COMMON FIXED POINTS FOR WEAK CONTRACTION OCCASIONALLY WEAKLY BIASED MAPPINGS

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Abstract. We discuss some common fixed point theorems for weakly contractive occasionally weakly biased mappings on metric spaces with illustrative examples.

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1. Introduction and Preliminaries

Let \((X, d)\) be a metric space. A mapping \(f : X \rightarrow X\), is called contraction if for each \(x, y \in X\), there exists a constant \(k \in [0, 1)\) such that

\[d(fx, fy) \leq kd(x, y)\]  

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Alber and Guerre-Delabriere[2] defined the concept of weakly contractive mapping on Hilbert spaces and proved the existence of fixed points. Rhoades [11] showed that most results of Alber and Guerre-Delabriere[2] are still true for any Banach space. Note that in Alber and Guerre-Delabriere[2], $\phi$ is assumed with an additional condition $\lim_{t \to \infty} \phi(t) = \infty$. However, Rhoades [11] obtained the result without using this additional condition. Following Rhoades [11], a mapping $f : (X,d) \to (X,d)$ is called a weakly contractive, if for each $x, y \in X$

\begin{equation}
\tag{1.2}
d(fx, fy) \leq d(x, y) - \phi(d(x, y))
\end{equation}

where $\phi : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing and positive on $(0, \infty)$ with $\phi(0) = 0$.

Let $f, g : (X, d) \to (X, d)$ be two mappings, then the mapping $f$ is called $g$-weakly contractive[15] if for each $x, y \in X$

\begin{equation}
\tag{1.3}
d(fx, fy) \leq d(gx, gy) - \phi(d(gx, gy)),
\end{equation}

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function from right such that $\phi$ is positive on $(0, \infty)$ and $\phi(0) = 0$. If $g = I$, an identical operator, then $f$ is reduced to weak contraction.

Further, if $g = I$ and $\phi(t) = (1 - k)t$ where $k \in (0, 1)$, then $g$-weakly contractive is reduced to inequality(1.1). If $\psi(t) = t - \phi(t)$ and $g = I$, then $\psi(t)$ is upper semi-continuous from right and inequality (1.3) reduces into contractive types of Boyd and Wong [4]. Thus

\begin{equation}
\tag{1.4}
d(fx, fy) \leq \psi(d(x, y))
\end{equation}

Further more, if $k(t) = 1 - \frac{\phi(t)}{t}$ for $t > 0$ and $k(0) = 0$ together with $g = I$, then inequality(1.3) is closely related to Reich type[10]. In fact, the classes of weak contractive are closely related to Boyd and Wong [4], and Reich[10] types (see also [16],[15]).

We denote $C(f, g) = \{x \in X : fx = gx\}$ and $F(f, g) = \{x \in X : fx = gx = x\}$.

In the sequel we need the following definitions.

**Definition 1.1**[13]. Mappings $f$ and $g$ are called weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$, for all $x \in X$. 
Definition 1.2[1](also see Sastry and Murthy[12]). Mappings \( f \) and \( g \) are called said to satisfy property (E.A) if there exists a sequences \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \).

Definition 1.3[9]. Mappings \( f \) and \( g \) are called weakly compatible if \( fgx = gf x \) for all \( x \in C(f, g) \).

Definition 1.4[8]. Mappings \( f \) and \( g \) are called weakly \( g \)-biased if \( d(gfx, gx) \leq d(fgx, fx) \) for all \( x \in C(f, g) \).

If the role of \( f \) and \( g \) are interchanged in above definition, then the mappings are called weakly \( f \)-biased. Note that weakly compatible mappings implies weakly biased mappings (i.e. both \( f \)- and \( g \)-biased) but the converse is not true in general[14].

Definition 1.5[3]. Mappings \( f \) and \( g \) are called occasionally weakly compatible(owc) if \( fgx = gf x \) for some \( x \in C(f, g) \).

From above definitions, one may agree that weakly compatible mappings pair implies owc but the converse may not be true in general ( also see [3]).

Definition 1.6[5]. Mappings \( f \) and \( g \) are called occasionally weakly \( g \)-biased if \( d(gfx, gx) \leq d(fgx, fx) \) for some \( x \in C(f, g) \).

