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A NOTE ON TRANSITIVE POINTS OF SET-VALUED DISCRETE SYSTEMS

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Abstract. This paper is devoted to the study of transitive points for the induced set-valued discrete systems, which is an extension of transitive points for original systems. Some properties of transitive points of set-valued discrete systems are investigated.

Keywords: transitive point; topological transitivity; set-valued discrete system.

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1. Introduction

Throughout this paper a topological dynamical system (abbreviated by TDS) is a pair (X, f), where X is a compact metric space with the metric d and $f : X \to X$ is a continuous map. When X is finite, it is a discrete space and there is no any non-trivial convergence. Hence, we assume that X contains infinitely many points. (X, f) induces a set-valued dynamical system $(\kappa(X), \bar{f})$ with the Hausdorff metric d_H, where $\kappa(X)$ is the space of all non-empty compact subsets of X, and $\bar{f}$ is the induced set-valued map defined by $\bar{f} : \kappa(X) \to \kappa(X)$, $\bar{f}(A) = f(A) = \{f(a) : a \in A\}$, $A \in \kappa(X)$. Let $\mathbb{N}$ denotes the set of all positive integers and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

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Topological transitivity, weak mixing and sensitive dependence on initial conditions (see [2, 5, 10, 12]) are global characteristics of topological dynamical systems. A continuous map $f: X \to X$ is called to be topologically transitive(transitive) if for every pair of non-empty open sets U and V there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$, f is point transitive if there exists a point $x_0 \in X$ such that the orbit of x_0 is dense in X, i.e., $\overline{orb(x_0)} = X$, x_0 is called a transitive point of X. By [5], if X is a compact metric space, then the two definitions are equivalent. (X, f) is topologically weakly mixing (weakly mixing) if for any non-empty open subsets U_1, U_2, V_1 and V_2 of X, there exists a $n \in \mathbb{N}$ such that $f^n(U_1) \cap V_1 \neq \emptyset$ and $f^n(U_2) \cap V_2 \neq \emptyset$. It follows from these definitions that weak mixing implies transitivity.

The properties of topological transitivity, weak mixing and sensitive on initial conditions for set-valued discrete systems were discussed (see [1, 4, 6, 7, 9, 11, 13, 14]). Also, we continue to discuss transitive points of set-valued discrete systems, give some properties of transitive points.

2. **Preliminaries**

A TDS (X, f) is point transitive if there exists a point $x_0 \in X$ with dense orbit, that is, $\overline{orb(x_0)} = X$, where $\overline{orb(x_0)}$ denotes the closure of $orb(x_0)$. Such a point x_0 is called transitive point of (X, f). If X is a compact metric space without isolated points, then topologically transitive and point transitive are equivalent (see [5]). A TDS (X, f) is minimal if $\overline{orb(x, f)} = X$ for every $x \in X$, thai is, every point is transitive point. A point x is called minimal if the subsystem $(\overline{orb(x, f)}, f)$ is minimal.

A point $p \in X$ is periodic for f if $f^k(p) = p$ for some $k \in \mathbb{N}$. An $x \in X$ is asymptotically periodic if there is a periodic point $p \in X$ satisfying $\lim_{n\to\infty} d(f^n(x), f^n(p)) = 0$. $y \in X$ is an $\omega-$ limit point of $x \in X$ if $\liminf_{n\to\infty} d(f^n(x), y) = 0$, i.e., the orbit of x accumulates at y. The set $\omega(x, f)$ of all $\omega-$ limit points of x is the $\omega-$limit set of x.

The distance from a point x to a non-empty set A in X is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Let $\kappa(X)$ be the family of all non-empty compact subsets of X. The Hausdorff metric on $\kappa(X)$ is defined by

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} \text{ for every } A,B \in \kappa(X).$$

It follows from Michael [8] and Engelking [3] that $\kappa(X)$ is a compact metric space. The Vietoris topology τ_υ on $\kappa(X)$ is generated by the base

$$\upsilon(U_1,U_2,\cdots,U_n) = \{F \in \kappa(X) : F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leq n\}$$

where $U_1,U_2,\cdots,U_n$ are open subsets of X.

Let $\bar{f}$ be the induced set-valued map defined by

$$\bar{f} : \kappa(X) \to \kappa(X),\ \bar{f}(F) = f(F), \text{ for every } F \in \kappa(X).$$

Then $\bar{f}$ is well defined. $(\kappa(X),\bar{f})$ is called a set-valued discrete system.

Banks and Peris established the following celebrating result between a given dynamical system and its induced set-valued discrete system.

Theorem 2.1. ([1, 9]) *Let X be a compact space, $\kappa(X)$ be equipped with the Vietoris topology. If $f : X \to X$ is a continuous map, then $\bar{f} : \kappa(X) \to \kappa(X)$ is continuous and (X, f) is weakly mixing $\Longleftrightarrow$ $(\kappa(X),\bar{f})$ is weakly mixing $\Longleftrightarrow$ $(\kappa(X),\bar{f})$ is topologically transitive.*

When the underlying space X is self-dense, the system is transitive if and only if it has transitive points as X as a compact metric space is of second category. Since X is infinite, the induced set-valued discrete system $(\kappa(X),\bar{f})$ is necessarily self-dense. Hence, $(\kappa(X),\bar{f})$ being transitive is equivalent to that has transitive points.

