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A NEW MONOTONE HYBRID ALGORITHM FOR A CONVEX FEASIBILIY PROBLEM FOR AN INFINITE FAMILY OF NONEXPANSIVE-TYPE MAPS, WITH APPLICATIONS

CHARLES E. CHIDUME[1,*], EMMANUEL E. OTUBO[1,2], CHINEDU G. EZEA[1,3], MARKJOE O. UBA[4]

[1]Department of Pure and Applied Mathematics, African University of Science and Technology, Abuja, Nigeria

[2]Department of Mathematics and Computer Science, Ebonyi State University, Abakaliki, Nigeria

[3]Department of Mathematics, Nnamdi Azikiwe University, Awka, Nigeria

[4]Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Abstract. Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. A new monotone hybrid method for finding a common element for a family of a general class of nonlinear nonexpansive maps is constructed and, the sequence of the method is proved to converge strongly to a common element of the family. Finally, application of this theorem complements, generalizes and extends some recent important results.

Keywords: nonexpansive maps; monotone maps; strong convergence.

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1. Introduction

The problem of finding a point in the intersection of a family of closed and convex subsets of a Banach space generally referred to as *the convex feasibility problem* appears frequently in

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*Corresponding author
E-mail address: cchidume@aust.edu.ng
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various areas of physical sciences and has been studied well in the framework of Hilbert spaces and has found applications in areas such as image restoration, computer tomography, radiation therapy treatment planning (see e.g., Combettes [15]). Significant research has also been done on the convex feasibility problem in real Banach spaces more general than Hilber space (see e.g., Kitahara and Takahashi [21], O'Hare *et al.* [28], Chang *et al.* [10], Qin *et al.* [29], Zhou and Tan [38], Wattanwitoon and Kumam [34], Li and Su [23], Takahashi and Zembayashi [33], Kikkawa and Takahashi [20], Aleyner and Reich [5], Sahu *et al.* [32], Ceng *et al.* [9], and the references contained in them).

Let C be a nonempty subset of a normed space. A map $T : C \to E$ is called *nonexpansive* if $||Tx-Ty|| \leq ||x-y||$ for all $x,y \in C$. Recently, Nakajo and Takahashi [27] studied the following algorithm for a nonexpansive self-map T of a nonempty closed and convex subset C of a Hilbert space, H:

$$(1) \qquad \begin{cases} x_0 \in C \; arbitrary, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{u \in C : ||u_n - u|| \leq ||x_n - u||\}, \\ Q_n = \{u \in C : \langle x_n - u, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They proved a strong convergence theorem if the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is bounded above by 1. Martinez-Yanes and Xu [25] introduced the following so-called Ishikawa scheme for a nonexpansive self-map T of a nonempty closed and convex subset C of a Hilbert space:

$$(2) \qquad \begin{cases} x_0 \in C \; arbitrary, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{u \in C : ||y_n - u||^2 \leq ||x_n - u||^2\}, \\ +(1-\alpha_n)(||z_n||^2 - ||x_n||^2 + 2\langle x_n - z_n, u \rangle \geq 0\}, \\ Q_n = \{u \in C : \langle x_n - u, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that under appropriate conditions on these sequences, the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$. Qin and Su [30] modified the algorithm of Nakajo and Takahashi [27] by introducing the monotone hybrid method for a nonexpansive map T still in a Hilbert space H as follows:

$$(3)\qquad \begin{cases} x_1 = x \in C,\ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{u \in C : ||u_n - u|| \le ||x_n - u||\}, \\ Q_n = \{u \in C : \langle x_n - u, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

for all $n \in \mathbb{N}$, $\alpha_n \in (0,1)$. Using this algorithm, they proved a strong convergence theorem under suitable conditions on α. Recently, Klin-earn, Suantai and Takahashi [22] presented a new and interesting monotone hybrid iterative method *for a convex feasibility problem* for a family of generalized nonexpansive maps in a Banach space more general than Hilbert spaces. They proved the following Theorem:

Theorem 1.1. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E and let T be a family of closed generalized nonexpansive mappings from C into E such that $\cap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with T. Let $\{x_n\}$ be the sequence generated by*

$$(4)\qquad \begin{cases} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \le \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz\rangle \ge 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}$$

for all $n \in N$, where J is the duality mapping on E and $\alpha_n \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

Let E be a real normed space with dual space E^*. A map $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, $\langle \eta - \nu, x - y \rangle \geq 0 \ \forall \ \eta \in Ax, \ \nu \in Ay$. Consider, for example, the following: Let $g : E \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of g, $\partial g : E \to 2^{E^*}$, is defined for each $x \in E$ by

$$\partial g(x) = \left\{ x^* \in E^* : g(y) - g(x) \geq \langle y - x, x^* \rangle \ \forall \ y \in E \right\}.$$

It is easy to check that ∂g is a *monotone map* on E, and that $0 \in \partial g(u)$ *if and only if u is a minimizer of g*. Setting $\partial g \equiv A$, it follows that solving the inclusion

$$0 \in Au, \tag{5}$$

in this case, is solving for a minimizer of g.

Let E be a real normed space with dual space E^*. A map $J : E \to 2^{E^*}$ defined by $Jx := \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|, \ \|x\| = \|x^*\| \right\}$ is called the *normalized duality map* on E. A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\langle \eta - \nu, j(x-y) \rangle \geq 0 \ \forall \ \eta \in Ax, \ \nu \in Ay$. Accretive operators have been studied extensively by numerous mathematicians (see e.g., Browder [7], Deimling [16], Kato [19], Ioana [17], Reich [31], and a host of other authors).

In studying the equation

$$Au = 0, \tag{6}$$

where A is an accretive operator on a Banach space E, Browder introduced an operator T defined by $T := I - A$ where I is the identity map on E. He called such an operator *pseudo-contractive*. It is clear that, fixed points of T correspond to solutions of $Au = 0$, if they exist. Consequently, approximating solutions of this equation when A is an accretive-type operator, *using fixed point techniques* has become a flourishing area of research for numerous mathematicians. This resulted in the publication of several monographs which presented in-depth coverage of the main ideas, concepts and most important results on iterative algorithms for appropriation of solutions of several nonlinear equations involving accretive-type maps (see e.g., Agarwal *et al.* [1];

Berinde [6]; Chidume [11]; Censor and Reich [8]; William and Shahzad [35], and the references contained in them).

Unfortunately, developing algorithms for approximating solutions of the equation (5) when $A : E \to 2^{E^*}$ is of *monotone-type* has not been very fruitful. The fixed point technique introduced by Browder for studing equation (6) when A is of the accretive type is not applicable in this case since A maps a space E to its dual space E^*, and so the map $T : I - A$, which he called pseudo-contractive does not make sense here.

Fortunately, a new concept of *fixed points for maps from a real normed space E to its dual space* E^* has very recently been introduced and studied (see Chidume and Idu [12], Liu [24], Zegeye [37]). This notion has been found to be quite natural and very applicable. For applications to approximation of zeros of maximal monotone maps, applications to proximal point algorithm, solutions of Hammerstein integral equations, and convex minimization problems, the reader is referred to Chidume and Idu [12]. These developments have provided fixed point theory for studying the equation $Au = 0$ where A maps a space E to its dual space E^*. In particular, this evolving fixed point theory for maps from a space E to its dual space E^* is suitable for studying, in particular, the equation $Au = 0$ where A is the subdifferential of a convex function. It is well known that most monotone operators on a normed space are subdifferentials of some convex function (see e.g., [17]). Furthermore, these developments have also generated considerable interest in *fixed point theory for maps from a real Banach space E to its dual space* E^* and their applications (see e.g., Chidume *et al.* [13], Liu [24]. Zegeye [37]).