If the role of mappings are interchanged, then the mappings pair is called occasionally weakly \( f \)-biased. Further, it may be noted that the notions of owc and weakly \( g \)-biased mappings are occasionally weakly \( g \)-biased but the converse does not hold in general(see [5]).

Example 1.7. Let \( X = [0, 1] \subset \mathbb{R} \) with usual metric \( d \). Define \( f, g : X \to X \) by \( fx = \frac{x}{3} + x, \, gx = \frac{1}{2} \), for \( x < \frac{1}{2}, f\frac{1}{2} = \frac{2}{3} = g\frac{1}{2}, \, fx = 1, \, gx = 1 - x \), for \( x > \frac{1}{2} \). Here, \( C(f, g) = \{ \frac{1}{6}, \frac{1}{2} \} \). Also, we have \( f\frac{1}{6} = \frac{1}{2} = g\frac{1}{6}, f\frac{1}{2} = \frac{2}{3} = g\frac{1}{2} \) and \( fg\frac{1}{6} = \frac{2}{3} = gf\frac{1}{6} \), but \( fg\frac{1}{2} = 1 \neq gf\frac{1}{2} = \frac{1}{3} \). The mappings pair \((f, g)\) is occasionally weakly compatible but not weakly compatible. However, the mappings are weakly biased and hence occasionally weakly biased.

Example 1.8. Let \( X = [0, 1] \subset \mathbb{R} \) with usual metric \( d \). Define \( f, g : X \to X \) by \( fx = 1, \, gx = \frac{1}{2} \), for \( x < \frac{1}{2}, f\frac{1}{2} = 0 = g\frac{1}{2}, \, fx = x, \, gx = 1 - x \), for \( x > \frac{1}{2} \). Here, \( C(f, g) = \{ \frac{1}{2} \} \). Also, we have \( f\frac{1}{2} = 0 = g\frac{1}{2} \) and \( |gf\frac{1}{2} - g\frac{1}{2}| = |\frac{1}{2} - 0| = \frac{1}{2} \leq |fg\frac{1}{2} - f\frac{1}{2}| = |1 - 0| = 1 \). The mappings pair
\((f, g)\) is weakly biased and hence occasionally weakly \(g\)-biased but neither weakly compatible nor owc.

**Example 1.9.** Let \(X = [0, 1] \subset \mathbb{R}\) with usual metric \(d\). Define \(f, g : X \to X\) by \(fx = 2x, gx = 1 - 2x\), for \(x \leq \frac{1}{4}\), \(fx = 1, gx = \frac{1}{2}\), for \(\frac{1}{4} < x \leq \frac{1}{2}\), \(fx = \frac{7}{8}, gx = \frac{1 + 8x}{8}\), for \(\frac{1}{2} < x \leq \frac{3}{4}\), \(fx = \frac{1}{6}, gx = \frac{3}{4}\), for \(\frac{3}{4} < x \leq 1\). Here, \(C(f, g) = \{\frac{1}{4}, \frac{3}{4}\}\). Also \(f\frac{1}{4} = g\frac{1}{4} = \frac{1}{2} = g\frac{3}{4}\) and \(f\frac{3}{4} = g\frac{3}{4} = \frac{7}{8} = g\frac{3}{4}\) implies that

\[
|gf\frac{1}{4} - g\frac{1}{4}| = \frac{1}{4} \leq |fg\frac{1}{4} - f\frac{1}{4}| = \frac{1}{2}
\]

and

\[
|gf\frac{3}{4} - g\frac{3}{4}| = \frac{1}{8} \leq |fg\frac{3}{4} - f\frac{3}{4}| = \frac{17}{24}
\]

Therefore, the pair \((f, g)\) is occasionally weakly \(g\)-biased, but it is neither weakly \(g\)-biased nor weakly compatible (resp. owc)

In this paper, we prove some common fixed point theorems for weak contraction occasionally weakly biased mappings pair on metric spaces.

### 2. Main Results

Song[15] proved the following theorem.

**Theorem 1.1 (Song[15]).** Let \((X, d)\) be a metric space and \(f, g : X \to X\) two self mappings with \(fX \subset gX\). Assume that either \(fX\) or \(gX\) is complete, and \(f\) is \(g\)-weakly contractive mapping, then \(C(f, g) \neq \emptyset\). If in addition, \((f, g)\) is weakly compatible, then \(F(f, g)\) is singleton.

