UNIQUE COMMON FIXED POINTS FOR PAIRS OF MULTI-VALUED MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we obtain a unique common fixed point theorems for pairs of multi-valued non-self mappings on a partial Hausdorff metric space without using any continuity or commutativity of the mappings. In doing so, we generalize a theorem by Rao and Rao.

Keywords: partial Hausdorff metric; multi-valued mapping; common fixed points; partial metric space; non-self mapping.

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1. Introduction


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Using the procedure described by Assad and Kirk [2], we extend the theorem by Rao and Rao [10] to apply to a pair of non-self multi-valued mappings.

2. Preliminaries

We now introduce preliminaries which will be of use in this paper.

Definition 2.1 [8] A partial metric on a non-empty set $X$ is a mapping $p : X \times X \to [0, +\infty)$, such that for all $x, y, z \in X$.

$P0$: $0 \leq p(x, x) \leq p(x, y)$,

$P1$: $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

$P2$: $p(x, y) = p(y, x)$ and

$P3$: $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair $(X, p)$ is said to be a partial metric space.

From Definition 2.1, we deduce the following:

$$p(x, y) = 0 \Rightarrow x = y.$$  \hfill (2.1)

Proof. If $p(x, y) = 0$, then $p(x, x) = 0$ because $0 \leq p(x, x) \leq p(x, y)$ from $P0$. Similarly, $p(x, y) = 0$ implies $p(y, y) = 0$ because $0 \leq p(y, y) \leq p(x, y)$. Hence $p(x, y) = 0$ implies $p(x, x) = p(x, y) = p(y, y) = 0$. From $P1$ this means that $x = y$.

From $P3$, we infer that

$$p(x, y) \leq p(x, z) + p(z, y).$$  \hfill (2.2)

Example 2.1 Let $X = \mathbb{R}^+$ and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base being the family of open balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 2.2 [8] Let $(X, p)$ be a partial metric space and $\{x_n\}$ be a sequence in $X$. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$.

(ii) $\{x_n\}$ is called a Cauchy sequence if only if there exists (and is finite) $\lim_{n, m \to +\infty} p(x_n, x_m)$. 

(iii) A partial metric space \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, with respect to \(\tau_p\), to a point \(x \in X\) such that

\[
p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m).
\]

**Lemma 2.1** [8] If \(p\) is a partial metric on \(X\), then the mapping \(p^s : X \times X \to [0, +\infty)\) given by

\[
p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\]

(2.3)

defines a metric on \(X\).

In this paper, we denote \(p^s\) as the metric derived from the partial metric \(p\).

**Lemma 2.2** [8]

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, p^s)\).

(b) \((X, p)\) is complete if and only if \((X, p^s)\) is complete. Furthermore \(\lim_{n \to +\infty} p(x_n, x) = 0\) if and only if

\[
p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0.
\]

It is easy to see that every closed subset of a complete partial metric space is complete [6].

We define a metrically convex metric space.

**Definition 2.3** [2] A complete metric space \((X, d)\) is said to be (metrically) convex if \(X\) has the property that for each \(x, y \in X\) with \(x \neq y\) there exists \(z \in X, x \neq z \neq y\), such that

\[
d(x, z) + d(z, y) = d(x, y).
\]

If \((X, d)\) is a metrically convex metric space, and \(x, y \in X\), we term

\[
\text{seg}[x, y] := \{z \in X : d(x, y) = d(x, z) + d(z, y)\}.
\]

(2.4)

We get the following lemma from Assad and Kirk [2].

**Lemma 2.3** [2] Let \(C\) be a closed subset of the complete and convex metric space \(X\). If \(x \in C\) and \(y \notin C\), then there exists a point \(z \in \partial C\) (the boundary of \(C\)) such that

\[
d(x, z) + d(z, y) = d(x, y).
\]

Using (2.4), we can rephrase Lemma 2.3 as follows:
Lemma 2.4 Let $C$ be a closed subset of the complete and convex metric space $X$. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of $C$) such that $z \in \text{seg}[x, y]$.

Now, we introduce the metrically convex partial metric space.

Definition 2.4 A partial metric space $(X, p)$ is said to be metrically convex if the corresponding metric space $(X, p^s)$ is metrically convex in the sense of Lemma 2.1, where

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \text{ for all } x, y \in X.$$ 

As an example, the partial metric space $(\mathbb{R}^+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$ is metrically convex because $(X, p^s)$ where $p^s(x, y) = |x - y|$ is the metric derived from the partial metric $p$, is metrically convex.

Lemma 2.5 Let $(X, p)$ be a metrically convex partial metric space. Let $x, y \in X$. If $z \in \text{seg}[x, y]$ then:

(i) $p(x, y) = p(x, z) - p(z, z) + p(z, y),$

(ii) $p(x, y) \geq p(x, z)$.

Proof. Applying (2.3) to Definition 2.3, if $z \in \text{seg}[x, y]$, then we have

$$p^s(x, y) = p^s(x, z) + p^s(z, y)$$

$$\Rightarrow 2p(x, y) - p(x, x) - p(y, y) = 2p(x, z) - p(x, x) - p(z, z)$$

$$+ 2p(z, y) - p(z, z) - p(y, y)$$

$$\Rightarrow p(x, y) = p(x, z) - p(z, z) + p(z, y).$$

As $(- p(z, z) + p(z, y)) \geq 0$, from P2 of Definition 2.1, we have $p(x, y) \geq p(x, z)$.

This completes the proof.

