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APPROXIMATING COMMON FIXED POINT FOR FAMILY OF MULTIVALUED MEAN NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we prove some fixed points properties and demiclosedness principle for multivalued mean nonexpansive mappings in uniformly convex hyperbolic spaces. We further propose an iterative scheme for approximating a common fixed point of multivalued mean nonexpansive mappings and establish some strong and $\triangle$-convergence theorems for such mappings in the frame work of uniformly convex hyperbolic spaces. Our results presented in this paper extend and generalize corresponding results in uniformly convex Banach spaces, CAT(0) spaces and many other results in this direction.

Keywords: Multivalued mean nonexpansive mappings; uniformly convex hyperbolic spaces; strong and $\triangle$-convergence.

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1. Introduction

Let (X,d) be a metric space and K be a nonempty closed and convex subset of X. A mapping $T : K \to K$ is said to be

(i) *nonexpansive* if

$$d(Tx,Ty) \leq d(x,y),\ \forall x,y \in K,$$

(ii) *Suzuki-generalized nonexpanpansive* (or said to satisfy condition (C)) if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq d(x,y),\ \forall x,y \in K,$$

(iii) *mean nonexpansive* if

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Ty),\ \forall x,y \in C,\ a,b \geq 0,\ a+b \leq 1.$$

Remarks: It is worth mentioning that nonexpansive mappings are Suzuki-generalized nonexpansive mappings. However Suzuki [19] gave an exmaple of a Suzuki-generalized nonexpansive mapping which is not nonexpansive. We also mention that every nonexpansive mapping is a mean nonexpansive mapping. However, the following examples show that there are some mean nonexpansive mappings which are not nonexpansive.

Example 1.1. [25, 24]. Let $T : [0,1] \to [0,1]$ be a mapping defined by

$$Tx = \begin{cases} \frac{x}{5}, & x \in [0, \frac{1}{2}), \\ \frac{x}{6}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then, T is a mean nonexpansive mapping with $a = \frac{1}{3}$ and $b = \frac{2}{3}$. However, we see clearly that T is not continuous at $x = \frac{1}{2}$. Therefore, T cannot be a nonexpansive mapping.

Although it was shown in [14] that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive mappings. However, Nakprasit [14] gave the following example of a mean nonexpansive mapping which is not a Suzuki-generalized nonexpansive mapping.

Example 1.2. Let $T:[0,5]\to[0,2]$ be a mapping defined by

$$Tx=\begin{cases}2 & \text{if } x\in[0,4],\\ 1 & \text{if } x\in(4,5],\\ 0 & \text{if } x=5.\end{cases}$$

Then T is mean nonexpansive. If $x=4$ and $y=5$, then we have that T is not Suzuku-generalized nonexpansive.

The class of mean nonexpansive mappings was first introduced by Zhang [23], who proved that a mean nonexpansive mapping has a fixed point in a weakly compact convex subset, C (with normal structure) of a Banach space. Since then, authors began to study the mean nonexpansive mappings in Banach spaces. For example, Zuo [25] studied some fixed point theorems for mean nonexpansive mappings in Banach spaces and proved that under certain conditions, a mean nonexpansive mapping has a fixed point in C, where C is a nonempty and closed subset of a Banach space. Furthermore, he proved that if T is a mean nonexpansive mapping and $\{x_n\}$ is a sequence in C, then $\{x_n-Tx_n\}$ converges strongly to 0 i.e T is regular. For other extensive studies on mean nonexpansive mappings in Banach spaces, see [7, 21, 22, 16] and the references contained therein. Recently, mean nonexpansive mappings was studied in CAT(0) by Zhou and Cui [24] by approximating its fixed point using the following Ishikawa iteration: For $x_1\in C,\ \{t_n\},\{s_n\}\in[0,1]$, define $\{x_n\}$ iteratively by

$$\begin{cases}y_n=(1-s_n)x_n\oplus s_nTx_n,\\ x_{n+1}=(1-t_n)x_n\oplus t_nTy_n,\ n=1,2,\ldots.\end{cases}\tag{1.1}$$

They proved both strong and Δ-convergence theorems for the sequence $\{x_n\}$ generated by the above algorithm.

Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is well known that Banach spaces with convex structures have been studied extensively. This is because of the fact that Banach spaces are vector space, thus it is easier to introduce a convex structure in them. However, metric spaces do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The notion of convex metric spaces was first introduced by Takahashi [20] who studied the fixed point theory for nonexpansive mappings in the settings of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced to hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [5, 10, 17]). Although the class of hyperbolic spaces defined by Kohlenbach [10] is slightly restrictive than the class of hyperbolic spaces introduced in [5], it is however, more general than the class of hyperbolic spaces introduced in [17]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [10]. Some other examples of this class of hyperbolic spaces includes Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and $\mathbb{R}$-trees. For more details on hyperbolic spaces see e.g., [5, 6, 10, 17] for more discussion and examples of hyperbolic spaces.

Recently, Alagoz *et al* [15] proved a strong convergence result for a finite family of nonexpansive multivalued mappings in hyperbolic spaces. They study the following problem

Let X be a hyperbolic space and K be a nonempty convex subset of X. Let $\{T_i : \ i = 1,2...,k\}$ be a family of multivalued mappings such that $T_i : K \to P(K)$ and $P_{T_i}(x) = \{y \in T_i x : d(x,y) = d(x,T_x)\}$ is a nonexpansive mapping. Suppose that $\alpha_{in} \in [0,1]$, for all $n = 1,2,...$ and $i = 1,2,...,k$ for $x_0 \in K$ and let $\{x_n\}$ be the sequence generated by the following;

$$(1.2) \qquad \begin{cases} x_{n+1} = W\left(u_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}\right) \\ y_{(k-1)n} = W\left(u_{(k-2)n}, y_{(k-2)n,} \alpha_{(k-1)n}\right) \\ \vdots \\ y_{2n} = W\left(u_{1n}, y_{1n}, \alpha_{2n}\right) \\ y_{1n} = W\left(u_{0n}, y_{0n}, \alpha_{1n}\right) \end{cases}$$

where $u_{in} \in P_{T_{i+1}}(y_{in})$, $i = 0,1,2,...,k-1$ and $y_{0n} = x_n$

It is worth mentioning that, as far as we know, no work has been done on fixed point problems for mean nonexpansive mappings in hyperbolic spaces. Therefore, it is necessary to extend results on fixed point problems for mean nonexpansive mappings from uniformly convex Banach spaces and CAT(0) spaces to uniformly convex hyperbolic spaces, since the class of uniformly convex hyperbolic spaces generalizes the class of uniformly convex Banach spaces as well as CAT(0) spaces.

In this manuscript, we introduce the notion of multivalued mean nonexpansive mappings in hyperbolic spaces. We also prove some properties of fixed point set and demiclosedness principle of this class of maps. Moreover, strong and Δ-convergence are also proved for approximation of fixed point of this class of maps.
Thus, our results presented in this paper extend and improve the results of Zuo [25], Zhou and Cui [24], Alagoz *et al* [15] and a host of other important results in this direction.

2. Preliminaries

Throughout this paper, we carry out all our study in the frame work of hyperbolic spaces introduced by Kohlenbach in [10].

Let (X,d) be a metric space and K be a nonempty subset of X. K is said to be proximinal if there exists an element $y \in K$ such that

$$d(x,y) = d(x,K) := \inf_{z \in K} d(x,z)$$

for each $x \in X$. The collection of all nonempty compact subset of K, the collection of all nonempty closed bounded subsets and nonempty Proximinal bounded subsets of K are denoted by $C(K)$, $CB(K)$ and $P(K)$ respectively. The hausdorff metric H on $CB(X)$ is defined by

$$H(A,B) := \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\}$$

for all $A,B \in CB(X)$. Let $T : K \to CB(K)$ be a multivalued mapping. A element $x \in K$ is said to be a fixed point of T if $x \in Tx$. A multivalued mapping $T : K \to CB(X)$ is said to be mean nonexpansive if

$$H(Tx,Ty) \leq ad(x,y)+bd(x,Tx), \forall x,y \in K \quad a,b, \geq 0,\ a+b \leq 1 \tag{2.1}$$

If $b=0$ and $a=1$ (2.1) reduces to multivalued nonexpansive mapping.

Note: If X is a metric space and $CB(X)$ be the family of all nonempty closed and bounded subsets of X. Suppose $A,B \in CB(X)$ and $a \in A$, then

$$d(a,B) \leq H(A,B).$$

Indeed, from the definition of Hausdorff distance, we have

$$\begin{aligned} H(A,B) &= \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} \\ &\geq \sup_{a\in A} d(a,B) \\ &\geq d(a,B), \end{aligned}$$

which implies $d(a,B) \leq H(A,B),\ \forall a \in A$.

