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FIXED POINT THEOREMS FOR SUM OF TWO MAPPINGS ON NOT NECESSARILY CONVEX SUBSET OF A LOCALLY CONVEX SPACE

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Abstract. In this paper, we prove some fixed point theorems for sum of two mappings in locally convex space. The results generalized the fixed point theorem of Cain and Nashed [2] for sum of two mappings on a convex subset of a locally convex space to sum of two mappings defined on almost convex subset as well as star-shaped subset of a locally convex space.

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1. Introduction

Let X be a nonempty closed convex and bounded subset of a Banach space E, and $T : X \to E$, a contraction mapping and $S : X \to E$, a compact mapping. Krasnoselskii, in his 1995 paper [9], proved the existence of a fixed point in X for the sum $T+S$ of the two mappings T and S which satisfy the condition $Tx+Sy \in X$ for all $x,y \in X$. Since then, many authors have generalized Krasnoselskii's result in different directions. For instance, Nashed and Wong [10] proved the

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existence of fixed point for the sum $T+S$ of a nonlinear contraction mapping $T : X \to E$ and a compact mapping $S : X \to E$.

In the setting of locally convex topological vector space, Cain and Nashed [2], extended Krasnoselskii's result to the sum $T+S$ of a contraction mapping $T : X \to E$ and a continuous mapping $S : X \to E$, where X is a nonempty complete convex subset of a locally convex space E. The fixed point result of Nashed and Wong [10] was proved in locally convex space setting when Sehgal and Singh [13] extended the result of Cain and Nashed [2] to a sum $T+S$ of a nonlinear contraction mapping $T : X \to E$ and a continuous mapping $S : X \to E$.

The classes of almost convex sets and star-shaped sets are wider than convex sets as every convex set is almost convex and star-shaped. The purpose of this paper is to prove some extensions of a result of Cain and Nashed [2] for sum of two mappings on a convex subset of a locally convex topological vector space to sum of two mappings defined on almost convex subsets as well as star-shaped subsets of a locally convex topological vector space. Throughout this paper, E denotes a Hausdorff locally convex topological vector space and $(p_\alpha)_{\alpha \in J}$, a family of seminorms which defines the topology on E with J an indexing set.

2. Preliminaries

Definition 2.1. Let X be a nonempty subset of E. A mapping $T : X \to E$ is called contraction if for each $\alpha \in J$, there is a real number λ_α with $0 < \lambda_\alpha < 1$ such that $p_\alpha(Tx - Ty) \leq \lambda_\alpha p_\alpha(x-y)$ for all $x, y \in X$.

Cain and Nashed [2] proved the following extension of the Banach contraction mapping principle.

Theorem 2.2. *Let X be a nonempty sequentially complete subset of E and $T : X \to X$, a contraction mapping, then T has a unique fixed point $\bar{x} \in X$ and $T^n x \to \bar{x}$ for all $x \in X$.*

We state the following Tychonoff's fixed point theorem [14] and consider some of its variants and generalizations.

Theorem 2.3. *Let X be a nonempty compact convex subset of E. If $T : X \to X$ is any continuous mapping, then T has a fixed point in X.*

The following is a variant of the Tychonoff's theorem, known as the Shauder-Tychonoff fixed point theorem(see [1] and [7]).

Theorem 2.4. *Let X be a nonempty convex subset of E and $T : X \to X$ a compact continuous mapping. Then T has a fixed point.*

Himmelberg [3] introduced the following notion of almost convex set.

Definition 2.5. A nonempty subset X of a topological vector space E is called almost convex if for any neighbourhood V of the origin 0 in E and for any finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$, there exists a finite set $\{z_1, z_2, \ldots, z_n\} \subseteq X$ such that for each $i \in \{1, 2, 3 \ldots, n\}$, $z_i - x_i \in V$ and $\mathrm{co}\{z_1, z_2, \ldots, z_n\} \subseteq X$.

Definition 2.6. In the above definition, "co" stands for the convex hull of a set. If X is a convex subset of E, then for every 0-neighbourhood V and any finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$, choose $z_i \in (x_i + V) \cap X$ for $i = 1, 2, 3, \ldots, n$ since $(x_i + V) \cap X \neq \emptyset$. Clearly, $z_i - x_i \in V$ and $co\{z_1, z_2, \ldots, z_n\} \subseteq co(X) = X$. Hence, X is almost convex. Therefore, every convex set is almost convex but the converse is not true in general.

Park and Tan [11] proved the following generalization of the Shauder-Tychonoff fixed point theorem.

Theorem 2.7. *Let X be a nonempty almost convex subset of E, and $T : X \to X$ a compact continuous mapping. Then T has a fixed point.*

If X is compact, then we have the following:

Theorem 2.8. *Let X be a nonempty compact almost convex subset of E, and $T : X \to X$ a continuous mapping. Then T has a fixed point.*

Definition 2.9. Let X be a subset of a vector space E. Then X is called star-shaped if there exists $p \in X$ such that $tp + (1-t)x \in X$ for all $x \in X$, $0 \leq t \leq 1$.

The point p is called a star-point and the set of all the star-points of X is called the star-core of X.

Clearly, the star-core is a convex subset of X.