Lemma 2.6 Let $C$ be a non-empty subset of a metrically convex partial metric space $(X, p)$ which is closed in $(X, p^s)$. If $x \in C$ and $y \in X \setminus C$, then there exists a point $z \in \partial C$ (the boundary of $C$ with respect to $(X, p^s)$) such that

$$p(x, y) + p(z, z) = p(x, z) + p(z, y).$$

Proof. From Definition 2.4, if the partial metric space $(X, p)$ is metrically convex, then $(X, p^s)$ is metrically convex. From Lemma 2.3, this means that if $x \in C$ and $y \in X \setminus C$ then there exists $z$
in $\partial C$, (the boundary of $C$), such that $s(x, y) = p^s(x, z) + p^s(z, y)$. Using (2.3), this means

$$p^s(x, y) = p^s(x, z) + p^s(z, y)$$

$$\Rightarrow 2p(x, y) - p(x, x) - p(y, y) = 2p(x, z) - p(x, x) - p(z, z)$$

$$+ 2p(z, y) - p(z, z) - p(y, y)$$

$$\Rightarrow 2p(x, y) = 2p(x, z) + 2p(z, y) - 2p(z, z)$$

$$\Rightarrow p(x, y) + p(z, z) = p(x, z) + p(z, y)$$

$$\Rightarrow p(x, z) + p(z, y) = p(x, y) + p(z, z).$$

This completes the proof.

### 3. The Partial Hausdorff Metric

Now, we describe the partial Hausdorff metric.

Let $CB^p$ be a family of all non-empty, closed and bounded subsets of a partial metric space $(X, p)$, induced by the partial metric $p$. The set $A$ is said to be a bounded subset in $(X, p)$ if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$.

**Definition 3.1** [3] For all $A, B \in CB^p(X)$ and $x \in X$, we define

(i) $p(x, A) = \inf \{p(x, a) : a \in A\}$,

(ii) $\delta_p(A, B) = \sup \{p(a, B) : a \in A\}$,

(iii) $\delta_p(B, A) = \sup \{p(b, A) : b \in B\}$,

(iv) $H_p(A, B) = \max \{\delta_p(A, B), \delta_p(B, A)\}$.

The mapping $H_p : CB^p \times CB^p \to [0, +\infty)$ is called the partial Hausdorff metric.

**Remark 3.1** [3] Let $(X, p)$ be a partial metric space and $A$ any non-empty set in $(X, p)$, then $a \in \bar{A}$ if and only if $p(a, A) = p(a, a)$, where $\bar{A}$ denotes the closure of $A$ with respect to the partial metric $p$.

We now state some properties of mappings $\delta_p$ and $H_p$.

**Lemma 3.1** [3] Let $(X, p)$ be a partial metric space. For any $A, B \in CB^p(X)$ we have
(i) \( \delta_p(A,A) = \sup\{p(a,a) : a \in A\} \);
(ii) \( \delta_p(A,A) \leq \delta(A,B) \);
(iii) \( \delta_p(A,B) = 0 \) implies that \( A \subseteq B \);
(h1) \( H_p(A,A) \leq H_p(A,B) \);
(h2) \( H_p(A,B) = H_p(B,A) \);
(h3) \( H_p(A,B) = 0 \) implies \( A = B \).

We will utilize the following lemma in our proofs.

**Lemma 3.2** [3] Let \((X, p)\) be a partial metric space, \( A, B \in CB^p(X) \) and \( K > 1 \). For any \( a \in A \), there exists \( b = b(a) \in B \) such that

\[
p(a, b) \leq KH_p(A, B).
\]

The following definitions will be used in the course of our proofs.

Let \( T : C \to X \) be a multi-valued mapping, where \( C \subseteq X \). We say that \( T \) is a **self mapping** if \( C = X \), otherwise \( T \) is called a **non-self mapping**. If there is an element \( x \in C \) such that \( x \in Tx \), we say that \( x \) is a **fixed point** of \( T \) in \( X \).

Suppose we have two multi-valued mappings \( S, T : C \to X \), with \( C \subseteq X \). If there is an element \( x \in C \) such that \( x \in (Sx \cap Tx) \) then we call \( x \) a **common fixed point** of \( S \) and \( T \) in \( X \).

We now prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [2], as it is necessary for our work.

**Lemma 3.3** Consider a sequence \( \{w_n\}_{n \in \mathbb{N}} \in \mathbb{R}_+ \) such that, for all \( n \geq 2 \) we have

\[
w_n \leq k \max\{w_{n-2}, w_{n-1}\}, k \in (0, 1),
\]

then

\[
w_n \leq k^{n/2}k^{-1/2} \max\{w_0, w_1\}.
\]

**Proof.** We prove the lemma by the induction. First we show that Lemma 3.3 holds for \( n = 2 \).

We note that \( k \in (0, 1) \) implies \( k < k^{1/2} \). Hence if \( n = 2 \), then (3.1) leads to

\[
w_2 \leq k \max\{w_0, w_1\} \leq k^{1/2} \max\{w_0, w_1\} = k^{2/2}k^{-1/2} \max\{w_0, w_1\}.
\]
We then show that the lemma holds for $n = 3$. If $n = 3$, then (3.1) leads to $w_3 \leq k \max\{w_1, w_2\}$.

If $w_1 \geq w_2$, then we get

$$w_3 \leq k \max\{w_1, w_2\}$$

$$\Rightarrow w_3 \leq kw_1$$

$$\leq k \max\{w_0, w_1\}$$

$$= k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}.$$  

If however $w_1 < w_2$, we get

$$w_3 \leq k \max\{w_1, w_2\}$$

$$\Rightarrow w_3 \leq kw_2$$

$$\Rightarrow w_3 \leq k \times k^{2/2}k^{-1/2} \max\{w_0, w_1\}, \text{ from (3.3)}$$

$$\leq k^{3/2} \max\{w_0, w_1\}$$

$$\leq k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}, \text{ because } k^{-1/2} \geq 1.$$  

We now show that, if Lemma 3.3 holds for $1 \leq n \leq j$ where $j \geq 2$, then it must be hold for $j + 1$. Hence we have from (3.1)

$$w_{j+1} \leq k \max\{w_{j-1}, w_j\}. \quad (3.4)$$

We consider two cases.