Definition 2.1. A hyperbolic space (X,d,W) is a metric space (X,d) together with a convexity mapping $W : X^2 \times [0,1] \to X$ satisfying

(1) $d(u,W(x,y,\alpha)) \leq \alpha d(u,x)+(1-\alpha)d(u,y)$;

(2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta|d(x,y)$;

(3) $W(x,y\alpha) = W(y,x,1-\alpha)$;

(4) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$;

for all $w,x,y,z \in X$ and $\alpha,\beta \in [0,1]$.

Example 2.2. [18] Let X be a real Banach space which is equipped with norm $||.||$. Define the function

$d : X^2 \to [0,\infty)$ by

$$d(x,y) = ||x-y||$$

as a metric on X. Let C be a nonempty bounded closed and convex subset of Banach space. Then, we have that (X,d) is a hyperbolic space with mapping $W : X^2 \times [0,1] \to X$ defined by $W(x,y,\alpha) = (1-\alpha)x+\alpha y$.

Definition 2.3. (see [18]) Let X be a hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$.

(i) A hyperbolic space is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $x,y,z \in X$

$$d(W(x,y,\frac{1}{2}),z) \leq (1-\delta)r,$$

provided $d(x,z) \leq r, d(y,z) \leq r$ and $d(x,y) \geq \varepsilon r$.

(ii) A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ε).

Definition 2.4. Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, we define a continuous functional $r(\cdot,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x).$$

The asymptotic radius $r(C,\{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x,\{x_n\}) = \inf\{r(y,\{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C,\{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively.

It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed convex subsets.

Definition 2.5. [11]. A sequence $\{x_n\}$ in X is said to be $\triangle$-convergence to $x \in X$, if x is the unique asymptotic center of $\{x_{nk}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$.

Remark 2.6. [12]. We note that the $\triangle$-convergence coincides with the usually weak convergence known in Banach spaces with the Opial property.

Lemma 2.7 ([13]). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Lemma 2.8 ([3]). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{nk}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{nk}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Lemma 2.9 ([8]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x^*) \leq c$, $\limsup_{n\to\infty} d(y_n,x^*) \leq c$ and $\lim_{n\to\infty} d(W(x_n,y_n,t_n),x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Definition 2.10. Let C be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1},x) \leq d(x_n,x).$$

Lemma 2.11 ([2]). *Let C be a nonempty closed subset of a complete metric space X and $\{x_n\}$ be a Fejér monotone sequence with respect to C. Then $\{x_n\}$ converges to some $x^* \in C$ if and only if $\lim_{n\to\infty} d(x_n,C) = 0$.*

Proposition 2.12. *Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect of C, then we have the following:*

(i) $\{x_n\}$ is bounded.

(ii) The sequence $\{d(x_n,x^*)\}$ is decreasing and converges for all $x^* \in F(T)$.

(iii) $\lim_{n\to\infty} d(x_n,F(T))$ exists.

3. **Main Result**

3.1. **Some Properties of Fixed Point Set of Multivalued Mean Nonexpansive Mappings.**

Theorem 3.1. *Let K be a nonempty closed and convex subset of a complete hyperbolic space X. Let $T : K \to CB(K)$ be a multivalued mean nonexpansive mapping with $b < 1$ and $F(T) \neq \emptyset$ and $Tx^* = \{x^*\}$ for each $x^* \in F(T)$, then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$ as follows:
Observe that

$$\begin{aligned} d(x_n,Ty) \leq H(Tx_n,Ty) &\leq ad(x_n,y)+bd(x_n,Ty), \\ \text{which implies } d(x_n,Ty) &\leq \frac{a}{1-b}d(x_n,y) \\ &\leq d(x_n,y). \end{aligned}$$

Taking $\lim_{n\to\infty}$ of both sides, we have

$$\lim_{n\to\infty} d(x_n,Ty) \leq \lim_{n\to\infty} d(x_n,y) = 0.$$

Then, by the uniqueness of limit, we have that

$$y \in Ty.$$

Hence, $F(T)$ is closed.
Next, we show that $F(T)$ is convex. Let $x,y \in F(T)$ and $\alpha \in [0,1]$. Then, we have

$$\begin{aligned} d(x,T(W(x,y,\alpha))) \leq H(Tx,T(W(x,y,\alpha))) &\leq ad(x,W(x,y,\alpha))+bd(x,T(W(x,y,\alpha))), \\ \text{which implies } d(x,T(W(x,y,\alpha))) &\leq \frac{a}{1-b}d(x,W(x,y,\alpha)) \\ &\leq d(x,W(x,y,\alpha)) \qquad (3.1) \end{aligned}$$

