}\\
&\le k \max \big\{ G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0),\\
&\qquad G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0) \big\}\\
&= k \big[G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0) + 2G(f^{n-1}x_0, f^n x_0, f^n x_0) \big]\\
&\le \big(\tfrac{k}{1-2k}\big) G(f^{n-2}x_0, f^{n-1}x_0, f^{n-1}x_0),
\end{aligned}$$

from which, by induction, it follows that

$$G(f^{n-1}x_0, f^n x_0, f^n x_0) \le \big(\tfrac{k}{1-2k}\big)^{n-1} G(x_0, fx_0, fx_0), n \ge 1.$$

Substituting this in (4.8), we get

$$G(f^n x_0, p, p) \le k \left[\big(\tfrac{k}{1-2k}\big)^{n-1} G(x_0, fx_0, fx_0) + 2G(f^{n-1}x_0, p, p) \right], n \ge 1,$$

which, again by induction, gives

$$\begin{aligned}
G(f^n x_0, p, p) \le& k\big[1 + (2k)^2 + \cdots + (2k)^{n-1}\big] \big(\tfrac{k}{1-2k}\big)^{n-1} G(x_0, fx_0, fx_0)\\
&+ (2k)^n G(x_0, p, p)\\
=& k \left[\tfrac{1-(2k)^n}{1-2k}\right] \big(\tfrac{k}{1-2k}\big)^{n-1} G(x_0, fx_0, fx_0)\\
&+ (2k)^n G(x_0, p, p) \text{ for all } n \ge 1.
\end{aligned} \tag{4.9}$$

Note that $2k < 1$. Therefore, applying the limit as $n \to \infty$ in (4.9), we see that $G(f^n x_0, p, p) \to 0$ or $f^n x_0 \to p$ as $n \to \infty$. Since x_0 is arbitrary, we conclude that p is a G-contractive fixed point. □

Conclusion: A new class Φ_α of auxiliary functions has been introduced and then the generalizations of the fixed point theorems of Mustafa and Sims [2], Mohanta [1] and of Vats et al [7] have been proved. Also, G-contractive fixed points are obtained for self-maps satisfying the contractive type conditions (3.20), (3.22) and (3.26).

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Mohanta, S. K., Some fixed point theorems in G-Metric Spaces, An. St. Univ. Ovidius Constanta 20(1) (2012), 285-306.

[2] Mustafa, Z., Sims, B., A new approach to generalized metric spaces, Journal of Nonlinear and Convex Anal. 7(2) (2006), 289-297.

[3] Mustafa, Z., Sims, B., Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Appl., 2009 (2009) Article ID 917175, 10 pages.

[4] Phaneendra, T., Kumara Swamy, K., Unique fixed point in G-metric space through greatest lower bound properties, NoviSad J. Math., 43(2) (2013), 107-115.

[5] Phaneendra, T., Saravanan,S., On G-contractive fixed points, Jnanabha, in press.

[6] Solmon Leader, Fixed Points for a General Contraction in Metric Space, *Math. Japonica.* 24(1) (1979), 17-24.

[7] Vats, R. K., Kumar, S., Sihag,V., Fixed Point Theorems in Complete G-metric space, Fasc. Math., 47 (2011), 127-138.