FIXED POINT THEOREMS FOR SUM OF TWO MAPPINGS ON NOT NECESSARILY CONVEX SUBSET OF A LOCALLY CONVEX SPACE

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Abstract. In this paper, we prove some fixed point theorems for sum of two mappings in locally convex space. The results generalized the fixed point theorem of Cain and Nashed [2] for sum of two mappings on a convex subset of a locally convex space to sum of two mappings defined on almost convex subset as well as star-shaped subset of a locally convex space.

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1. Introduction

Let $X$ be a nonempty closed convex and bounded subset of a Banach space $E$, and $T : X \to E$, a contraction mapping and $S : X \to E$, a compact mapping. Krasnoselskii, in his 1995 paper [9], proved the existence of a fixed point in $X$ for the sum $T + S$ of the two mappings $T$ and $S$ which satisfy the condition $Tx + Sy \in X$ for all $x, y \in X$. Since then, many authors have generalized Krasnoselskii’s result in different directions. For instance, Nashed and Wong [10] proved the
existence of fixed point for the sum $T + S$ of a nonlinear contraction mapping $T : X \to E$ and a compact mapping $S : X \to E$.

In the setting of locally convex topological vector space, Cain and Nashed [2], extended Krasnoselskii’s result to the sum $T + S$ of a contraction mapping $T : X \to E$ and a continuous mapping $S : X \to E$, where $X$ is a nonempty complete convex subset of a locally convex space $E$. The fixed point result of Nashed and Wong [10] was proved in locally convex space setting when Sehgal and Singh [13] extended the result of Cain and Nashed [2] to a sum $T + S$ of a nonlinear contraction mapping $T : X \to E$ and a continuous mapping $S : X \to E$.

The classes of almost convex sets and star-shaped sets are wider than convex sets as every convex set is almost convex and star-shaped. The purpose of this paper is to prove some extensions of a result of Cain and Nashed [2] for sum of two mappings on a convex subset of a locally convex topological vector space to sum of two mappings defined on almost convex subsets as well as star-shaped subsets of a locally convex topological vector space. Throughout this paper, $E$ denotes a Hausdorff locally convex topological vector space and $(p_\alpha)_{\alpha \in J}$, a family of seminorms which defines the topology on $E$ with $J$ an indexing set.

2. Preliminaries

Definition 2.1. Let $X$ be a nonempty subset of $E$. A mapping $T : X \to E$ is called contraction if for each $\alpha \in J$, there is a real number $\lambda_\alpha$ with $0 < \lambda_\alpha < 1$ such that $p_\alpha(Tx - Ty) \leq \lambda_\alpha p_\alpha(x - y)$ for all $x, y \in X$.

Cain and Nashed [2] proved the following extension of the Banach contraction mapping principle.

Theorem 2.2. Let $X$ be a nonempty sequentially complete subset of $E$ and $T : X \to X$, a contraction mapping, then $T$ has a unique fixed point $\bar{x} \in X$ and $T^n x \to \bar{x}$ for all $x \in X$.

We state the following Tychonoff’s fixed point theorem [14] and consider some of its variants and generalizations.

Theorem 2.3. Let $X$ be a nonempty compact convex subset of $E$. If $T : X \to X$ is any continuous mapping, then $T$ has a fixed point in $X$. 
The following is a variant of the Tychonoff’s theorem, known as the Schauder-Tychonoff fixed point theorem (see [1] and [7]).

**Theorem 2.4.** Let $X$ be a nonempty convex subset of $E$ and $T : X \to X$ a compact continuous mapping. Then $T$ has a fixed point.

Himmelberg [3] introduced the following notion of almost convex set.

**Definition 2.5.** A nonempty subset $X$ of a topological vector space $E$ is called almost convex if for any neighbourhood $V$ of the origin $0$ in $E$ and for any finite set \( \{x_1, x_2, \ldots, x_n\} \subseteq X \), there exists a finite set \( \{z_1, z_2, \ldots, z_n\} \subseteq X \) such that for each $i \in \{1, 2, 3, \ldots, n\}$, $z_i - x_i \in V$ and $\text{co}\{z_1, z_2, \ldots, z_n\} \subseteq X$.

**Definition 2.6.** In the above definition, “co” stands for the convex hull of a set. If $X$ is a convex subset of $E$, then for every 0-neighbourhood $V$ and any finite set \( \{x_1, x_2, \ldots, x_n\} \subseteq X \), choose $z_i \in (x_i + V) \cap X$ for $i = 1, 2, 3, \ldots, n$ since $(x_i + V) \cap X \neq \emptyset$. Clearly, $z_i - x_i \in V$ and $\text{co}\{z_1, z_2, \ldots, z_n\} \subseteq \text{co}(X) = X$. Hence, $X$ is almost convex. Therefore, every convex set is almost convex but the converse is not true in general.

Park and Tan [11] proved the following generalization of the Schauder-Tychonoff fixed point theorem.

**Theorem 2.7.** Let $X$ be a nonempty almost convex subset of $E$, and $T : X \to X$ a compact continuous mapping. Then $T$ has a fixed point.

If $X$ is compact, then we have the following:

**Theorem 2.8.** Let $X$ be a nonempty compact almost convex subset of $E$, and $T : X \to X$ a continuous mapping. Then $T$ has a fixed point.

**Definition 2.9.** Let $X$ be a subset of a vector space $E$. Then $X$ is called star-shaped if there exists $p \in X$ such that $tp + (1 - t)x \in X$ for all $x \in X$, $0 \leq t \leq 1$.

The point $p$ is called a star-point and the set of all the star-points of $X$ is called the star-core of $X$.

Clearly, the star-core is a convex subset of $X$.