Let \(\varphi : [0, \infty) \to [0, \infty)\) be a lower semi-continuous function with \(\varphi(t) = 0\) if and only it \(t = 0\).

Let \(f\) and \(g\) be two self mappings on a metric space \((X, d)\). We denote

\[
M(x, y) = \max \left\{ d(gx, gy), d(fx, gy), d(fy, gx), \frac{1}{2} [d(fx, gx) + d(fy, gy)] \right\}
\]

and

\[
N(x, y) = \max \left\{ d(gx, fy), d(fx, fy), d(gx, gy), \frac{1}{2} [d(fx, gx) + d(fy, gy)] \right\}
\]
**Theorem 2.2.** Let $f$ and $g$ be two self mappings of a metric space $(X,d)$ satisfying the following inequality

\[(2.3) \quad d(fx, fy) \leq M(x, y) - \varphi(M(x, y)), \forall x, y \in X\]

If $(f, g)$ satisfies property-(E.A) and $gX$ is closed in $X$, then $C(f, g) \neq \phi$. Further, if $(f, g)$ is occasionally weakly $g$-biased, then $F(f, g)$ is singleton.

**Proof.** Since $f$ and $g$ satisfy property (E.A), there exists a sequence in $X$ such that $fx_n, gx_n \to t$ for some $t \in X$. As $gX$ is closed and $t \in X$, there exists $u \in X$ such that $t = gu$. We claim that $fu = gu$. By (2.1) and (2.3), we obtain

\[d(fx_n, fu) \leq M(x_n, u) - \varphi(M(x_n, u))\]

and

\[M(x_n, u) = \max \left\{ d(gx_n, gu), d(fx_n, gu), d(fu, gx_n), \frac{1}{2}[d(fx_n, gx_n) + d(fu, gu)] \right\} \]

On letting $n \to \infty$, we obtain

\[d(gu, fu) \leq \max \left\{ 0, 0, d(fu, gu), \frac{1}{2}d(fu, gu) \right\} - \varphi\left( \max \left\{ 0, 0, d(fu, gu), \frac{1}{2}d(fu, gu) \right\} \right) = d(fu, gu) - \varphi(d(fu, gu))\]

which gives $fu = gu$. Therefore, $C(f, g) \neq \phi$. Since $(f, g)$ is occasionally weakly $g$-biased mappings, then $fu = gu$ for some $u \in C(f, g)$ and

\[(2.4) \quad d(gfu, gu) \leq d(fgu, fu).\]
Also, \(f u = gu\) yields \(ffu = fg u\) and \(gfu = ggu\). Now we show that \(ffu = fu\), otherwise by (2.1), (2.3) and (2.4), we obtain

\[
d(ffu, fu) \leq M(fu, u) - \varphi \left( M(fu, u) \right)
\]

\[
= \max \left\{ d(gfu, gu), d(ffu, gu), \frac{1}{2} [d(ffu, gfu) + d(fu, gu)] \right\}
\]

\[
- \varphi \left( \max \left\{ d(gfu, gu), d(ffu, gu), \frac{1}{2} [d(ffu, gfu) + d(fu, gu)] \right\} \right)
\]

\[
\leq d(ffu, fu) - \varphi(d(ffu, fu))
\]

which gives \(\varphi(d(ffu, fu)) = 0 \Rightarrow ffu = fu\). By occasionally weakly \(g\)-biased of \(f\) and \(g\), we obtain

\[
d(gfu, gu) \leq d(fgu, fu) = d(ffu, fu) = 0,
\]

which in turn gives \(gfu = fu\). Therefore, \(fu = z\) is a common fixed point of \(f\) and \(g\). For the uniqueness, let \(z \neq z' \in X\) such that \(fz = gz = z\) and \(fz' = gz' = z'\), then by (2.1) and (2.3), we obtain

\[
d(z, z') = d(fz, f z')
\]

\[
\leq M(z, z') - \varphi \left( M(z, z') \right)
\]

\[
= d(z, z') - \varphi(d(z, z'))
\]

which yields \(\varphi(d(z, z')) = 0\) and \(z = z'\). This completes the proof.

The following example illustrate the validity of above theorem.

