A NOTE ON TRANSITIVE POINTS OF SET-VALUED DISCRETE SYSTEMS

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Abstract. This paper is devoted to the study of transitive points for the induced set-valued discrete systems, which is an extension of transitive points for original systems. Some properties of transitive points of set-valued discrete systems are investigated.

Keywords: transitive point; topological transitivity; set-valued discrete system.

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1. Introduction

Throughout this paper a topological dynamical system (abbreviated by TDS) is a pair $(X, f)$, where $X$ is a compact metric space with the metric $d$ and $f : X \to X$ is a continuous map. When $X$ is finite, it is a discrete space and there is no any non-trivial convergence. Hence, we assume that $X$ contains infinitely many points. $(X, f)$ induces a set-valued dynamical system $(\kappa(X),\tilde{f})$ with the Hausdorff metric $d_H$, where $\kappa(X)$ is the space of all non-empty compact subsets of $X$, and $\tilde{f}$ is the induced set-valued map defined by $\tilde{f} : \kappa(X) \to \kappa(X)$, $\tilde{f}(A) = f(A) = \{f(a) : a \in A\}, A \in \kappa(X)$. Let $\mathbb{N}$ denotes the set of all positive integers and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

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Topological transitivity, weak mixing and sensitive dependence on initial conditions (see [2, 5, 10, 12]) are global characteristics of topological dynamical systems. A continuous map \( f : X \to X \) is called to be topologically transitive (transitive) if for every pair of non-empty open sets \( U \) and \( V \) there exists a positive integer \( n \) such that \( f^n(U) \cap V \neq \emptyset \), \( f \) is point transitive if there exists a point \( x_0 \in X \) such that the orbit of \( x_0 \) is dense in \( X \), i.e., \( \overline{\text{orb}(x_0)} = X \), \( x_0 \) is called a transitive point of \( X \). By [5], if \( X \) is a compact metric space, then the two definitions are equivalent. \((X, f)\) is topologically weakly mixing (weakly mixing) if for any non-empty open subsets \( U_1, U_2, V_1 \) and \( V_2 \) of \( X \), there exists a \( n \in \mathbb{N} \) such that \( f^n(U_1) \cap V_1 \neq \emptyset \) and \( f^n(U_2) \cap V_2 \neq \emptyset \).

It follows from these definitions that weak mixing implies transitivity.

The properties of topological transitivity, weak mixing and sensitive on initial conditions for set-valued discrete systems were discussed (see [1, 4, 6, 7, 9, 11, 13, 14]). Also, we continue to discuss transitive points of set-valued discrete systems, give some properties of transitive points.

2. Preliminaries

A TDS \((X, f)\) is point transitive if there exists a point \( x_0 \in X \) with dense orbit, that is, \( \overline{\text{orb}(x_0)} = X \), where \( \overline{\text{orb}(x_0)} \) denotes the closure of \( \text{orb}(x_0) \). Such a point \( x_0 \) is called transitive point of \((X, f)\). If \( X \) is a compact metric space without isolated points, then topologically transitive and point transitive are equivalent (see [5]). A TDS \((X, f)\) is minimal if \( \overline{\text{orb}(x, f)} = X \) for every \( x \in X \), that is, every point is transitive point. A point \( x \) is called minimal if the subsystem \( \overline{\text{orb}(x, f)} \) is minimal.

A point \( p \in X \) is periodic for \( f \) if \( f^k(p) = p \) for some \( k \in \mathbb{N} \). An \( x \in X \) is asymptotically periodic if there is a periodic point \( p \in X \) satisfying \( \lim_{n \to \infty} d(f^n(x), f^n(p)) = 0 \). \( y \in X \) is an \( \omega \)–limit point of \( x \in X \) if \( \lim \inf_{n \to \infty} d(f^n(x), y) = 0 \), i.e., the orbit of \( x \) accumulates at \( y \). The set \( \omega(x, f) \) of all \( \omega \)–limit points of \( x \) is the \( \omega \)–limit set of \( x \).

The distance from a point \( x \) to a non-empty set \( A \) in \( X \) is defined by

\[
d(x, A) = \inf_{a \in A} d(x, a).
\]
Let \( \kappa(X) \) be the family of all non-empty compact subsets of \( X \). The Hausdorff metric on \( \kappa(X) \) is defined by

\[
d_H(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \} \quad \text{for every} \quad A, B \in \kappa(X).
\]

It follows from Michael [8] and Engelking [3] that \( \kappa(X) \) is a compact metric space. The Vietoris topology \( \tau_\nu \) on \( \kappa(X) \) is generated by the base

\[
\nu(U_1, U_2, \cdots, U_n) = \{ F \in \kappa(X) : F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leq n \}
\]

where \( U_1, U_2, \cdots, U_n \) are open subsets of \( X \).