Using similar argument, we have

$$d(y,T(W(x,y,\alpha))) \leq d(y,W(x,y,\alpha)). \qquad (3.2)$$

Using (3.1) and (3.2), we have

$$
\begin{aligned}
d(x,y) &\leq d(x,T(W(x,y,\alpha))) + d(T(W(x,y,\alpha)),y) \\
&\leq d(x,W(x,y,\alpha)) + d(W(x,y,\alpha),y) \\
&\leq (1-\alpha)d(x,x) + \alpha d(x,y) + (1-\alpha)d(x,y) + \alpha d(y,y) \\
&\leq d(x,y).
\end{aligned}
\tag{3.3}
$$

Hence, we conclude that (3.1) and (3.2) are $d(x,T(W(x,y,\alpha))) = d(x,W(x,y,\alpha))$ and $d(y,T(W(x,y,\alpha))) = d(y,W(x,y,\alpha))$ respectively. Because if $d(x,T(W(x,y,\alpha))) < d(x,W(x,y,\alpha))$ or $d(y,T(W(x,y,\alpha))) < d(y,W(x,y,\alpha))$, then the inequality in (3.3) becomes strictly less than $(<)$, which therefore gives us a contradiction, that is, $d(x,y) < d(x,y)$. Hence, we have that

$$T(W(x,y,\alpha)) = W(x,y,\alpha) \ \forall x,y \in F(T) \text{ and } \alpha \in [0,1].$$

Thus, $W(x,y,\alpha) \in F(T)$, which implies that $F(T)$ is convex. □

Corollary 3.2. *Let K be a nonempty closed and convex subset of complete uniformly convex hyperbolic space X. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping and $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Then $F(T)$ is closed and convex.*

We now establish the demiclosedness principle for mean nonexpansive mappings in hyperbolic spaces.

Theorem 3.3. *Let K be a nonempty closed and convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T : K \to CB(K)$ be a multivalued mean nonexpansive mapping with $b < 1$ and $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x^*$. Then $x^* \in F(T)$.*

Proof. Since $\{x_n\}$ is a bounded sequence in X, we have from Lemma 2.7 that $\{x_n\}$ has a unique asymptotic center in K. Also, since $\Delta - \lim_{n\to\infty} x_n = x^*$, we have that $A(\{x_n\}) = \{x^*\}$.

Now,

$$\begin{aligned} d(x_n,Tx^*) &\le d(x_n,Tx_n)+d(Tx_n,Tx^*) \\ &\le d(x_n,Tx_n)+H(Tx_n,Tx^*) \\ &\le d(x_n,Tx_n)+ad(x_n,x^*)+bd(x_n,Tx^*) \end{aligned}$$

which implies $d(x_n,Tx^*) \le \frac{1}{1-b}[d(x_n,Tx_n)+ad(x_n,x^*)]$.

Taking $\limsup_{n\to\infty}$ of both sides, we have

$$r(Tx^*,\{x_n\}) = \limsup_{n\to\infty} d(x_n,Tx^*) \le \frac{1}{1-b}\limsup_{n\to\infty}[d(x_n,Tx_n)+ad(x_n,x^*)] \le \limsup_{n\to\infty} d(x_n,x^*) = r(x^*,\{x_n\}).$$

By the uniqueness of the asymptotic center of $\{x_n\}$, we have $Tx^* = x^*$. Hence, $x^* \in F(T)$. □

Corollary 3.4. *Let K be a nonempty closed and convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with $b < 1$ and $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x^*$. Then $x^* \in F(T)$.*

3.2. **Strong and Δ-Converges Theorems for Multi-valued Mean Nonexpansive Mappings.**

Lemma 3.5. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X. Let $T_i : K \to CB(K)$ $(i = 1,2,...,k)$ be finite family of multivalued mean nonexpansive mapping, such that $\Upsilon := \cap_{i=1}^n F(T_i) \ne \emptyset$. Suppose that $T_i(p) = \{p\}$ $\forall \in \cap_{i=1}^n F(T_i)$, $\alpha_{in} \in [0,1]$, for all $n = 1,2,...$ and $i = 1,2,...,k$ for $x_0 \in K$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_{n+1} = W\left(u_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}\right) \\ y_{(k-1)n} = W\left(u_{(k-2)n}, y_{(k-2)n}, \alpha_{(k-1)n}\right) \\ \vdots \\ y_{2n} = W\left(u_{1n}, y_{1n}, \alpha_{2n}\right) \\ y_{1n} = W\left(u_{0n}, y_{0n}, \alpha_{1n}\right) \end{cases} \tag{3.4}$$

where $u_{in} \in T_{i+1}(y_{in})$, $i = 0,1,2,...,k-1$ and $y_{0n} = x_n$, then

(i) $d(y_{in},p) \le d(x_n,p)$, $i = 1,2,...,k-1$.