If A is a transitive point of $(\kappa(X),\bar{f})$, then we have the following theorem. The proof of this result is straightforward.

Theorem 2.2. *A is a transitive point of $(\kappa(X),\bar{f})$ if and only if for any finitely many non-empty open sunsets $U_1,U_2,\cdots,U_p$ of X, there exists $m \in \mathbb{N}$ such that $f^m(A) \cap U_i \neq \emptyset$ for $i = 1,2,\cdots,p$, and $f^m(A) \subseteq \bigcup_{i=1}^{p} U_i$.*

3. Main results

Let (X,f) be any compact and infinite dynamical system with metric d. By Theorem 2.1, $(\kappa(X),\bar{f})$ has a transitive point if and only if (X,f) is weakly mixing. The space X of an infinite weak mixing system is self-dense; when X is self-dense, so is $\kappa(X)$.

Proposition 3.1. *If A is a transitive point of $(\kappa(X),\bar{f})$, then for any $m \in \mathbb{Z}_+$, $f^m(A)$ is again a transitive point of $(\kappa(X),\bar{f})$.*

Proof. By the assumption and the paragraphy at the beginning of this section, X is self-dense, and so is $\kappa(X)$. Since A is a transitive point of $(\kappa(X),\bar{f})$, it follows that the orbit $\{\bar{f}^n(A) : n \in \mathbb{Z}_+\}$ is dense in $\kappa(X)$. Hence, every tail orbit $\{\bar{f}^n(A) : n \geq m\}$ remains dense in $\kappa(X)$, $m \in \mathbb{N}$, i.e., $\bar{f}^m(A)$ is again a transitive point of $\bar{f}$ for any $m \in \mathbb{N}$. Noting that $\bar{f}^m(A) = f^m(A)$, further, $f^m(A)$ is a transitive point of $(\kappa(X),\bar{f})$.

Proposition 3.2. *If A is a transitive point of $(\kappa(X),\bar{f})$, then A is an infinite subset of X. Furthermore, for any $m \in \mathbb{N}$, $f^m(A)$ is an infinite subset of X.*

Proof. Since X is infinite, for any $l \in \mathbb{N}$ there exist l pairwise disjoint non-empty open subsets of X, V_i, $1 \leq i \leq l$. By Theorem 2.2, there exists $m \in \mathbb{Z}_+$ such that $f^m(A) \cap V_i \neq \emptyset$ for $i = 1,2,\cdots,l$(and $f^m(A) \subseteq \bigcup_{i=1}^{l} V_i$), implying $card(f^m(A)) \geq l$. Hence, $card(A) \geq l$. As l is arbitrary, A is necessarily an infinite subset of X.

Furthermore, for any $m \in \mathbb{N}$, $f^m(A)$ is again a transitive point of $(\kappa(X),\bar{f})$ by Proposition 3.1, thus an infinite subset of X.

Proposition 3.3. *The set of all transitive points of $(\kappa(X),\bar{f})$ is a dense G_δ set of $\kappa(X)$, i.e., the intersection of countably many dense open subsets.*

Proof. By the assumption, $(\kappa(X),\bar{f})$ is transitive. For a transitive dynamical system, the set of all transitive points is a dense G_δ set [2, 5, 10, 12].

Proposition 3.4. *If A is a transitive point, then $\omega(A,\bar{f}) = \kappa(X)$.*

proof. By Proposition 3.1, X is self-dense, so is $\kappa(X)$. Since $\{\bar{f}^n(A) : n \in \mathbb{Z}_+\}$ is dense in $\kappa(X)$, we have $\omega(A,\bar{f}) = \kappa(X)$.

Proposition 3.5. *If A is a transitive point of $(\kappa(X), \bar{f})$, then A is a proper subset of X.*

Proof. By Theorem 2.1, (X, f) is weakly mixing. Hence, f is surjective. If $A = X$, then for any $m \in \mathbb{N}$, we have $f^m(A) = X$, implying $\bar{f}^m(A) = X$ for every $m \in \mathbb{N}$. Therefore, the orbit of A under $\bar{f}$ would be a single element. This contradicts to the orbit of A under $\bar{f}$ to be dense in $\kappa(X)$.