**Case (i):** Suppose $w_{j-1} \leq w_j$. Then (3.4) becomes

$$w_{j+1} \leq kw_j$$

$$\leq k \cdot k^{j/2}k^{-1/2} \max\{w_0, w_1\} \text{ from (3.2)} \quad (3.5)$$

$$= k^{(j+2)/2}k^{-1/2} \max\{w_0, w_1\}.$$  

**Case (ii):** Suppose $w_{j-1} > w_j$. Then (3.4) becomes

$$w_{j+2} \leq kw_{j-1}$$

$$\leq k \cdot k^{(j-1)/2}k^{-1/2} \max\{w_0, w_1\} \text{ from (3.2)} \quad (3.6)$$

$$= k^{(j+1)/2}k^{-1/2} \max\{w_0, w_1\}.$$
We note that for $j \geq 2$ and $k \in (0, 1)$ we have $k^{(j+1)/2} > k^{(j+2)/2}$. Hence (3.5) and (3.6) imply that

$$w_{j+1} \leq k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}.$$  

This completes the proof.

Aydi et al. proved the following theorem.

**Theorem 3.1.** [3] Let $(X, p)$ be a complete partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$ we have

$$H_p(Tx, Ty) \leq kp(x, y),$$  

where $k \in (0, 1)$, then $T$ has a fixed point.

### 4. Main Results

We start by proving an extension of Theorem 3.1 which will then be used to establish Theorem 4.3.

**Theorem 4.1** Let $(X, p)$ be a complete metrically convex partial metric space and $C$ a non-empty closed subset of $X$, the closure being with respect to $(X, p^s)$. Let $\partial C$, the boundary of $C$ with respect to $(X, p^s)$, be non-empty. Let $S, T : C \to CB^p(X)$ be multi-valued mappings such that for all $x, y \in C$ we have

$$H_p(Tx, Ty) \leq kp(x, y),$$  

where $k \in (0, 1)$. Furthermore, let $x \in \partial C$ imply $Tx \subset C$ and $Sx \subset C$. Then there exists a common fixed point $x^* \in C$ and $p(x^*, x^*) = 0$.

**Proof.** We commence with an arbitrary $x_0 \in \partial C$. This implies from the assumption that we can choose $x_1 \in Tx_0 \subset C$. By Lemma 3.2 with $K = \frac{1}{\sqrt{k}}$, there exists $y_2 \in Sx_1$ such that

$$p(x_1, y_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Sx_1).$$  

If $y_2 \in C$, we set $x_2 = y_2$. Thus (4.2) becomes

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Sx_1).$$  

If $y_2 \in C$, we set $x_2 = y_2$. Thus (4.2) becomes

$$p(x_1, x_2) \leq \frac{1}{\sqrt{k}} H_p(Tx_0, Sx_1).$$
By (4.1), we have $H_p(Tx_0, Sx_1) \leq kp(x_0, x_1)$. This means
\[
p(x_1, x_2) \leq \sqrt{k}p(x_0, x_1).
\]

If $y_2 \notin C$, then by Lemma 2.4, there is $x_2 \in \partial C$ such that $x_2 \in \text{seg}[x_1, y_2]$. Using Lemma 2.5 (ii), we get
\[
p(x_1, x_2) \leq p(x_1, y_2) = p(y_1, y_2), \text{ because } x_1 = y_1
\]
\[\leq \frac{1}{\sqrt{k}}H_p(Tx_0, Sx_1)
\]
\[\leq \sqrt{k}p(x_0, x_1).
\]

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way, using $K = \frac{1}{\sqrt{k}} > 1$:

(4.i) $x_0 \in \partial C, y_1 \in Tx_0 \subset C$.
(4.ii) For all $n \geq 1, y_{2n} \in Sx_{2n-1}, y_{2n+1} \in Tx_{2n}$.
(4.iii) Here we apply Lemma 3.2. For all $n \geq 1$, we choose $y_{2n+1}$ such that
\[
p(y_{2n+1}, y_{2n}) \leq \frac{1}{\sqrt{k}}H_p(Tx_{2n}, Sx_{2n-1}).
\]
Similarly we choose $y_{2n+2}$ such that
\[
p(y_{2n+1}, y_{2n+2}) \leq \frac{1}{\sqrt{k}}H_p(Tx_{2n}, Sx_{2n+1}).
\]
(4.iv) For all $n \geq 1$, if $y_n \in C$, then $x_n = y_n$. However if $y_n \notin C$, then applying Lemma 2.4, we choose $x_n \in \partial C$ such that $x_n \in \text{seg}[x_{n-1}, y_n]$.

Let us partition the elements in the sequence $\{x_n\}$ into two sets $P$ and $Q$, where
\[P = \{x_i \in \{x_n\} : x_i = y_i\} \text{ and } Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.
\]

We consider the following cases

**Case 4.1** Consider the case where $(x_n, x_{n+1}) \in P \times P, n \geq 1$. Suppose $n$ is even, that is $n = 2m$ for some $m \in \mathbb{N}$. Then, from (4.iv) we have $x_{2m} = y_{2m}$ and $x_{2m+1} = y_{2m+1}$. Applying (4.iii) we
have

\[ p(x_{2m}, x_{2m+1}) = p(y_{2m}, y_{2m+1}) \]
\[ = p(y_{2m+1}, y_{2m}) \]
\[ \leq \frac{1}{\sqrt{k}} H_p(Tx_{2m}, Sx_{2m-1}), \text{ by (4.iii)} \]
\[ \leq \frac{1}{\sqrt{k}} \times kp(x_{2m}, x_{2m-1}) \text{ by (4.1)} \]
\[ = \sqrt{k} p(x_{2m-1}, x_{2m}). \]

Using a similar argument, when \( n \) is odd, that is, when \( n = 2m + 1 \) for some \( m \in \mathbb{N} \), we get

\[ p(x_{2m+1}, x_{2m+2}) \leq \sqrt{k} p(x_{2m}, x_{2m+1}). \]

Thus in general, when \((x_n, x_{n+1}) \in P \times P, n \geq 1\), we have

\[ p(x_n, x_{n+1}) \leq \sqrt{k} p(x_{n-1}, x_n). \tag{4.4} \]

**Case 4.2** Let us now consider the situation where \((x_n, x_{n+1}) \in P \times Q, n \geq 1\). Suppose \( n \) is even, that is \( n = 2m \) for some \( m \in \mathbb{N} \). Then, from (4.iv) we have \( x_{2m} = y_{2m} \).