Let \( \bar{f} \) be the induced set-valued map defined by

\[
\bar{f} : \kappa(X) \to \kappa(X), \quad \bar{f}(F) = f(F), \quad \text{for every} \quad F \in \kappa(X).
\]

Then \( \bar{f} \) is well defined. \((\kappa(X), \bar{f})\) is called a set-valued discrete system.

Banks and Peris established the following celebrating result between a given dynamical system and its induced set-valued discrete system.

**Theorem 2.1.** ([1, 9]) Let \( X \) be a compact space, \( \kappa(X) \) be equipped with the Vietoris topology. If \( f : X \to X \) is a continuous map, then \( \bar{f} : \kappa(X) \to \kappa(X) \) is continuous and \((X, f)\) is weakly mixing \( \iff \) \((\kappa(X), \bar{f})\) is weakly mixing \( \iff \) \((\kappa(X), \bar{f})\) is topologically transitive.

When the underlying space \( X \) is self-dense, the system is transitive if and only if it has transitive points as \( X \) as a compact metric space is of second category. Since \( X \) is infinite, the induced set-valued discrete system \((\kappa(X), \bar{f})\) is necessarily self-dense. Hence, \((\kappa(X), \bar{f})\) being transitive is equivalent to that has transitive points.

If \( A \) is a transitive point of \((\kappa(X), \bar{f})\), then we have the following theorem. The proof of this result is straightforward.

**Theorem 2.2.** A is a transitive point of \((\kappa(X), \bar{f})\) if and only if for any finitely many non-empty open sunsets \( U_1, U_2, \cdots, U_p \) of \( X \), there exists \( m \in \mathbb{N} \) such that \( f^m(A) \cap U_i \neq \emptyset \) for \( i = 1, 2, \cdots, p \), and \( f^m(A) \subseteq \bigcup_{i=1}^{p} U_i \).
3. Main results

Let \((X,f)\) be any compact and infinite dynamical system with metric \(d\). By Theorem 2.1, \((\kappa(X),\tilde{f})\) has a transitive point if and only if \((X,f)\) is weakly mixing. The space \(X\) of an infinite weak mixing system is self-dense; when \(X\) is self-dense, so is \(\kappa(X)\).

**Proposition 3.1.** If \(A\) is a transitive point of \((\kappa(X),\tilde{f})\), then for any \(m \in \mathbb{Z}_+\), \(f^m(A)\) is again a transitive point of \((\kappa(X),\tilde{f})\).

**Proof.** By the assumption and the paragraphy at the beginning of this section, \(X\) is self-dense, and so is \(\kappa(X)\). Since \(A\) is a transitive point of \((\kappa(X),\tilde{f})\), it follows that the orbit \(\{\tilde{f}^n(A) : n \in \mathbb{Z}_+\}\) is dense in \(\kappa(X)\). Hence, every tail orbit \(\{\tilde{f}^n(A) : n \geq m\}\) remains dense in \(\kappa(X)\), \(m \in \mathbb{N}\), i.e., \(\tilde{f}^m(A)\) is again a transitive point of \(\tilde{f}\) for any \(m \in \mathbb{N}\). Noting that \(\tilde{f}^m(A) = f^m(A)\), further, \(f^m(A)\) is a transitive point of \((\kappa(X),\tilde{f})\).

**Proposition 3.2.** If \(A\) is a transitive point of \((\kappa(X),\tilde{f})\), then \(A\) is an infinite subset of \(X\). Furthermore, for any \(m \in \mathbb{N}\), \(f^m(A)\) is an infinite subset of \(X\).

**Proof.** Since \(X\) is infinite, for any \(l \in \mathbb{N}\) there exist \(l\) pairwise disjoint non-empty open subsets of \(X\), \(V_i, 1 \leq i \leq l\). By Theorem 2.2, there exists \(m \in \mathbb{Z}_+\) such that \(f^m(A) \cap V_i \neq \emptyset\) for \(i = 1, 2, \ldots, l\) (and \(f^m(A) \subseteq \bigcup_{i=1}^l V_i\)), implying \(\text{card}(f^m(A)) \geq l\). Hence, \(\text{card}(A) \geq l\). As \(l\) is arbitrary, \(A\) is necessarily an infinite subset of \(X\).

Furthermore, for any \(m \in \mathbb{N}\), \(f^m(A)\) is again a transitive point of \((\kappa(X),\tilde{f})\) by Proposition 3.1, thus an infinite subset of \(X\).

**Proposition 3.3.** The set of all transitive points of \((\kappa(X),\tilde{f})\) is a dense \(G_\delta\) set of \(\kappa(X)\), i.e., the intersection of countably many dense open subsets.

**Proof.** By the assumption, \((\kappa(X),\tilde{f})\) is transitive. For a transitive dynamical system, the set of all transitive points is a dense \(G_\delta\) set [2, 5, 10, 12].