In this paper, we continue the study of fixed point theory for maps from a real Banach space E to its dual space E^*. An analogue of Theorem 1.1 is proved for an infinite family of nonexpansive-type maps from a normed space E into its dual space E^*. Finally, application of our theorem in a real Hilbert space complements and extends the result of Nakajo and Takahashi [27], Martinez-Yanese and Xu [25], Qin and Su [30] and results of a host of other authors.

2. Preliminaries

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E, $\rho_E : [0,\infty) \to [0,\infty)$, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}.$$

A normed linear space E is called *uniformly smooth* if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$. A Banach space E is said to be *strictly convex* if $\|x\| = \|y\| = 1, \quad x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \varepsilon = \|x-y\|\right\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S := \{u \in E : \|u\| = 1\}$, $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$, exists and is attained uniformly for $y \in E$. We list some properties of the normalized duality map (defined earlier) which are well known (see e.g., Cioranescu [14]).

- $J(0) = 0$,
- For $x \in E$, Jx is nonempty closed and convex,
- If E is strictly convex, then $J$is one-to-one, i.e., if $x \neq y$, then $Jx \cap Jy = \emptyset$,
- If E is reflexive, then J is onto,
- If E is smooth, then J is single-valued,
- If E is uniformly smooth, then J is uniformly continuous on bounded subsets of E.

In the sequel, we shall need the following definitions and results.

Definition 2.1 (*J-fixed point*)**.** Let E be an arbitrary normed space and E^* be its dual. Let $T : E \to E^*$ be any mapping. A point $x \in E$ will be called a *J-fixed point* of T if and only if $Tx = Jx$.

Definition 2.2 (*J-pseudocontractive mappings*)**.** Let E be a normed space. A mapping $T : E \to E^*$ is called *J-pseudocontractive* if for every $x, y \in E$,

$$\langle Tx - Ty, x - y \rangle \le \langle Jx - Jy, x - y \rangle.$$

Remark 1. The $T := J - A$ is J-pseudocontractive if and only if A is monotone (see Chidume and Idu [12]).

This notion has been applied to construct an iterative scheme for the approximation of zeros of bounded maximal monotone maps. In particular, the following theorem has been proved

Theorem 2.3. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \to 2^{E^*}$ be a multi-valued maximal monotone and bounded map such that $A^{-1}0 \neq \emptyset$. For fixed $u, x_1 \in E$, let a sequence $\{x_n\}$ be iteratively defined by:*

$$x_{n+1} = J^{-1}\left[Jx_n - \lambda_n \mu_n - \lambda_n \theta_n (Jx_n - Ju)\right],\ n \ge 1,\ \mu_n \in Ax_n. \tag{7}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$. Then, the sequence $\{x_n\}$ converges strongly to a zero of A.

Let E be a smooth real Banach space with dual E^*. The Lyapounov functional $\phi : E \times E \to \mathbb{R}$, defined by:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \text{ for } x, y \in E, \tag{8}$$

where J is the normalized duality map from E into E^* will play a central role in the sequel. It was introduced by Alber and was first studied by Alber [2], Alber and Guerre-Delabriere [3], Kamimura and Takahashi [18], Reich [31] and a host of other authors. If $E = H$, a real Hilbert space, then equation (8) reduces to $\phi(x,y) = \|x-y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{9}$$

Definition 2.4. Let C be a nonempty closed and convex subset of a real Banach space E and T be a map from C to E.The map T is called *generalized nonexpansive* if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and $\phi(Tx, p) \le \phi(x, p)$ for all $x \in C, p \in F(T)$. A map R from E onto C is said to be a retraction if $R^2 = R$. The map R is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \leq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction R frm E onto C. We now list some lemmas which will be used in the sequel.

Lemma 2.5. *(see e.g., Alber* [2]*) Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, the following are equivalent.*
(i) C is a sunny generalized nonexpansive retract of E,
(ii) C is a generalized nonexpansive retract of E,
(iii) JC is closed and convex.