We also have \( x_{2m+1} \in \partial C \) and \( x_{2m+1} \in \text{seg}[y_{2m}, y_{2m+1}] \). From Lemma 2.5 (ii), we note that

\[ p(x_{2m}, x_{2m+1}) = p(y_{2m}, x_{2m+1}) \leq p(y_{2m}, y_{2m+1}). \]

Applying (4.iii) we have

\[ p(x_{2m}, x_{2m+1}) \leq p(y_{2m}, y_{2m+1}) \]
\[ \leq \sqrt{k} p(x_{2m-1}, x_{2m}), \]

using the argument in Case 4.1.

Using a similar procedure, we can show that

\[ p(x_{2m+1}, x_{2m+2}) \leq \sqrt{k} p(x_{2m}, x_{2m+1}). \]

In general, when \((x_n, x_{n+1}) \in P \times Q, n \geq 1\), we have

\[ p(x_n, x_{n+1}) \leq \sqrt{k} p(x_{n-1}, x_n). \tag{4.5} \]

**Case 4.3** We consider the situation where \((x_n, x_{n+1}) \in Q \times P, n \geq 1\). In this case, we can show by contradiction that \( x_{n-1} \in P \).
We assume \( x_{n-1} \in Q \). This implies \( x_{n-1} \in \partial C \). This in turn implies that 

\[ x_n = y_n \in T_{x_{n-1}} \subseteq C, \text{ implying } x_n \in P, \text{ which is a contradiction. Hence } x_{n-1} \in P, \text{ implying } x_{n-1} = y_{n-1}. \]

Let us consider when \( n \) is even, that is \( n = 2m \) for some \( m \in \mathbb{N} \). Then, from (4.iv), we have 

\[ x_{2m+1} = y_{2m+1}. \]

We also have \( x_{2m} \in \partial C \) and \( x_{2m} \in \text{seg}[y_{2m-1}, y_{2m}] \). Hence

\[ p(x_{2m}, x_{2m+1}) = p(x_{2m}, y_{2m+1}) \]
\[ \leq p(x_{2m}, y_{2m}) + p(y_{2m}, y_{2m+1}), \text{ according to (2.2),} \]
\[ \leq p(y_{2m-1}, y_{2m}) + p(y_{2m}, y_{2m+1}), \text{ using Lemma 2.5 (ii),} \]
\[ \leq \frac{1}{\sqrt{k}} H_p(T_{x_{2m-2}}, S_{x_{2m-1}}) + \frac{1}{\sqrt{k}} H_p(S_{x_{2m-1}}, T_{x_{2m}}), \text{ by (4.iii)} \]
\[ = \frac{1}{\sqrt{k}} H_p(T_{x_{2m-2}}, S_{x_{2m-1}}) + \frac{1}{\sqrt{k}} H_p(T_{x_{2m}}, S_{x_{2m-1}}) \]
\[ \leq \frac{1}{\sqrt{k}} \times k \left( p(x_{2m-2}, x_{2m-1}) + p(x_{2m}, x_{2m-1}) \right), \text{ by (4.1)} \]
\[ = \sqrt{k} \left( p(x_{2m-2}, x_{2m-1}) + p(x_{2m-1}, x_{2m}) \right) \]
\[ \leq 2 \sqrt{k} \max \{ p(x_{2m-2}, x_{2m-1}), p(x_{2m-1}, x_{2m}) \}. \]

We get a similar result when \( n \) is odd.

In general, when \( (x_n, x_{n+1}) \in Q \times P \), and \( n \geq 2 \), then we have

\[ p(x_n, x_{n+1}) \leq 2 \sqrt{k} \max \{ p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n) \}. \tag{4.6} \]

The case where \( (x_n, x_{n+1}) \in Q \times Q \) is not possible.

Thus in all cases, according to (4.4), (4.5) and (4.6), we have

\[ p(x_n, x_{n+1}) \leq t \max \{ p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n) \}, \tag{4.7} \]

where \( t = 2 \sqrt{k} < 1 \), implying \( k < \frac{1}{4} \).

According to Lemma 3.3, (4.7) implies

\[ p(x_n, x_{n+1}) \leq t^{n/2} \delta, \tag{4.8} \]

where \( \delta = t^{-1/2} \max \{ p(x_0, x_1), p(x_1, x_2) \} \).
Consider \( n, m \in \mathbb{N} \) with \( n > m \). Then, we have inductively from (2.2)

\[
\begin{align*}
p(x_m, x_n) & \leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) \\
& \leq \sum_{i=m}^{n-1} t^{i/2} t^{-1/2} \delta \\
& \leq t^{-1/2} \delta \sum_{i=m}^{+\infty} t^{i/2} \\
& = \delta \frac{t^{m/2}}{1 - t^{1/2}} t^{-1/2}.
\end{align*}
\]

As \( m, n \to +\infty \) we get

\[
\lim_{m,n \to +\infty} p(x_m, x_n) = 0 < +\infty.
\]

From Definition 2.2 (ii), this shows that the sequence \( \{x_n\} \in C \) is a Cauchy sequence. Because

\( C \) is closed in \( (X, p^s) \), it is complete in \( (X, p^s) \) and hence is complete in \( (X, p) \).

This means, according to Lemma 2.2, there is \( x^* \in C \) such that

\[
\lim_{m,n \to +\infty} p(x_m, x_n) = \lim_{n \to +\infty} p(x^*, x_n) = p(x^*, x^*) = 0.
\]

We now show that \( x^* \) is a fixed point of \( S \) and \( T \).