Definition 2.10. A mapping T on a convex set X is called affine if it satisfies the identity

$$T(tx + (1-t)y) = tTx + (1-t)Ty$$

where $0 < t < 1$, $x, y \in X$.

Hu [4,5] showed that:

Theorem 2.11. *If X is a compact star-shaped subset of E and C is the corresponding star-core of X. Then C is a compact convex subset of X.*

Hu and Heng [6] proved the following results.

Theorem 2.12. *Let X be a nonempty compact star-shaped subset of a topological vector space E. Then every decreasing chain of nonempty compact and star-shaped subsets of X has a nonempty intersection that is compact and star-shaped.*

Theorem 2.13. *Suppose X is a star-shaped subset of a topological vector space E and $T : X \to X$ a surjective mapping that is affine on X. Then the star-core of X is invariant under T.*

Applying the above results, we have the following:

Theorem 2.14. *Let X be a nonempty compact and star-shaped subset of a Hausdorff locally convex space E. If $T : X \to X$ is an affine continuous mapping, then T has a fixed point in X.*

Proof. Since affine maps preserve star-shapedness and continuous maps preserve compactness, we define a decreasing chain of nonempty, compact and star-shaped subsets of X by $X_1 = X$ and $X_{n+1} = TX_n$, $n = 1,2,3,...$ Clearly, $TX_1 \subseteq X_1$. Suppose $TX_n \subseteq X_n$. Then

$$TX_{n+1} = T(TX_n) \subseteq TX_n = X_{n+1}$$

Hence by induction $TX_n \subseteq X_n \; \forall \; n$.

Applying theorem 2.12 and Zorn's lemma, we get a minimal nonempty, compact and star-shaped subset M of X which is invariant under T. We claim that $TM = M$. Suppose that $TM = S \subset M$. Since T is affine and continuous, S is nonempty compact and star-shaped and $TS \subseteq TM = S$. That is, S is a nonempty compact and star-shaped subset of X which is invariant under T. This contradicts the minimality of M. Hence, $TM = M$, that is, $T : M \to M$ is surjective.

Now, let C be the star-core of M. By theorems 2.11 and 2.13, C is a compact convex subset of M and $T : C \to C$. Hence, by the Tychonoff fixed point theorem, T has a fixed point in $C \subset X$.

Theorems 2.8 and 2.14 generalize the Tychonoff's theorem [14] to almost convex and star-shaped subsets of E respectively.

3. Main results

The following are extensions of a result of Cain and Nashed [2](theorem 3.1) to a sum of a contraction mapping and a continuous mapping defined on an almost convex subset and star-shaped subset of a Hausdorff locally convex space. The proofs follow the same line of argument as in [2].

Theorem 3.1. *Let X be a nonempty compact almost convex subset of E. Let $T,S : X \to E$ be mappings such that $Tx+Sy \in X$ for all $x,y \in X$. If T is a contraction and S is continuous, then there is a point $\bar{x} \in X$ such that $T\bar{x}+S\bar{x} = \bar{x}$.*

Proof. For each $y \in X$, we define a mapping $F : X \to X$ by

$$Fx = Tx + Sy$$

For $x_1, x_2 \in X$ and $\alpha \in J$, we have

$$\begin{aligned} p_\alpha(Fx_1 - Fx_2) &= p_\alpha(Tx_1 + Sy - Tx_2 - Sy) \\ &= p_\alpha(Tx_1 - Tx_2) \\ &\leq \lambda_\alpha p_\alpha(x_1 - x_2) \end{aligned}$$

Hence F is a contraction on X. By theorem 2.2, F has a unique fixed point in X. Denote this fixed point by Hy. That is,

$$Hy = F(Hy) = T(Hy) + Sy$$

Thus for all $u_1, u_2 \in X$, we have

$$\begin{aligned} Hu_1 - Hu_2 &= T(Hu_1) + Su_1 - T(Hu_2) - Su_2 \\ &= T(Hu_1) - T(Hu_2) + Su_1 - Su_2 \end{aligned}$$

So that

$$\begin{aligned} p_\alpha(Hu_1 - Hu_2) &\leq p_\alpha(T(Hu_1) - T(Hu_2)) + p_\alpha(Su_1 - Su_2) \\ &\leq \lambda_\alpha p_\alpha(Hu_1 - Hu_2) + p_\alpha(Su_1 - Su_2) \end{aligned}$$

This implies

$$p_\alpha(Hu_1 - Hu_2) \leq (1 - \lambda_\alpha)^{-1} p_\alpha(Su_1 - Su_2)$$

As S is continuous, it follows that H is continuous. By theorem 2.8, H has a fixed point $\bar{x} \in X$ and

$$\bar{x} = H\bar{x} = T(H\bar{x}) + S\bar{x}$$

$$= T\bar{x} + S\bar{x}$$

This completes the proof.

Mimicking the proof above and applying theorems 2.2 and 2.14 we establish the following:

Theorem 3.2. *Let X be a nonempty compact complete star-shaped subset of E. Let $T, S : X \to E$ be mappings such that $Tx + Sy \in X$ for all $x, y \in X$. If T is a contraction mapping and S is an affine continuous mapping, then there is a point $\bar{x} \in X$ such that $T\bar{x} + S\bar{x} = \bar{x}$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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