Lemma 2.6. *(see e.g., Alber* [2]*) Let C be a nonempty closed and convex subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C.Then, the following hold.*
(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z),\ z \in C,\ x \in E.$

Lemma 2.7. *(see e.g., Xu* [36]*) Let E be a uniformly convex Banach space. Let $r > 0$. Then, there exists a strictly increasing continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and the following unequality holds:*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda ||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

for all $x, y \in B_r(0)$, where $B_r(0) := \{v \in E : ||v|| \leq r\}$ and $\lambda \in [0, 1]$.

Lemma 2.8. *(see e.g., Kamimura and Takahashi* [18]*) Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{x_n\}$ be sequences in E such that either $\{x_n\}$ or $\{x_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

2.1. **NST-condition.** Let C be a closed subset of a Banach space E. Let $\{T_n\}$ and Γ be two families of generalized nonexpansive maps of C into E such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of $\{T_n\}$ and $F(\Gamma)$ is the set of common fixed points of Γ.

Definition 2.9. The sequence $\{T_n\}$ satisfies the NST-condition (see e.g., Nakajo, Shimoji and Takahashi [26]) with Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||x_n - T_n x_n|| = 0 \Rightarrow \lim_{n\to\infty} ||x_n - T x_n|| = 0, \ for\ all\ T \in \Gamma.$$

Example 1. If $\Gamma = \{T\}$ a singleton, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. If $T_n = T$ for all $n \geq 1$, then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$.

Example 2. Let C be a closed subset of a uniformly smooth and uniformly convex Banach space E and let S and T be generalized nonexpaansive mappings from C into E with $F(S) \cap F(T) \neq \emptyset$. Let $\{\beta_n\} \subset [0,1]$ satisfy $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. For $n \in \mathbb{N}$, define the mapping T_n from C into E by

$$T_n x = \beta_n S x + (1-\beta_n) T x,$$

for all $x \in C$. Then, $\{T_n\}$ is a countable family of generalized nonexpansive mappings satisfying the NST-condition with $\tau = \{S, T\}$.

Proof. See [22]. □

3. **Main Results**

Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space with dual space E^*. Let J be the normalized duality map on E and J_* be the normalized duality map on E^*. Observe that under this setting, J^{-1} exists and $J^{-1} = J_*$. With these notations, we have the followung definitions.

Let C be a nonempty subset of a real normed space E with dual space E^*.

Definition 3.1. A map $T : C \to E^*$ is called J_**-closed* if $(J_* o T) : C \to E$ is a closed map, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \to x$ and $(J_* o T)x_n \to y$, then $(J_* o T)x = y$.

Definition 3.2. A point $x^* \in C$ is called a *J-fixed point of T* if $Tx^* = Jx^*$. The set of J-fixed points of T will be denoted by $F_J(T)$.

Definition 3.3. A map $T : C \to E^*$ will be called *quasi-ϕ-J-nonexpansive* if $F_J(T) \neq \emptyset$, and $\phi(p,(J_* oT)x) \leq \phi(p,x)$ for all $x \in C$ and for all $p \in F_J(T)$.

Let C be a closed subset of a real Banach space E. Let $\{T_n\}$ and Γ be two families of quasi-ϕ-J-nonexpansive maps of C into E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$, where $F_J(\Gamma)$ denotes the set of common J-fixed points of Γ.

Definition 3.4. A sequence $\{T_n\}$ of maps from C to E^* will be said to *satisfy the NST-condition with* Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||Jx_n - T_n x_n|| = 0 \Rightarrow \lim_{n\to\infty} ||Jx_n - Tx_n|| = 0, \text{ for every } T \in \Gamma.$$

We now prove the following theorem.

Theorem 3.5. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^* such that JC is closed and convex. Let $T_n : C \to E^*, n = 1,2,3,...$ be an infinite family of quasi-ϕ-J-nonexpansive maps and Γ be a family of J_*-closed and quasi-ϕ-J-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:*

$$\left\{\begin{array}{l} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT_n)x_n), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u,u_n) \leq \phi(u,x_n)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{array}\right. \tag{10}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(\Gamma)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(\Gamma)$.

Proof. The proof is given in 4 steps.