(ii) $\lim_{n\to\infty} d(x_n, p)$ *exist.*

Proof.

$$\begin{aligned}
d(y_{1n},p) &= d\left(W(u_{0n},y_{0n},\alpha_{1n}),p\right) \\
&\le (1-\alpha_{1n})d(u_{0n},p)+\alpha_{1n}d(y_{0n},p) \\
&\le (1-\alpha_{1n})H\left(T_1(y_{0n}),T_1(p)\right)+\alpha_{1n}d(y_{0n},p) \\
&= (1-\alpha_{1n})\left[ad(y_{0n},p)+bd(y_{0n},p)\right]+\alpha_{1n}d(y_{0n},p) \\
&= \left[a(1-\alpha_{1n})+b(1-\alpha_{1n})+\alpha_{1n}\right]d(y_{0n},p) \\
&= \left[(a+b)(1-\alpha_{1n})+\alpha_{1n}\right]d(y_{0n},p) \\
&\le d(y_{0n},p) \\
&\le d(x_n,p). \qquad (3.5)
\end{aligned}$$

Assume that $d(y_{in},p)\le d(x_n,p)$ holds for some $1\le j\le k-2$. Then

$$\begin{aligned}
d(y_{(j+1)n},p) &= d\left(W(u_{jn},y_{jn},\alpha_{(j+1)n},p\right) \\
&\le (1-\alpha_{(j+1)n})d(u_{jn},p)+\alpha_{(j+1)n}d(y_{in},p) \\
&\le (1-\alpha_{(j+1)})H\left(T_{(j+1)n}y_{in},T_{(j+1)n}(p)\right)+\alpha_{(i+1)n}d(y_{in},p) \\
&\le d(y_{in},p) \\
&= d(x_n,p) \qquad \text{(from our assumption)}. \qquad (3.6)
\end{aligned}$$

We now show that $d(y_{in},p)\le d(x_n,p)$ for $j=k-1$.

$$\begin{aligned}
d(y_{(k-1)n},p) &= d\left(W(u_{(k-2)n},y_{(k-2)n},\alpha_{(k-1)n}),p\right) \\
&\le (1-\alpha_{(k-1)n})d(u_{(k-2)n},p)+\alpha_{(k-1)n}d(y_{(k-2)n},p) \\
&\le (1-\alpha_{(n-1)n})H\left(T_{k-1}y_{(k-2)n},T_{k-1}(p)\right)+\alpha_{(k-1)n}d(y_{(k-2)n},p)
\end{aligned}$$

(3.7)

$$
\begin{aligned}
&\leq (1-\alpha_{(k-1)n})\left[ad(y_{(k-2)n},p)+bd(y_{(k-2)n},p)\right]+\alpha_{(k-1)n}d(y_{(k-2)n},p)\\
&= (1-\alpha_{(k-1)n})\left[(a+b)d(y_{(k-2)n},p)\right]+\alpha_{(k-1)n}d(y_{(k-2)n},p)\\
&\leq (1-\alpha_{(k-1)n})d(y_{(k-2)n},p)+\alpha_{(k-1)n}d(y_{(k-2)n},p)\\
&= d(y_{(k-2)n},p)\leq d(x_n,p). \qquad (3.8)
\end{aligned}
$$

Thus, by induction, we obtain

$$d(y_{in},p) \quad \leq \quad d(x_n,p) \quad \forall \;\; i=1,2,...,k-1. \qquad (3.9)$$

Also, from (3.4) and (3.9), we obtain

$$
\begin{aligned}
d(x_{n+1},p) &= d\left(W(u_{(k-1)n},y_{(k-1)n},\alpha_{kn}),p\right)\\
&\leq (1-\alpha_{kn})d(u_{(k-1)n},p)+\alpha_{kn}d(y_{(k-1)n},p)\\
&\leq (1-\alpha_{kn})H\left(T_k(y_{(k-1)n}),T_k(p)\right)+\alpha_{kn}d(y_{(k-1)n},p)\\
&\leq (1-\alpha_{kn})\left[ad(y_{(k-1)n},p)+bd(y_{(k-1)n},p)\right]+\alpha_{kn}d(y_{(k-1)n},p)\\
&= (1-\alpha_{kn})\left[(a+b)d(y_{(k-1)n},p)\right]+\alpha_{kn}d(y_{(k-1)n},p)\\
&\leq (1-\alpha_{kn})d(y_{(k-1)n},p)+\alpha_{kn}d(y_{(k-1)n},p)\\
&= d(y_{(k-1)n},p)\\
&\leq d(x_n,p). \qquad (3.10)
\end{aligned}
$$