Consider a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) for which each \( x_{n_j} \in P \). If \( n_j \) is odd, that is

\( n_j = 2m_j + 1 \), then we have from the assumption,

\[
H_p(Tx^*, Sx_{2m_j+1}) \leq kp(x^*, x_{2m_j+1}).
\]

This implies

\[
\lim_{j \to +\infty} H_p(Tx^*, Sx_{2m_j+1}) = p(x^*, x^*) = 0. \tag{4.9}
\]

Now consider \( n_j \) being an even number, that is \( n_j = 2m_j \) for some \( m_j \). Because \( x_{2m_j} \in Sx_{2m_j-1} \), we have

\[
p(Tx^*, x_{2m_j}) \leq \delta_p(Tx^*, Sx_{2m_j-1}) \leq H_p(Tx^*, Sx_{2m_j-1}). \tag{4.10}
\]

Taking \( j \to +\infty \) in (4.10) and applying (4.9), we get

\[
\lim_{j \to +\infty} p(Tx^*, x_{2m_j}) \leq \lim_{j \to +\infty} H_p(Tx^*, Sx_{2m_j-1}) = 0
\]

\[
\Rightarrow p(Tx^*, x^*) = 0 = p(x^*, x^*)
\]

\[
\Rightarrow x^* \in Tx^*. \tag{4.11}
\]
This shows that $x^*$ is a fixed point of $T$. Using a similar argument we conclude that $x^*$ is also a fixed point of $S$.

Rao and Rao [10] proved the following fixed point theorem (Theorem 2.8) involving the Hausdorff partial metric for a pair of multi-valued self mappings.

**Theorem 4.2.** [10] Let $(X, p)$ be a complete partial metric space and let $S, T : X \to CB^p(X)$ be mappings satisfying

$$H_p(Sx, Ty) \leq \alpha \max \left\{ p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2} [p(x, Ty) + p(y, Sx)] \right\}$$

for all $x, y \in X$ and $0 < \alpha < 1$. Then $S$ and $T$ have a common fixed point in $X$. Further, if we assume that $p(x, y) \leq p(y, Sx)$ or $p(x, y) \leq p(y, Tx)$ for all $x, y \in X$, then $S$ and $T$ have a unique common fixed point in $X$.

In this research, we modify the Theorem 4.2 so that it applies to a pair of non-self multi-valued mappings in a metrically convex partial metric space.

We provide a proof for the following assumption.

**Theorem 4.3.** Let $(X, p)$ be a complete metrically convex partial metric space and $C$ a non-empty closed subset of $X$, the closure being with respect to $(X, p')$. Let $\partial C$, the boundary of $C$ with respect to $(X, p')$, be non-empty. Let $S, T : C \to CB^p(X)$ be mappings satisfying

$$H_p(Sx, Ty) \leq \alpha \max \left\{ p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2} [p(x, Ty) + p(y, Sx)] \right\}$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{4}$. Let the following conditions apply:

(i) $x \in \partial C$ implies $Tx \subset C$.

(ii) $x \in \partial C$ implies $Sx \subset C$.

Then $S$ and $T$ have a common fixed point in $X$. Further, if we assume that $p(x, y) \leq p(y, Sx)$ or $p(x, y) \leq p(y, Tx)$ for all $x, y \in X$, then $S$ and $T$ have a unique common fixed point $z$ in $C$ with $p(z, z) = 0$.

**Proof.** We construct sequences $\{x_n\} \in C$ and $\{y_n\} \in X$ in the following way.

We commence by choosing an arbitrary $x_0 \in \partial C$. According to (i), we choose $x_1 \in C$ such that $x_1 \in Tx_0$. We set $y_1 = x_1$. Because $\alpha \in (0, \frac{1}{4})$ implies $\frac{1}{\sqrt{\alpha}} > 1$, by Lemma 3.2, there exists
$y_2 \in Sx_1$ such that

$$p(y_1, y_2) \leq \frac{1}{\sqrt{\alpha}} H_p(Tx_0, Sx_1).$$

If $y_2 \in C$, then we set $x_2 = y_2$.

If however $y_2 \not\in C$, then, according to Lemma 2.4, there is $x_2 \in \partial C$ such that $x_2 \in \text{seg}[x_1, y_2]$. Using Lemma 3.2, and recalling that $y_2 \in Sx_1$, we choose $y_3 \in Tx_2$ such that

$$p(y_3, y_2) \leq \frac{1}{\sqrt{\alpha}} H_p(Tx_2, Sx_1).$$

From (i) in the assumption, we have $y_3 \in C$.

In general, the sequences $\{x_n\} \subset C$ and $\{y_n\}_{n \geq 1} \subset X$ are constructed in the same way as we did when proving Theorem 4.1.

We partition the elements of $\{x_n\}$ into sets $P$ and $Q$ such that

$P = \{x_i \in \{x_n\} : x_i = y_i\}$ and $Q = \{x_i \in \{x_n\} : x_i \neq y_i\}$.

Now for $n \geq 2$, we consider the following cases.

**Case 4.4** Consider $x_n \in P \times P$. This means $x_n = y_n$.

If $n$ is even, that is, if $n = 2m$ for some $m \in \mathbb{N}$, we have $x_n = x_{2m} = y_{2m}$. As $x_{2m} = y_{2m} \in Sx_{2m-1}$, from (4.ii), we can choose $y_{2m+1} \in Tx_{2m}$ such that

$$p(x_{2m}, y_{2m+1}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}). \quad (4.12)$$

We consider two scenarios.