Step 1: We establish that the sequence $\{x_n\}$ is well defined. We begin by showing that JC_n and JQ_n are closed and convex. First, we observe that since J is one to one, we have that $J(Q_{n-1} \cap C_{n-1}) = JQ_{n-1} \cap JC_{n-1}$. From the definitions of JC_n and JQ_n, it is easy to see that JQ_n

and JC_n are closed. We show they are convex. Proceeding by induction, given that $JQ_0 = JC_0$ is convex, assume that JQ_{n-1} and JC_{n-1} are convex. Let $w_1, w_2 \in JQ_n$ and $\lambda \in [0,1]$. Then, there exist $u_1, u_2 \in Q_n$ such that $w_1 = Ju_1$ and $w_2 = Ju_2$. Set $w = J^{-1}(\lambda w_1 + (1-\lambda)w_2)$. By definition of JQ_n we have that $JQ_n \subset JQ_{n-1} \cap JC_{n-1}$. Hence by induction hypothesis, we have that $\lambda w_1 + (1-\lambda)w_2 = Jw \in J(Q_{n-1} \cap C_{n-1})$. Furthermore, we have

$$\begin{aligned}\langle x - x_n, Jx_n - Jw\rangle &= \langle x - x_n, Jx_n - \lambda w_1 - (1-\lambda)w_2\rangle \\ &= \lambda\langle x - x_n, Jx_n - Ju_1\rangle + (1-\lambda)\langle x - x_n, Jx_n - Ju_2\rangle \\ &\geq \lambda \cdot 0 + (1-\lambda)\cdot 0 = 0.\end{aligned}$$

Thus we have that $w \in Q_n$, i.e., $\lambda w_1 + (1-\lambda)w_2 \in JQ_n$. Hence JQ_n is convex. Observing that the condition $\phi(u, u_n) \leq \phi(u, x_n)$ is equivalent to

$$\|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle,$$

and following similar arguement, we conclude that JC_n is also convex.

Next, we show $F_J(\Gamma) \subset C_n \cap Q_n \forall n \in \mathbb{N}$. Since $T_n : C \to E^*, n = 1,2,3,...$ is an infinite family of quasi-ϕ-J-nonexpansive maps such that $F_J(\Gamma) \neq \emptyset$, let $p \in F_J(\Gamma)$. We compute as follows:

$$\begin{aligned}\phi(p, u_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT_n)x_n) \\ &\leq \alpha_n\big[||p||^2 - 2\langle p, Jx_n\rangle + ||x_n||^2\big] + (1-\alpha_n)\big[||p||^2 - 2\langle p, J(J_* oT_n)x_n\rangle \\ &+ ||T_n x_n||^2\big] - \alpha_n(1-\alpha_n)g(||Jx_n - J(J_* oT_n x_n)x_n||) \\ &= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, (J_* oT_n)x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - T_n x_n||),\end{aligned}$$

which yields

$$\phi(p, u_n) \leq \phi(p, x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - T_n x_n||). \tag{11}$$

Hence, $\phi(p, u_n) \leq \phi(p, x_n) \forall n \in \mathbb{N}$ so that

$$F_J(\Gamma) \subset C_n \forall n \in \mathbb{N}. \tag{12}$$

Now, we show $F_J(\Gamma) \subset C_n \cap Q_n \forall n \in \mathbb{N}$. We use induction. Since J is one-to-one, we have, $J(C_n \cap Q_n) = JC_n \cap JQ_n\ \forall n \in \mathbb{N}$ and so $J(C_n \cap Q_n)$ is closed and convex $\forall n \in \mathbb{N}$. By Lemma 2.5,