**Proposition 3.4.** If \(A\) is a transitive point, then \(\omega(A,\tilde{f}) = \kappa(X)\).

**Proof.** By Proposition 3.1, \(X\) is self-dense, so is \(\kappa(X)\). Since \(\{\tilde{f}^n(A) : n \in \mathbb{Z}_+\}\) is dense in \(\kappa(X)\), we have \(\omega(A,\tilde{f}) = \kappa(X)\).
**Proposition 3.5.** If $A$ is a transitive point of $(\kappa(X), \bar{f})$, then $A$ is a proper subset of $X$.

**Proof.** By Theorem 2.1, $(X, f)$ is weakly mixing. Hence, $f$ is surjective. If $A = X$, then for any $m \in \mathbb{N}$, we have $f^m(A) = X$, implying $\bar{f}^m(A) = X$ for every $m \in \mathbb{N}$. Therefore, the orbit of $A$ under $\bar{f}$ would be a single element. This contradicts to the orbit of $A$ under $\bar{f}$ to be dense in $\kappa(X)$.

**Theorem 3.1.** If $A$ is a transitive point of $(\kappa(X), \bar{f})$, then for any $m \in \mathbb{N}$, there exist $m$ pairwise disjoint non-empty compact subsets $A_i (1 \leq i \leq m)$ satisfying $A = \bigcup_{i=1}^{m} A_i$.

**Proof.** Since $X$ is infinite set, we can choose pairwise disjoint non-empty open subsets $V_i (1 \leq i \leq m)$ of $X$. As $A$ is a transitive point of $(\kappa(X), \bar{f})$, by Theorem 2.2 there exists $n \in \mathbb{Z}_+$ satisfying $f^n(A) \subseteq \bigcup_{i=1}^{m} V_i$ and $f^n(A) \cap V_i \neq \emptyset (1 \leq i \leq m)$. For $i = 1, 2, \cdots, m$, put $A_i = A \cap f^{-n}(V_i \cap f^n(A))$.

Since $V_i (1 \leq i \leq m)$ are pairwise disjoint non-empty open subsets with $f^n(A) \subseteq \bigcup_{i=1}^{m} V_i$, $V_i \cap f^n(A)$ ($1 \leq i \leq m$) are compact, thus $f^{-n}(V_i \cap f^n(A)) (1 \leq i \leq m)$ are compact. Moreover, we can check that the constructed $A_i$’s are pairwise disjoint non-empty compact subsets with $A = \bigcup_{i=1}^{m} A_i$.

**Corollary 3.1.** If $A$ is a transitive point of $(\kappa(X), \bar{f})$, then $A$ is a disconnected compact subset of $X$.

**Theorem 3.2.** If $A$ is a transitive point of $(\kappa(X), \bar{f})$, then for any non-empty open subsets $U_i (1 \leq i \leq m)$, there exists a strictly increasing sequence of non-negative integers $n_k$ satisfying $f^{n_k}(A) \subseteq \bigcup_{i=1}^{m} U_i$ and $f^{n_k}(A) \cap U_i \neq \emptyset$ for $i = 1, 2, \cdots, m$.

**Proof.** Since $A$ is a transitive point of $(\kappa(X), \bar{f})$, by Theorem 2.1, $(X, f)$ is weakly mixing. Furthermore, $X$ is self-dense, so is $\kappa(X)$. From Proposition 3.4, $\omega(A, \bar{f}) = \kappa(X)$. As

$$\nu(U_1, U_2, \cdots, U_m) = \{F \in \kappa(X) : F \subseteq \bigcup_{i=1}^{m} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leq m\}$$

is a non-empty open subset of $\kappa(X)$, there exists a strictly increasing sequence of non-negative integers $n_k$ satisfying $\bar{f}^{n_k}(A) \in \nu(U_1, U_2, \cdots, U_m)$, i.e., $f^{n_k}(A) \subseteq \bigcup_{i=1}^{m} U_i$ and $f^{n_k}(A) \cap U_i \neq \emptyset$ for $i = 1, 2, \cdots, m$.

**Corollary 3.2.** If $A$ is a transitive point of $(\kappa(X), \bar{f})$, then for any non-empty subsets $U$ of $X$, there exists a strictly increasing sequence of non-negative integers $n_k$ satisfying $f^{n_k}(A) \subseteq U$. 
Corollary 3.2 implies that every point of $A$ is a transitive point of $(X,f)$. (Corollary 3.3)

**Corollary 3.3.** If $A$ is a transitive point of $(\kappa(X),\bar{f})$, then every $x \in A$ is a transitive point of $(X,f)$. Moreover, $\omega(x,f) = X$.

**Conflict of Interests**
The authors declare that there is no conflict of interests.

**Authors’ Contributions**
Lei Liu (the first author) carried out the study of transitive points of set-valued discrete systems. Dongmei Peng (the second author) helped to draft the manuscript. All authors read and approved the final manuscript.

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