**Example 2.3.** Let \(X = [0, 1] \subset \mathbb{R}\) with usual metric \(d(x, y) = |x - y|\). Define \(f, g : X \to X\) by \(fx = \frac{x}{2}\), for \(0 \leq x \leq \frac{1}{2}\); \(fx = \frac{1}{4}\), for \(\frac{1}{2} < x < 1\) and \(gx = \frac{1}{2} (1 + x)\), for \(0 \leq x < \frac{1}{2}\); \(g \frac{1}{2} = \frac{1}{2}\), \(gx = \frac{3}{4}\), for \(\frac{1}{2} < x < 1\). Here, \(fX = \left\{ \frac{1}{4}, \frac{1}{2} \right\}\) is not contained in \(gX = \left[ \frac{1}{2}, \frac{3}{4} \right]\), and \(gX\) is closed in \(X\). Mappings \(f\) and \(g\) satisfy property (E.A), to verify this, let \(\{x_n\}\) be a sequence in \(X\), \(x_n > 0, n = 1, 2, 3, \ldots\) such that \(x_n \to 0\) as \(n \to \infty\) then \(fx_n, gx_n \to \frac{1}{2} \in X\). One can also verify that \((f, g)\) satisfies inequality (2.3) for every \(x, y \in X\) taking with \(\varphi(t) = \frac{t}{4}\). Also, \(C(f, g) = \left\{ 0, \frac{1}{2} \right\}\) and \(f0 = \frac{1}{2} = g0\) which implies \(f\) and \(g\) are occasionally weakly \(g\)-biased mappings. Thus, all the conditions of the theorem are satisfied and \(\frac{1}{2}\) is the unique common point.
Corollary 2.4 Let $f$ and $g$ be two self mappings of a metric space $(X, d)$ satisfying the following: for every $x, y \in X$,

$$d(fx, fy) \leq \psi(M(x, y))$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function such that $0 < \psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. If $(f, g)$ satisfies the property (E.A) and $gX$ is closed in $X$, then $C(f, g) \neq \phi$. Further, if $(f, g)$ is occasionally weakly $g$-biased, then $F(f, g)$ is singleton.

Proof. Letting $\varphi(t) = t - \psi(t)$, then $0 < \psi(t) = t - \varphi(t) < t$ for $t > 0$ (by definition of $\psi$) and inequality (2.5) implies that

$$d(fx, fy) \leq M(x, y) - \varphi(M(x, y))$$

Therefore, the result follows from Theorem 2.2.

Corollary 2.5 Let $f$ and $g$ be two self mappings of a metric space $(X, d)$ such that for every $x, y \in X$

$$d(fx, fy) \leq \alpha(M(x, y))M(x, y)$$

where $\alpha : [0, \infty) \to [0, 1)$ is a function. If $(f, g)$ satisfies the property (E.A) and $gX$ is closed in $X$, then $C(f, g) \neq \phi$. Further, if $(f, g)$ is occasionally weakly $g$-biased, then $F(f, g)$ is singleton.

Proof. Setting $\varphi(t) = [1 - \alpha(t)]t$, then equation (2.6) implies that

$$d(fx, fy) \leq M(x, y) - \varphi(M(x, y))$$

The result follows from Theorem 2.2.

Theorem 2.6. Let $f$ and $g$ be two self mappings of a metric space $(X, d)$ satisfying

$$d(fx, gy) \leq N(x, y) - \varphi(N(x, y)), \forall x, y \in X$$

If $(f, g)$ satisfies the property-(E.A) and $fX$ is closed in $X$, then $C(f, g) \neq \phi$. Further, if $(f, g)$ is occasionally weakly $g$-biased, then $F(f, g)$ is singleton.

Proof. Since $f$ and $g$ satisfy property-(E.A), there exists a sequence $\{x_n\}$ in $X$ such that $fx_n, gx_n \to t$ for some $t \in X$. As $fX$ is closed and $t \in X$, there exists $u \in X$ such that $t = fu$. We
claim that $fu = gu$. By (2.2) and (2.7), we obtain
\[ d(f x_n, gu) \leq N(x_n, u) - \varphi(N(x_n, u)) \]
and
\[ N(x_n, u) = \max \left\{ d(g x_n, fu), d(f x_n, fu), d(g x_n, gu), \frac{1}{2}[d(f x_n, g x_n) + d(fu, gu)] \right\} \]
On letting $n \to \infty$, we obtain
\[ d(gu, fu) \leq \max \left\{ 0, 0, d(fu, gu), \frac{1}{2}d(fu, gu) \right\} - \varphi \left( \max \left\{ 0, 0, d(fu, gu), \frac{1}{2}d(fu, gu) \right\} \right) \]
\[ = d(fu, gu) - \varphi(d(fu, gu)) \]
which gives $fu = gu$. Therefore, $C(f, g) \neq \emptyset$.