$C_n \cap Q_n$ is a sunny generalized nonexpansive retract of E. Clearly, $F_J(\Gamma) \subset C_0 \cap Q_0$. Assume $F_J(\Gamma) \subset C_{n-1} \cap Q_{n-1}$ for some $n \in \mathbb{N}$. Take $u \in F_J(\Gamma)$. Then, $u \in C_{n-1} \cap Q_{n-1}$. Since $x_n = R_{C_{n-1} \cap Q_{n-1}} x$, it follows from Lemma 2.6(i) that $\langle x - x_n, Jx_n - Ju \rangle \geq 0 \; \forall u \in C_{n-1} \cap Q_{n-1}$. Hence, $u \in Q_n$. This implies,

$$F_J(\Gamma) \subset Q_n \forall n \in \mathbb{N}. \tag{13}$$

From inclusions (12) and (13), we obtain that $F_J(\Gamma) \subset C_n \cap Q_n \forall n \in \mathbb{N}$. Hence, $x_{n+1} := R_{C_n \cap Q_n} x$ is well defined.

Step 2. $\lim_{n\to\infty} ||x_n - u_n|| = 0$.

We first prove that the sequences $\{x_n\}, \{u_n\}$, and $\{T_n x_n\}$ are bounded. From the definition of Q_n and by Lemma 2.6(ii) we have that $x_n = R_{Q_n} x$ and

$$\phi(x, x_n) = \phi(x, R_{Q_n} x) \leq \phi(x, u) - \phi(R_{Q_n} x, x_n) \leq \phi(x, u) \; \forall u \in F_J(\Gamma) \subset Q_n.$$

This implies that $\{\phi(x, x_n)\}$ is bounded. Hence, $\{x_n\}, \{u_n\}$, and $\{T_n x_n\}$ are bounded. Since $x_{n+1} := R_{C_n \cap Q_n} x \in C_n \cap Q_n$, and $x_n = R_{Q_n} x$, we have from Lemma 2.6(ii) that $\phi(x, x_n) \leq \phi(x, x_{n+1}) \; \forall n \in \mathbb{N}$. So, $lim_{n\to\infty} \phi(x, x_n)$ exists. Using Lemma 2.6(ii) and $x_n = R_{Q_n} x$, we obtain that for arbitrary positive integers $m, n, \; m > n$,

$$\begin{aligned} \phi(x_n, x_m) &= \phi(R_{Q_n} x, x_m) \leq \phi(x, x_m) - \phi(x, R_{Q_n} x) \\ &= \phi(x, x_m) - \phi(x, x_n) \to 0 \text{ as } m, n \to \infty. \end{aligned} \tag{14}$$

Hence, $lim_{n,m\to\infty} \phi(x_n, x_m) = 0$. By Lemma 2.8, we conclude that $||x_n - x_m|| \to 0, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in C, and so, there exists $x^* \in C$ such that $x_n \to x^*$. Observe that $x_{n+1} \in Q_n \cap C_n \subset C_n$. Hence, $\phi(x_n, u_n) \leq \phi(x_n, x_m) \to 0$ as $n \to \infty$. By Lemma 2.8, we have that $||x_n - u_n|| \to 0$ as $n \to \infty$, completing proof of Step 2.

Step 3: $\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0 \; \forall \; T \in \Gamma$.

Observe first that since J is uniformly continuous on bounded subsets of E, it follows from Step 2 that $||Ju_n - Jx_n|| \to 0$ as $n \to \infty$. From inequality (11) and for some constant $M > 0$, we obtain that:

$$\alpha_n(1-\alpha_n)g(||Jx_n - T_n x_n||) \leq \phi(p,x_n) - \phi(p,u_n) \leq 2||p||.||Ju_n - Jx_n|| + ||u_n - x_n||M.$$

Using $\liminf \alpha_n(1-\alpha_n) = a > 0$, there exists $n_0 \in \mathbb{N}$:

$$0 < \frac{a}{2} < \alpha_n(1-\alpha_n) \text{ for all } n \geq n_0.$$

Thus, we have

$$0 \leq \frac{a}{2} g(||Jx_n - T_n x_n||) \leq 2||p||.||Ju_n - Jx_n|| + ||u_n - x_n||M \ \forall\ n \geq n_0.$$

Using step 2, and properties of g, we obtain that $\lim_{n\to\infty} ||Jx_n - T_n x_n|| = 0$. Since $\{T_n\}_{n=1}^{\infty}$ satisfies the NST condition with Γ, we have that $\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0 \ \forall\ T \in \Gamma$, completing proof of Step 3.