which implies that $\lim_{n\to\infty} d(x_n,p)$ exists for $p\in\Upsilon$. □

Lemma 3.6. *Let K be a nonempty closed subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T_i : K \to CB(K)$, $i=1,2,...,k$ be a family of multivalued mean nonexpansive mapping such that $\Upsilon := \cap_{i=1}^{k} F(T_i) \neq \emptyset$. suppose that $T_i(p)=\{p\}$ for each $p\in\Upsilon$. Let $\{x_n\}$ be defined iteratively by Algorithm* (3.4)*, then* $\lim_{n\to\infty} d(x_n,T_ix_n)=0$, $i=1,2,...,k$

Proof. By Lemma 3.5, $\lim\limits_{n\to\infty} d(x_n,p)$ exists for each $p\in\Upsilon$. Thus $\{d(x_n,p)\}$ converges to c, for some $C\geq 0$. If $c=0$, then the proof is complete. So, we suppose that $c>0$. That is

$$\lim_{n\to\infty} d(x_n,p)=c>0. \qquad (3.11)$$

Taking lim sup on both sides of (3.9), we have from (3.11) that Also for $i = 1,2,...,k$, we obtain

$$\limsup_{n\to\infty} d(y_{in}, p) \leq c, \quad i = 1,2,...,k-1. \tag{3.12}$$

$$\begin{aligned} d(u_{(i-1)n}, p) &\leq H\left(T_i(y_{(i-1)n}), T_i(p)\right) \\ &\leq ad(y_{(i-1)n}, p) + bd(y_{(i-1)n}, p) \\ &\leq d(y_{(i-1)n}, p). \end{aligned}$$

Which implies that

$$\limsup_{n\to\infty} d(u_{(i-1)n}, p) \leq c, \quad i = 1,2,...,k. \tag{3.13}$$

From (3.11), we have that $\lim_{n\to\infty} d(x_{n+1}, p) = c$, hence

$$\lim_{n\to\infty} d\left(W(u_{(k-1)n,}, y_{(k-1)n}, \alpha_{kn})\right) = c. \tag{3.14}$$

From (3.12), (3.13), (3.14) and Lemma (2.9) we have

$$\lim_{n\to\infty} d(y_{(k-1)n}, u_{(k-1)n}) = 0. \tag{3.15}$$

From (3.7), we have

$$d(x_{n+1}, p) \leq d(y_{(k-1)n}, p) \leq d(y_{(k-2)n}, p) \leq d(y_{(k-3)n}, P) \leq ... \leq d(y_{in}, p) \leq d(y_{1n}, p)$$

which implies

$$d(x_{n+1}, p) \leq d(y_{in}, p), \quad i = 1,2,...,k-1.$$

Therefore, $c \leq \liminf d(y_{in}, p)$, $i = 1,2,...,k-1$ which implies

$$c \leq \liminf_{n\to\infty} d(y_{(i-n)n}, p), \; i = 1,2,... \tag{3.16}$$

From Algorithm (3.4), we have

$$d\left(W(u_{(i-2)n}, y_{(i-2)n}, \alpha_{(i-1)n}), p\right) = d(y_{(i-1)n}, p), \; i = 1,2,...,k. \tag{3.17}$$

From (3.12),(3.16) and (3.17), we have

$$\lim_{n\to\infty} d\left(W(u_{(i-2)n}, y_{(i-2)n}, \alpha_{(i-1)n}, p\right) = \lim_{n\to\infty} d(y_{(i-1)n}, p) = c. \tag{3.18}$$

Also, from (3.12),(3.13) (3.18), we have

$$\lim_{n\to\infty} d(u_{(i-2)n}, y_{(i-2)n}) = 0. \tag{3.19}$$

Thus, by induction

$$\lim_{n\to\infty} d(y_{(i-1)n}, u_{(i-1)n}) = 0, \text{for } \ i = 1,2,...,k. \tag{3.20}$$