Since $(f, g)$ is occasionally weakly $g$-biased mappings, then $fu = gu$ for some $u \in C(f, g)$ and
\[(2.8) \quad d(gfu, gu) \leq d(fgu, fu). \]
Also, $fu = gu$ yields $ffu = fgu$ and $gfu = ggu$. Now we show that $ffu = fu$, otherwise by (2.7), (2.2) and (2.8), we obtain
\[ d(ffu, fu) = d(ffu, gu) \]
\[ \leq N(fu, u) - \varphi \left( N(fu, u) \right) \]
\[ = \max \left\{ d(gfu, fu), d(ffu, fu), \frac{1}{2}d(ffu, gfu) \right\} - \varphi \left( \max \left\{ d(gfu, fu), d(ffu, fu), \frac{1}{2}d(ffu, gfu) \right\} \right) \]
\[ \leq \max \{ d(ffu, fu), d(ffu, fu), d(fgu, fu) \} - \varphi \left( \max \{ d(ffu, fu), d(ffu, fu), d(fgu, fu) \} \right) \]
\[ = d(ffu, fu) - \varphi(d(ffu, fu)) \]
which gives $\varphi(d(ffu, fu)) = 0 \Rightarrow ffu = fu$.

By occasionally weakly $g$-biased of $f$ and $g$, we obtain
which in turn gives $gfu = fu$. Therefore, $fu = z$ is a common fixed point of $f$ and $g$. For the uniqueness, let $z \neq z' \in X$ such that $fz = gz = z$ and $fz' = gz' = z'$, then by (2.2) and (2.7), we obtain

$$d(z, z') = d(fz, gz') \leq N(z, z') - \varphi(N(z, z')) = d(z, z') - \varphi(d(z, z'))$$

which yields $\varphi(d(z, z')) = 0$ and $z = z'$. This completes the proof.

The validity of above theorem is illustrated by the following example.

**Example 2.7.** Let $X = [0, 1) \subset \mathbb{R}$ with usual metric $d$. Define $f, g : X \rightarrow X$ by $fx = \frac{1}{2}$, for $0 \leq x \leq \frac{1}{2}$, $fx = 0$, for $x > \frac{1}{2}$ and $gx = \frac{1}{2}(1 + x)$, for $0 \leq x < \frac{1}{2}$, $g\frac{1}{2} = \frac{1}{2}$, $gx = \frac{3}{2}$, for $x > \frac{1}{2}$. Here, $fX = \{0, \frac{1}{2}\}$ is not contained in $gX = [\frac{1}{2}, \frac{3}{2})$, and $fX$ is closed in $X$. Mappings $f$ and $g$ satisfy property (E.A), to verify this, let $\{x_n\}$ be a sequence in $X$, $x_n > 0$, $n = 1, 2, 3, \ldots$ such that $x_n \rightarrow 0$ as $n \rightarrow \infty$ then $fx_n, gx_n \rightarrow \frac{1}{2} \in X$. One can also verify that $f$ and $g$ satisfy inequality (2.7) for every $x, y \in X$ taking with $\varphi(t) = \frac{t}{2}$. Also, $C(f, g) = \{0, \frac{1}{2}\}$ and $f0 = \frac{1}{2} = g0$ implies $f$ and $g$ are occasionally weakly $g$-biased mappings. Thus, all the conditions of the theorem are satisfied and $f0 = \frac{1}{2}$ is the unique common point.

**Corollary 2.8.** Let $f$ and $g$ be two self mappings of a metric space $(X, d)$ satisfying

$$d(fx, gy) \leq N(x, y) - \varphi(N(x, y)), \forall x, y \in X$$

If $(f, g)$ satisfies the property-(E.A) and $fX$ is closed in $X$, then $C(f, g) \neq \phi$. Further, if $(f, g)$ is occasionally weakly compatible, then $F(f, g)$ is singleton.

**References**