Step 4: Finally, we prove $x^* = R_{F_J(\Gamma)}x$.

From Step 3, we know that $\lim_{n\to\infty} ||Jx_n - Tx_n|| = 0 \ \forall\ T \in \Gamma$. Also, we have proved that $x_n \to x^* \in C$. Assume now that $(J_* oT)x_n \to y^*$. Since T is J_*-closed, we have $y^* = (J_* oT)x^*$. Furthermore, by the uniform continuity of J on bounded subsets of E, we have: $Jx_n \to Jx^*$ and $J(J_* oT)x_n \to Jy^*$ as $n \to \infty$. Hence,

$$lim_{n\to\infty} ||Jx_n - Tx_n|| = lim_{n\to\infty} ||Jx_n - J(J_* oT)x_n|| = 0,$$

which implies, $||Jx^* - Jy^*|| = ||Jx^* - J(J_* oT)x^*|| = ||Jx^* - Tx^*|| = 0$, and so, $x^* \in F_J(\Gamma)$. From Lemma 2.6(ii), we obtain that

$$\phi(x, R_{F_J(\Gamma)}x) \leq \phi(x, R_{F_J(\Gamma)}x) + \phi(R_{F_J(\Gamma)}x, x^*) \leq \phi(x, x^*). \tag{15}$$

Again, using Lemma 2.6(ii), definition of x_{n+1}, and $x^* \in F_J(\Gamma) \subset C_n \cap Q_n$, we compute as follows:

$$\begin{aligned} \phi(x, x_{n+1}) &\leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_{F_J(\Gamma)}x) \\ &= \phi(x, R_{C_n \cap Q_n}x) + \phi(R_{C_n \cap Q_n}x, R_{F_J(\Gamma)}x) \leq \phi(x, R_{F_J(\Gamma)}x). \end{aligned}$$

Since $x_n \to x^*$, taking limits on both sides of the last inequality, we obtain:

$$\phi(x,x^*) \leq \phi(x,R_{F_J(\Gamma)}x). \tag{16}$$

From inequalities (15) and (16), we obtain that $\phi(x,x^*) = \phi(x,R_{F_J(\Gamma)}x)$. By the uniqueness of $R_{F_J(\Gamma)}$, we obtain that $x^* = R_{F_J(\Gamma)}x$. This completes proof of the theorem □

4. Applications

We prove the following theorem in classical Banach spaces.

Theorem 4.1. *Let $E = L_p,\ l_p, or\ W_p^m(\Omega), 1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $T_n : C \to E^*, n = 1,2,3,...$ be an infinite family of quasi-ϕ-J-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:*

$$\begin{cases} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J(J_* o T_n) x_n), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u,u_n) \leq \phi(x_n,u)\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx - Ju \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases} \tag{17}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ such that $\liminf \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $R_{F_J(\Gamma)}x$, where R is the sunny generalized nonexpansive retraction of E onto $F_J(\Gamma)$.

Proof. E is uniformly smooth and uniformly convex. The result follows from Theorem 3.5. □

Corollary 4.2. *Let $E = L_p,\ l_p, or\ W_p^m(\Omega), 1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. Let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $T : C \to E^*$ be a quasi-ϕ-J-nonexpansive map such that $F_J(T) \neq \emptyset$. Let $\{x_n\}$ be generated* *by:*

$$
(18)\qquad \begin{cases} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J(J_* oT)x_n), \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : \phi(u, u_n) \le \phi(u, x_n\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx - Ju\rangle \ge 0, \\ x_{n+1} = R_{C_n \cap Q_n} x, \end{cases}
$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ *such that* $\liminf \alpha_n(1-\alpha_n) > 0$. *Then,* $\{x_n\}$ *converges strongly to* $R_{F_J(\Gamma)}x$*, where* R *is the sunny generalized nonexpansive retraction of* E *onto* $F_J(\Gamma)$*.*