**(4.4.1)** If $y_{2m+1} \in P$, then $x_{2m+1} = y_{2m+1}$. Hence, (4.12) becomes

$$p(x_{2m}, x_{2m+1}) = p(y_{2m}, y_{2m+1})$$

$$\leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m})$$

$$\leq \frac{1}{\sqrt{\alpha}} \times \alpha \max \left\{ p(x_{2m-1}, x_{2m}), p(x_{2m-1}, Sx_{2m-1}), p(x_{2m}, Tx_{2m}), \frac{1}{2} [p(x_{2m-1}, Tx_{2m}) + p(x_{2m}, Sx_{2m-1})] \right\}$$

$$\leq \sqrt{\alpha} \max \left\{ p(x_{2m-1}, x_{2m}), p(x_{2m-1}, y_{2m}), p(x_{2m}, y_{2m+1}), \frac{1}{2} [p(x_{2m-1}, y_{2m+1}) + p(x_{2m}, y_{2m})] \right\}$$
\[ = \sqrt{\alpha} \max \left\{ p(x_{2m-1}, x_{2m}), p(x_{2m-1}, x_{2m}), p(x_{2m}, x_{2m+1}), \frac{1}{2} \left[ p(x_{2m-1}, x_{2m+1}) + p(x_{2m}, x_{2m}) \right] \right\} \]

\[ \Rightarrow p(x_{2m}, x_{2m+1}) \leq \sqrt{\alpha} \max \left\{ p(x_{2m-1}, x_{2m}), \frac{1}{2} \left[ p(x_{2m-1}, x_{2m}) + p(x_{2m}, x_{2m+1}) \right] \right\} \]

(4.13)

If \( p(x_{2m-1}, x_{2m}) < \frac{1}{2} [p(x_{2m-1}, x_{2m}) + p(x_{2m}, x_{2m+1})] \) implying \( p(x_{2m-1}, x_{2m}) < p(x_{2m}, x_{2m+1}) \), then we have

\[ p(x_{2m}, x_{2m+1}) \leq \frac{\sqrt{\alpha}}{2} [p(x_{2m-1}, x_{2m}) + p(x_{2m}, x_{2m+1})] \]

\[ \leq \frac{\sqrt{\alpha}}{2 - \sqrt{\alpha}} p(x_{2m-1}, x_{2m}) \]

\[ < p(x_{2m-1}, x_{2m}), \text{ as } \frac{\sqrt{\alpha}}{2 - \sqrt{\alpha}} < 1. \]

This is a contradiction.

Hence \( p(x_{2m-1}, x_{2m}) \geq \frac{1}{2} [p(x_{2m-1}, x_{2m}) + p(x_{2m}, x_{2m+1})] \) implying

\[ p(x_{2m}, x_{2m+1}) \leq \sqrt{\alpha} p(x_{2m-1}, x_{2m}). \]

(4.4.2) If \( y_{2m+1} \in Q \), then \( x_{2m+1} \neq y_{2m+1} \). From the construction of proof, we have \( x_{2m+1} \in \text{seg}[x_{2m}, y_{2m+1}] \). Using Lemma 2.5 (ii), we get

\[ p(x_{2m}, x_{2m+1}) \leq p(x_{2m}, y_{2m+1}) \]

\[ = p(y_{2m}, y_{2m+1}) \]

\[ \leq \sqrt{\alpha} p(x_{2m-1}, x_{2m}), \]

using the argument in (4.4.1).

We get the following similar result when \( n \) is odd, that is, when \( n = 2m + 1 \) for some \( m \in \mathbb{N} \),

\[ p(x_n, x_{n+1}) = p(x_{2m+1}, x_{2m+2}) \leq \sqrt{\alpha} p(x_{2m}, x_{2m+1}). \]

Thus, for \( x_n \in P \), we have

\[ p(x_n, x_{n+1}) \leq \sqrt{\alpha} p(x_{n-1}, x_n). \]

(4.14)

Case 4.5 Consider the case where \( (x_n, x_{n+1}) \in Q \times P \). We claim that for \( n \geq 1, x_n \in Q \) implies \( x_{n-1} \in P \).

Let \( x_{n-1} \in Q \), then \( x_{n-1} \in \partial C \). This means, according to (ii), \( x_n = y_n \in C \). This implies \( x_n \in P \), which is a contradiction.
Hence we have
\[ x_{n-1}, x_{n+1} \in P \text{ and } x_n \in \text{seg}[x_{n-1}, y_n]. \]

Consider when \( n \) is even, that is, when \( n = 2m \) for some \( m \in \mathbb{N} \). According to (4.iii), \( y_{2m+1} \in T_{x_{2m}} \subset C \) was chosen in such a way that

\[ p(y_{2m}, y_{2m+1}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}). \quad (4.15) \]

We apply (2.2) and get

\[ p(x_{2m}, x_{2m+1}) = p(x_{2m}, y_{2m+1}) \]

\[ \leq p(x_{2m}, y_{2m}) + p(y_{2m}, y_{2m+1}) \]

\[ \Rightarrow p(x_{2m}, x_{2m+1}) \leq 2 \max\{p(x_{2m}, y_{2m}), p(y_{2m}, y_{2m+1})\}. \quad (4.16) \]

If \( p(x_{2m}, y_{2m}) \geq p(y_{2m}, y_{2m+1}) \), (4.16) becomes

\[ p(x_{2m}, x_{2m+1}) \leq 2p(x_{2m}, y_{2m}) \]

\[ \leq 2p(x_{2m-1}, y_{2m}), \text{ as per Lemma 2.5 (ii)} \quad (4.17) \]

\[ = 2p(y_{2m-1}, y_{2m}), \text{ as } x_{2m-1} = y_{2m-1} \]

\[ \leq 2\sqrt{\alpha} p(x_{2m-2}, x_{2m-1}), \]

using the argument in (4.4.2).