Also we have

$$\begin{aligned} d(y_{in}, y_{(i-1)n}) &= d\left(W(u_{(i-1)n}, y_{(i-1)n}, \alpha_{(i-1)n}\right) \\ &\leq (1-\alpha_{in})d(u_{(i-1)n}, y_{(i-1)n}) + \alpha_{in}d(y_{(i-1)n}, y_{(i-1)n}), \end{aligned}$$

which implies from (3.20) that

$$\lim_{n\to\infty} d(y_{in}, y_{(i-1)n}) = 0, \quad i = 1,2,...,k. \tag{3.21}$$

Again from (3.4), we have

$$\begin{aligned} d(x_n, y_{1n}) &= d(x_n, W(u_{0n}, y_{0n}, \alpha_{in})) \\ &\leq (1-\alpha_{1n})d(x_n, u_{0n}) + \alpha_{in}d(x_n, y_{0n}) \\ &= (1-\alpha_{1n})d(x_n, u_{0n}) + \alpha_{in}d(x_n, x_n) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_n, y_{1n}) = 0. \tag{3.22}$$

Using triangular inequality, we obtain

$$d(x_n, y_{in}) \leq d(x_n, y_{1n}) + d(y_{1n}, y_{12}) + d(y_{1n}, y_{12}) + ... + d(y_{(i-1)n}, y_{in}), \quad i = 1,2,...,k-1.$$

From (3.20) and (3.21), we have

$$\lim_{n\to\infty} d(x_n, y_{in}) = 0, \quad i = 1,2,...,k-1. \tag{3.23}$$

Now, we estimate $d(x_n, T_i x_n)$.

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + d(u_{(i-1)n}, T_i x_n) \\ &\leq d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + H\left(T_i(y_{(i-1)n}), T_i x_n\right) \\ &\leq d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + ad(y_{(i-1)n}, x_n) + bd(y_{(i-1)n}, T_i x_n) \\ &\leq d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + ad(y_{(i-1)n}, x_n) + bd(y_{(i-1)n}, x_n) + bd(x_n, T_i x_n), \end{aligned}$$

which implies

$$\begin{aligned} d(x_n, T_i x_n) &\leq \frac{(1+b)}{1-b} d(x_n, y_{(i-1)n}) + \frac{a}{1-b} d(x_n, y_{(i-1)n}) + \frac{1}{1-b} d(y_{(i-1)n}, u_{(i-1)n}) \\ &\leq 2d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}). \end{aligned} \tag{3.24}$$

Hence from (3.20), (3.23) and (3.24), we obtain

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \;\; i = 1, 2, ..., k. \tag{3.25}$$

Theorem 3.7. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic spaces X which is monotone modulus of uniform convexity η. Let T_i, $\cap_{i=1}^{k} F(T_i)$ be as in Lemma 3.6. Then $\{x_n\}$ defined by Algorithm* (3.4) Δ- *converges to $p \in \cap_{i=1}^{k} F(T_i)$.*

Let $p \in \Upsilon$ Then $p \in F(T_i)$ for $i = 1, 2, ..., k$ By lemma 3.5 $\{x_n\}$ is bounded and also $\lim\limits_{n\to\infty} d(x_n, p)$ exists. The $\{x_n\}$ has a unique asymptotic center. In other words, we have $A(\{x_n\}) = \{x\}$. Let $\{w_n\}$ be a subsequence of $\{x_n\}$ subsequence of $\{x_n\}$ such that $A(\{w_n\}) = \{x^*\}$. From Lemma 3.6 we get $\lim_{n\to\infty}(w_n, T_i(w_n) = 0$. We claim that x^* is a fixed point T_1

To prove this, we take another sequence $\{v_m\}$ in $T_1(x^*)$. Then,

$$\begin{aligned} r(v_m,\{w_n\}) &= \limsup_{n\to\infty} d(v_m,w_n) \\ &\leq \lim_{n\to\infty}\{d(v_m,T_1(w_n))+d(T_1(w_n),w_n)\} \\ &\leq \lim_{n\to\infty}\{H(T_1(x^*),T_1(w_n))+d(T_1(w_n),w_n))\} \\ &\leq \lim_{n\to\infty}[ad(x^*,w_n)+bd(x^*,T_1(w_n))] \\ &\leq \lim_{n\to\infty}[ad(x^*,w_n)+bd(x^*,(w_n)+bd(w_n,T_1(w_n))] \\ &\leq \lim_{n\to\infty}[(a+b)d(x^*,w_n)+bd(w_n,T_1(w_n))] \\ &\leq \limsup_{n\to\infty} d(x^*,w_n) \\ &= r(x^*,\{w_n\}) \end{aligned}$$

so we have $|r(v_m,\{w_n\})-r(x^*,\{w_n\})|\to 0$. for $m\to\infty$. By Lemma 2.3 we get $\lim_{n\to\infty} v_m = x^*$. Hence $T_1(x^*)$ is either closed or bounded. Consequently $\lim_{n\to\infty} v_m = x^* \in T_1(x^*)$. Similarly $x^*\in T_2(x^*)$, $x^*\in T_3(x^*)$,...,$x^*\in T_k(x^*)$. □