Proof. Again, E is uniformly smooth and unformly convex. Furthermore, set $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Theorem 4.1. □

Remark 2. (see e.g., Alber and Ryazantseva, [4]; p. 36) The analytical representations of duality maps are known in a number of Banach spaces. For instance, in the spaces l_p, $L_p(G)$ and $W_m^p(G)$, $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l_p}^{2-p} y \in l_q,\ y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$

$$J^{-1}x = ||x||_{l_q}^{2-q} y \in l_p,\ y = \{|x_1|^{q-2}x_1, |x_2|^{q-2}x_2, ...\},\ x = \{x_1, x_2, ...\},$$

$$Jx = ||x||_{L_p}^{2-p} |x(s)|^{p-2}x(s) \in L_q(G),\ s \in G,$$

$$J^{-1}x = ||x||_{L_q}^{2-q} |x(s)|^{q-2}x(s) \in L_p(G),\ s \in G,\ \textit{and}\ ,$$

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha(|D^\alpha x(s)|^{p-2} D^\alpha x(s)) \in W_{-m}^q(G), m > 0, s \in G.$$

Corollary 4.3. *Let* $E = H$*, a real Hilbert space. Let* C *be a nonempty closed and convex subset of* H*. Let* $T_n : C \to H, n = 1,2,3,...$ *be an infinite family of generalized nonexpansive maps and* Γ *be a family of closed and generalized nonexpansive maps from* C *to* H *such that* $\cap_{n=1}^\infty F(T_n) = F(\Gamma) \neq \emptyset$*. Assume that* $\{T_n\}$ *satisfies the NST-condition with* Γ*. Let* $\{x_n\}$ *be* *generated by:*

$$\begin{cases} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : ||u - u_n|| \leq ||u - x_n||, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, x - u \rangle \geq 0, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ *such that* $\liminf \alpha_n(1-\alpha_n) > 0$. *Then,* $\{x_n\}$ *converges strongly to* $P_\Gamma x$, *where* P *is the metric projection of* E *onto* $F(\Gamma)$.

Proof. In a Hilbert space, J is the identity operator and $\phi(x,y) = ||x-y||^2 \; \forall x, y \in H$. The result follows from Theorem 3.5. □

Corollary 4.4. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $T : C \to H$ *be a quasi nonexpansive map such that* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be generated by:*

$$\begin{cases} x_1 = x \in C; C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{u \in C_{n-1} \cap Q_{n-1} : ||u - u_n|| \leq ||u - x_n||, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, x - u \rangle \geq 0, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases} \tag{19}$$

for all $n \in \mathbb{N}, \alpha_n \in (0,1)$ *such that* $\liminf \alpha_n(1-\alpha_n) > 0$. *Then,* $\{x_n\}$ *converges strongly to* $P_\Gamma x$, *where* P *is the metric projection of* E *onto* $F(\Gamma)$.

Proof. Set $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Corollary 4.3. □

Remark 3. Theorem 3.5 is a complementary analogue of Theorem 1.1 in the sense that, while in Theorem 1.1 the family $\{T_n\}$ maps from a subset $C \subset E$ to the space E while in Theorem 3.5 the family $\{T_n\}$ maps from a subset $C \subset E$ to the dual E^*. Furthermore, in Hilbert spaces, both theorems virtually agree and yield the same conclusion.

Remark 4. Corollary 4.4 is an improvement and extension of the result of Nakajo and Takahashi [27], Martinez-Yanese and Xu [25], Qin and Su [30] in the following sense:

- The algorithm of Corollary 4.4 is more efficient than that of Martinez-Yanese and Xu [25] which requires more arithmetic at each stage to implement because of the extra equation y_n involved in the algorithm (2).
- Corollary 4.4 extends the results in Nakajo and Takahashi [27], Martinez-Yanese and Xu [25] and, Qin and Su [30] from *a nonexpansive self-map* to *a generalized nonexpansive non-self map*.

Conflict of Interests

The authors declare that there is no conflict of interests.

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