If \( p(x_{2m}, y_{2m}) < p(y_{2m}, y_{2m+1}) \), (4.16) becomes

\[ p(x_{2m}, x_{2m+1}) \leq 2p(y_{2m}, y_{2m+1}). \quad (4.18) \]

Let us consider the term \( p(y_{2m}, y_{2m+1}) \). From (4.15) and Theorem 3.1 we have

\[ p(y_{2m}, y_{2m+1}) \leq \frac{1}{\sqrt{\alpha}} H_p(Sx_{2m-1}, Tx_{2m}) \]

\[ \leq \sqrt{\alpha} \max\{p(x_{2m-1}, x_{2m}), p(x_{2m-1}, Sx_{2m-1}), p(x_{2m}, Tx_{2m}), \]

\[ \frac{1}{2} [p(x_{2m-1}, Tx_{2m}) + p(x_{2m}, Sx_{2m-1})]\} \]

\[ \Rightarrow p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} \max\{p(x_{2m-1}, x_{2m}), p(x_{2m-1}, y_{2m}), p(x_{2m}, y_{2m+1}), \]

\[ \frac{1}{2} [p(x_{2m-1}, y_{2m+1}) + p(x_{2m}, y_{2m})]\}. \quad (4.19) \]
As $x_{2m} \in \text{seg}[x_{2m-1}, y_{2m}]$, from Lemma 2.5 (ii), we have

$$p(x_{2m-1}, y_{2m}) \geq p(x_{2m-1}, x_{2m}).$$

From P3 of Definition 2.1, we also have

$$\leq [p(x_{2m-1}, x_{2m}) + p(x_{2m}, y_{2m+1}) - p(x_{2m}, x_{2m}) + p(x_{2m}, y_{2m})]$$

$$= [p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})].$$

(4.20)

The expression (4.20) is because $x_{2m} \in \text{seg}[x_{2m-1}, y_{2m}]$ and Lemma 2.5 (i). Hence (4.19) becomes

$$p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} \max\{p(x_{2m-1}, y_{2m}), p(y_{2m}, y_{2m+1}), p(x_{2m}, y_{2m+1}), \frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})]\}.\quad (4.21)$$

Suppose $p(x_{2m-1}, y_{2m}) < p(x_{2m}, y_{2m+1})$, implying

$$p(x_{2m}, y_{2m+1}) > \frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})].$$

Then (4.21) becomes

$$p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} p(x_{2m}, y_{2m+1}).\quad (4.22)$$

We continue from (4.18) and get

$$p(x_{2m}, x_{2m+1}) \leq 2p(y_{2m}, y_{2m+1})$$

$$\leq 2\sqrt{\alpha} p(x_{2m}, y_{2m+1})$$

$$= 2\sqrt{\alpha} p(x_{2m}, x_{2m+1}), \text{ as } x_{2m+1} = y_{2m+1}$$

$$< p(x_{2m}, x_{2m+1}), \text{ because } 2\sqrt{\alpha} < 1.$$ This is a contradiction.

Hence $p(x_{2m-1}, y_{2m}) \geq p(x_{2m}, y_{2m+1})$, implying

$$p(x_{2m-1}, y_{2m}) \geq \frac{1}{2}[p(x_{2m-1}, y_{2m}) + p(x_{2m}, y_{2m+1})].$$

Then (4.21) becomes

$$p(y_{2m}, y_{2m+1}) \leq \sqrt{\alpha} p(x_{2m-1}, y_{2m}).\quad (4.23)$$
We continue from (4.18) and get
\[ p(x_{2m}, x_{2m+1}) \leq 2p(y_{2m}, y_{2m+1}) \leq 2\sqrt{\alpha}p(y_{2m-1}, y_{2m}) = 2\sqrt{\alpha}p(y_{2m-1}, y_{2m}), \text{ because } x_{2m-1} = y_{2m-1} \leq 2\sqrt{\alpha} \times \sqrt{\alpha}p(x_{2m-2}, x_{2m-1}), \text{ as per (4.4.2) } \]
\[ \Rightarrow p(x_{2m}, x_{2m+1}) \leq \sqrt{\alpha}p(x_{2m-2}, x_{2m-1}), \text{ because } 2\sqrt{\alpha} < 1. \] (4.24)

Hence, in observing (4.17) and (4.24), when \( x_{2m} \in Q \), we have
\[ p(x_{2m}, x_{2m+1}) \leq 2\sqrt{\alpha}p(x_{2m-2}, x_{2m-1}). \] (4.25)

Using a similar argument, we can show that, when \( n \) is odd, that is, when \( n = 2m + 1 \) for some \( m \in \mathbb{N} \), we have
\[ p(x_{2m+1}, x_{2m+2}) \leq 2\sqrt{\alpha}p(x_{2m-1}, x_{2m}). \]

Hence in general, when \((x_n, x_{n+1}) \in P \times Q\) we have
\[ p(x_n, x_{n+1}) \leq 2\sqrt{\alpha}p(x_{n-2}, x_{n-1}). \]

The case of \((x_n, x_{n+1}) \in Q \times Q\) is not possible.

For all cases 4.4 and 4.5 we have
\[ p(x_n, x_{n+1}) \leq t \max \{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\}, \] (4.26)

where
\[ t = 2\sqrt{\alpha} < 1. \]

According to Lemma 3.3, (4.27) implies
\[ p(x_n, x_{n+1}) \leq t^{n/2}t^{-1/2} \max \{p(x_0, x_1), p(x_1, x_2)\}. \] (4.27)

Using the same argument used during the proof of Theorem 4.1, (4.27) shows that there is \( z \in C \) such that
\[ \lim_{m,n \to +\infty} p(x_m, x_n) = \lim_{n \to +\infty} p(z, x_n) = p(z, z) = 0. \]

We now prove that \( z \) is a fixed point of both \( S \) and \( T \).
Consider the subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) each of whose terms is in \( P \). This means \( x_{n_j} = y_{n_j} \) for \( j = 1, 2, \ldots \). Consider the case where \( n_j \) is odd, that is \( n_j = 2m_j + 1 \) for some \( m_j \in \mathbb{N} \).