**Definition 2.10.** A mapping $T$ on a convex set $X$ is called affine if it satisfies the identity

$$T(tx + (1 - t)y) = tTx + (1 - t)Ty$$

where $0 < t < 1, x, y \in X$. 
Hu [4,5] showed that:

**Theorem 2.11.** If \( X \) is a compact star-shaped subset of \( E \) and \( C \) is the corresponding star-core of \( X \). Then \( C \) is a compact convex subset of \( X \).

Hu and Heng [6] proved the following results.

**Theorem 2.12.** Let \( X \) be a nonempty compact star-shaped subset of a topological vector space \( E \). Then every decreasing chain of nonempty compact and star-shaped subsets of \( X \) has a nonempty intersection that is compact and star-shaped.

**Theorem 2.13.** Suppose \( X \) is a star-shaped subset of a topological vector space \( E \) and \( T : X \to X \) a surjective mapping that is affine on \( X \). Then the star-core of \( X \) is invariant under \( T \).

Applying the above results, we have the following:

**Theorem 2.14.** Let \( X \) be a nonempty compact and star-shaped subset of a Hausdorff locally convex space \( E \). If \( T : X \to X \) is an affine continuous mapping, then \( T \) has a fixed point in \( X \).

**Proof.** Since affine maps preserve star-shapedness and continuous maps preserve compactness, we define a decreasing chain of nonempty, compact and star-shaped subsets of \( X \) by \( X_1 = X \) and

\[
X_{n+1} = TX_n, \quad n = 1, 2, 3, \ldots
\]

Clearly, \( TX_1 \subseteq X_1 \). Suppose \( TX_n \subseteq X_n \). Then

\[
TX_{n+1} = T(TX_n) \subseteq TX_n = X_{n+1}
\]

Hence by induction \( TX_n \subseteq X_n \ \forall \ n \).

Applying theorem 2.12 and Zorn’s lemma, we get a minimal nonempty, compact and star-shaped subset \( M \) of \( X \) which is invariant under \( T \). We claim that \( TM = M \). Suppose that \( TM = S \subseteq M \). Since \( T \) is affine and continuous, \( S \) is nonempty compact and star-shaped and \( TS \subseteq TM = S \). That is, \( S \) is a nonempty compact and star-shaped subset of \( X \) which is invariant under \( T \). This contradicts the minimality of \( M \). Hence, \( TM = M \), that is, \( T : M \to M \) is surjective.

Now, let \( C \) be the star-core of \( M \). By theorems 2.11 and 2.13, \( C \) is a compact convex subset of \( M \) and \( T : C \to C \). Hence, by the Tychonoff fixed point theorem, \( T \) has a fixed point in \( C \subseteq X \).

Theorems 2.8 and 2.14 generalize the Tychonoff’s theorem [14] to almost convex and star-shaped subsets of \( E \) respectively.

### 3. Main results


The following are extensions of a result of Cain and Nashed [2](theorem 3.1) to a sum of a contraction mapping and a continuous mapping defined on an almost convex subset and star-shaped subset of a Hausdorff locally convex space. The proofs follow the same line of argument as in [2].

**Theorem 3.1.** Let $X$ be a nonempty compact almost convex subset of $E$. Let $T, S : X \rightarrow E$ be mappings such that $Tx + Sy \in X$ for all $x, y \in X$. If $T$ is a contraction and $S$ is continuous, then there is a point $\bar{x} \in X$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

**Proof.** For each $y \in X$, we define a mapping $F : X \rightarrow X$ by

$$Fx = Tx + Sy$$

For $x_1, x_2 \in X$ and $\alpha \in J$, we have

$$p_\alpha(Fx_1 - Fx_2) = p_\alpha(Tx_1 + Sy - Tx_2 - Sy)$$

$$= p_\alpha(Tx_1 - Tx_2)$$

$$\leq \lambda_\alpha p_\alpha(x_1 - x_2)$$

Hence $F$ is a contraction on $X$. By theorem 2.2, $F$ has a unique fixed point in $X$. Denote this fixed point by $Hy$. That is,

$$Hy = F(Hy) = T(Hy) + Sy$$

Thus for all $u_1, u_2 \in X$, we have

$$Hu_1 - Hu_2 = T(Hu_1) + Su_1 - T(Hu_2) - Su_2$$

$$= T(Hu_1) - T(Hu_2) + Su_1 - Su_2$$

So that

$$p_\alpha(Hu_1 - Hu_2) \leq p_\alpha(T(Hu_1) - T(Hu_2)) + p_\alpha(Su_1 - Su_2)$$

$$\leq \lambda_\alpha p_\alpha(Hu_1 - Hu_2) + p_\alpha(Su_1 - Su_2)$$

This implies

$$p_\alpha(Hu_1 - Hu_2) \leq (1 - \lambda_\alpha)^{-1} p_\alpha(Su_1 - Su_2)$$
As $S$ is continuous, it follows that $H$ is continuous. By theorem 2.8, $H$ has a fixed point $\bar{x} \in X$ and

$$\bar{x} = H\bar{x} = T(H\bar{x}) + S\bar{x}$$

$$= T\bar{x} + S\bar{x}$$

This completes the proof.

Mimicking the proof above and applying theorems 2.2 and 2.14 we establish the following:

**Theorem 3.2.** Let $X$ be a nonempty compact complete star-shaped subset of $E$. Let $T, S : X \to E$ be mappings such that $Tx + Sy \in X$ for all $x, y \in X$. If $T$ is a contraction mapping and $S$ is an affine continuous mapping, then there is a point $\bar{x} \in X$ such that $T\bar{x} + S\bar{x} = \bar{x}$.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