Theorem 3.8. *Let K be a nonempty closed convex subset of a hyperbolic space X and Let T_i and Υ be as defined in Lemma 3.6 and let $\{x_n\}$ be the iterative process defined in* (3.4)*, then $\{x_n\}$ converges to p in F if and only if* $\lim_{n\to\infty} d(x_n,F)=0$.

Proof. If $\{x_n\}$ converges to $p\in\Upsilon$, then $\lim_{n\to\infty} d(x_n,p)=0$. since $0\leq d(x_n,\Upsilon)\leq d(x_n,p)$, we have $\lim_{n\to\infty} d(x_n,\Upsilon)=0$. Conversely, suppose that $\lim_{n\to\infty} d(x_n,\Upsilon)=0$. By Lemma 3.5, we have

$$d(x_{n+1},p)\leq d(x_n,\Upsilon)$$

which implies

$$d(x_{n+1},\Upsilon)\leq d(x_n,\Upsilon).$$

This implies that $\lim_{n\to\infty} d(x_n,\Upsilon)$ exists. Therefore, by the hypothesis of our Theorem , $\liminf_{n\to\infty} d(x_n,\Upsilon)=0$. Thus we have that $\lim_{n\to\infty} d(x_n,\Upsilon)=0$. Now we show that $\{x_n\}$ is a Cauchy sequence in K. Let $m,n\in\mathbb{N}$ and assume $m>n$. Then it follows that $d(x_m,p)\leq d(x_n,p)$ for all $p\in\Upsilon$. Thus we get,

$$d(x_m,x_n)\leq d(x_m,p)+d(x_n,p)\leq 2d(x_n,p).$$

Taking inf on the set Υ, we have $d(x_m,x_n) \leq d(x_n,\Upsilon)$. Now as $n,m \to \infty$ in the inequality $d(x_m,x_n) \leq d(x_n,\Upsilon)$ we have that it converges to a point $q \in K$. Next we show that $q \in F(T_1)$. Indeed by $d(x_n,F(T_1)) = \inf_{y \in F(T_1)} d(x_n,y)$. So for each $\varepsilon > 0$, there exists $p_n^{(\varepsilon)} \in F(T_1)$ such that,

$$d(x_n, p_n^{(\varepsilon)}) < d(x_n,F(T_1)) + \frac{\varepsilon}{2}$$

This implies $\lim_{n\to\infty} d(x_n, p_n^{(\varepsilon)}) \leq \frac{\varepsilon}{2}$. Since $d(p_n^{(\varepsilon)},q) \leq d(x_n, p_n^{(\varepsilon)}) + d(x_n,q)$ it follows that $\lim\limits_{n\to\infty} d(p_n^{(\varepsilon)},q) \leq \frac{\varepsilon}{2}$. Hence we obtain

$$\begin{aligned}
d(T_1(q),q) &\leq d(q,p_n^{(\varepsilon)}) + d(p_n^{(\varepsilon)},T_1(q)) \\
&\leq d(q,p_n^{(\varepsilon)}) + H(T_1(p_n^{(\varepsilon)}),T_1(q)) \\
&\leq d(q,p_n^{(\varepsilon)}) + ad(p_n^{(\varepsilon)},q) + bd(p_n^{(\varepsilon)},T_1(q)) \\
&= d(q,p_n^{(\varepsilon)}) + ad(p_n^{(\varepsilon)},q) + bd(p_n^{(\varepsilon)},q) \\
&= d(q,p_n^{(\varepsilon)}) + (a+b)d(p_n^{(\varepsilon)},q) \\
&\leq 2d(p_n^{(\varepsilon)},q)
\end{aligned}$$

which shows that $d(T_1(q),q) < \varepsilon$. So, $d(T_1(q),q) = 0$. since ε was arbitrary chosen. Similarly we get for any $i = 1,2,...,k$ we obtain $d(T_i(q),q) = 0$. Since Υ is closed, then $q \in \Upsilon$. □

Conflict of Interest

The authors declare that there is no conflict of interests.

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