As \( x_{2m_j+1} \in Tx_{2m_j} \), we have

\[
p(z, Tx_{2m_j}) \leq p(z, x_{2m_j+1}).
\]

This implies \( \lim_{j \to +\infty} p(z, Tx_{2m_j}) = 0 \).

Using a similar argument, we can show that \( \lim_{j \to +\infty} p(z, Sx_{2m_j+1}) = 0 \).

Now consider

\[
p(z, Sz) \leq p(z, Tx_{2m_j}) + p(Tx_{2m_j}, Sz)
\]

\[
\leq p(z, Tx_{2m_j}) + \alpha \max \{ p(x_{2m_j}, z), p(x_{2m_j}, Tx_{2m_j}) \},
\]

\[
p(z, Sz), \frac{1}{2}[p(x_{2m_j}, Sz) + p(z, Tx_{2m_j})]\}.
\]

Taking \( j \to +\infty \), we have

\[
p(z, Sz) \leq 0 + \alpha \max \{ 0, 0, p(z, Sz), \frac{1}{2}[p(z, Sz)] \}
\]

\[
= \alpha p(z, Sz)
\]

\[
\leq p(z, Sz), \text{ because } \alpha < 1
\]

\[
\Rightarrow p(z, Sz) = 0.
\]

This implies \( z \in Sz \) meaning \( z \) is a fixed point in \( S \). Using a similar argument, we have \( z \) is a fixed point in \( T \).

We show the uniqueness of the fixed point. Let \( z \) and \( y \) be fixed points of both \( S \) and \( T \). As \( z \in Sz \) we have

\[
p(y, Sz) = \inf_{a \in Sz} p(y, a) \leq p(y, z) = p(z, y).
\]

(4.28)

Suppose, as per assumption, we have \( p(z, y) \leq p(y, Sz) \). Then, (4.28) leads us to conclude that

\[
p(z, y) = p(y, Sz).
\]

(4.29)

Because \( y \in Ty \), we have

\[
p(z, y) = p(y, Sz) \leq H_p(Ty, Sz)
\]

\[
\leq \alpha \max \{ p(z, y), p(y, Ty), p(z, Sz), \frac{1}{2}[p(y, Sz) + p(z, Ty)] \}
\]
\[ \Rightarrow p(z,y) = \frac{\alpha}{2}[p(y,Sz) + p(z,Ty)] \quad (4.30) \]

\[ \Rightarrow p(z,y) \leq \frac{\alpha}{2 - \alpha}p(z,Ty) \]

\[ \leq p(z,Ty), \text{ as } \frac{\alpha}{2 - \alpha} < 1. \quad (4.31) \]

Let us consider (4.30). We also consider (4.29) which states that \( p(z,y) = p(y,Sz) \). We then have

\[ p(z,y) \leq \frac{\alpha}{2}[p(y,Sz) + p(z,Ty)] = \frac{\alpha}{2}[p(z,y) + p(z,Ty)] \]

\[ \leq \frac{\alpha}{2}[p(z,y) + p(z,y)], \text{ because } y \in Ty \]

\[ = \alpha p(z,y) \]

\[ \Rightarrow p(z,y) = 0, \text{ as } \alpha < 1 \]

\[ \Rightarrow z = y, \text{ by (2.1)}. \]

We will reach the same conclusion if we assume \( p(z,y) \leq p(z,Ty) \). This shows that the common fixed point \( z \) is unique. The proof has been completed.

**Remark 4.1** Theorem 4.3 is valid when we have \( S = T \).

**Remark 4.2** If we set \( S = T \), and assume \( C = X \), only (4.4.1) applies, and we get Theorem 4.2 by Rao and Rao [10].

When we set \( T = f \) where \( f \) is a single valued mapping we get the following corollary:

**Corollary 4.1** Let \((X,p)\) be a complete metrically convex partial metric space and \( C \) a non-empty closed subset of \( X \), the closure being with respect to \((X,p^s)\). Let \( \partial C \), the boundary of \( C \) with respect to \((X,p^s)\), be non-empty. Let \( S, f : C \rightarrow CB^p(X) \) be mappings satisfying

\[ p(Sx, fy) \leq \alpha \max \left\{ p(x,y), p(x,Sx), p(y,fy), \frac{1}{2}[p(x,fy) + p(y,Sx)] \right\} \]

for all \( x, y \in X \) and \( 0 < \alpha < \frac{1}{4} \). Let the following conditions apply:

(i) \( x \in \partial C \) implies \( fx \in C \),

(ii) \( x \in \partial C \) implies \( Sx \subset C \).

Then \( S \) and \( f \) have a common fixed point in \( X \). Further, if we assume that \( p(x,y) \leq p(y,Sx) \) or
For all $x, y \in X$, then $S$ and $f$ have a unique common fixed point $z$ in $C$ with $p(z, z) = 0$.

If we set $T = f$, $S = g$, where both $f$ and $g$ are single valued mappings we get the following corollary:

**Corollary 4.2** Let $(X, p)$ be a complete metrically convex partial metric space and $C$ a non-empty closed subset of $X$, the closure being with respect to $(X, p^s)$. Let $\partial C$, the boundary of $C$ with respect to $(X, p^s)$, be non-empty. Let $g, f : C \to X$ be mappings satisfying

$$
p(gx, fy) \leq \alpha \max \left\{ p(x, y), p(x, gx), p(y, fy), \frac{1}{2}[p(x, fy) + p(y, gx)] \right\} \text{ for all } x, y \in X \text{ and } 0 < \alpha < \frac{1}{4}.
$$

Let the following condition apply: $x \in \partial C$ implies $fx \in C$ and $gx \subset C$.

Then $g$ and $f$ have a common fixed point in $X$. Further, if we assume that $p(x, y) \leq p(y, gx)$ or $p(x, y) \leq p(y, fx)$ for all $x, y \in X$, then $g$ and $f$ have a unique common fixed point $z$ in $C$ with $p(z, z) = 0$.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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