Theorem 3.1. *If A is a transitive point of $(\kappa(X), \bar{f})$, then for any $m \in \mathbb{N}$, there exist m pairwise disjoint non-empty compact subsets $A_i (1 \leq i \leq m)$ satisfying $A = \bigcup\limits_{i=1}^{m} A_i$.*

Proof. Since X is infinite set, we can choose pairwise disjoint non-empty open subsets $V_i (1 \leq i \leq m)$ of X. As A is a transitive point of $(\kappa(X), \bar{f})$, by Theorem 2.2 there exists $n \in \mathbb{Z}_+$ satisfying $f^n(A) \subseteq \bigcup\limits_{i=1}^{m} V_i$ and $f^n(A) \cap V_i \neq \emptyset$ $(1 \leq i \leq m)$. For $i = 1, 2, \cdots, m$, put $A_i = A \cap f^{-n}(V_i \cap f^n(A))$. Since V_i $(1 \leq i \leq m)$ are pairwise disjoint non-empty open subsets with $f^n(A) \subseteq \bigcup\limits_{i=1}^{m} V_i$, $V_i \cap f^n(A)$ $(1 \leq i \leq m)$ are compact, thus $f^{-n}(V_i \cap f^n(A))$ $(1 \leq i \leq m)$ are compact. Moreover, we can check that the constructed A_i's are pairwise disjoint non-empty compact subsets with $A = \bigcup\limits_{i=1}^{m} A_i$.

Corollary 3.1. *If A is a transitive point of $(\kappa(X), \bar{f})$, then A is a disconnected compact subset of X.*

Theorem 3.2. *If A is a transitive point of $(\kappa(X), \bar{f})$, then for any non-empty open subsets $U_i (1 \leq i \leq m)$, there exists a strictly increasing sequence of non-negative integers n_k satisfying $f^{n_k}(A) \subseteq \bigcup\limits_{i=1}^{m} U_i$ and $f^{n_k}(A) \cap U_i \neq \emptyset$ for $i = 1, 2 \cdots, m$.*

Proof. Since A is a transitive point of $(\kappa(X), \bar{f})$, by Theorem 2.1, (X, f) is weakly mixing. Furthermore, X is self-dense, so is $\kappa(X)$. From Proposition 3.4, $\omega(A, \bar{f}) = \kappa(X)$. As

$$\upsilon(U_1, U_2, \cdots, U_m) = \{F \in \kappa(X) : F \subseteq \bigcup_{i=1}^{m} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leq m\}$$

is a non-empty open subset of $\kappa(X)$, there exists a strictly increasing sequence of non-negative integers n_k satisfying $\bar{f}^{n_k}(A) \in \upsilon(U_1, U_2, \cdots, U_m)$, i.e., $f^{n_k}(A) \subseteq \bigcup\limits_{i=1}^{m} U_i$ and $f^{n_k}(A) \cap U_i \neq \emptyset$ for $i = 1, 2 \cdots, m$.

Corollary 3.2. *If A is a transitive point of $(\kappa(X), \bar{f})$, then for any non-empty subsets U of X, there exists a strictly increasing sequence of non-negative integers n_k satisfying $f^{n_k}(A) \subseteq U$.*

Corollary 3.2 implies that every point of A is a transitive point of (X, f). (Corollary 3.3)

Corollary 3.3. *If A is a transitive point of $(\kappa(X), \bar{f})$, then every $x \in A$ is a transitive point of (X, f). Moreover, $\omega(x, f) = X$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

Authors' Contributions

Lei Liu (the first author) carried out the study of transitive points of set-valued discrete systems. Dongmei Peng (the second author) helped to draft the manuscript. All authors read and approved the final manuscript.

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References

[1] J. Banks, Chaos for induced hyperspace maps, *Chaos, Solitons & Fractals* **25** (2005), 681-685.

[2] L.S. Block and W.A. Coppel, *Dynamics in one dimension.* Lecture Notes in Mathematics, 1513, Springer Verlag, Berlin, 1992.

[3] R. Engelking, *General topology.* PWN, Warszawa, 1977.

[4] A. Fedeli, On chaotic set-valued discrete dynamical systems, *Chaos, Solitons & Fractals.* **23** (2005), 1381-1384.

[5] S. Kolyada and L. Snoha, Some aspects of topological transitivity-a survey, *Grazer Math. Ber, Bericht nr.* **334** (1997), 3-35.

[6] H. Liu, E. shi and G. Liao, Sensitive of set-valued discrete systems, *Nonlinear Anal.* **82** (2009), 985-992.

[7] L. Liu, S. Zhao, Martelli's chaos in inverse limit dynamical systems and hyperspace dynamical systems, Results Math. **63** (2013), 195-207.

[8] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152-182.

[9] A. Peris, Set-valued discrete chaos, *Chaos, Solitons & Fractals.* **26** (2005), 19-23.

[10] C. Robinson, *Dynamical systems: stability, symbolic dynamics, and chaos.* 2nd ed. Boca Raton, FL: CRc Press Inc; 1999.

[11] H. Roman-Flores, A note on transitivity in set-valued discrete systems, *Chaos, Solitons* & *Fractals.* **71** (2003) 99-104.

[12] P. Walters, *An introduction to ergodic theory*. Texts in Math, vol. 79, Springer-Verlag, New York, 1982.

[13] Y. Wang and G. Wei, Characterizing mixing, weak mixing and transitivity of induced hyperspace dynamical systems, *Top. Appl.* **155** (2007), 56-68.

[14] Y. Wang, G. Wei and H. Campbell, Sensitive dependence on intial conditions between dynamical systems and their induced hyperspace dynamical systems, *Top Appl.* **156** (2009), 803-811.