FIXED AND BEST PROXIMITY POINTS FOR CYCLIC WEAKLY
CONTRACTION MAPPINGS

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Abstract. In this paper we obtain some fixed and best proximity point theorems for cyclic \((\psi, \varphi)\)-weakly contraction mappings. The results obtained herein extend some recent results.

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1. Introduction and Preliminaries

Though this paper \(\mathbb{N}\) denotes the set of naturals and \(X\) a metric space \((X, d)\). Let \(A\) and \(B\) be nonempty subsets of a metric space \(X\). A mapping \(T : A \cup B \to A \cup B\) is called a cyclic mapping if \(T(A) \subseteq B\) and \(T(B) \subseteq A\). A point \(z \in A \cup B\) is said to be fixed point of \(T\) if \(Tz = z\) and a best proximity point of \(T\) if \(d(z, Tz) = d(A, B)\), where \(d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}\). All mappings do not have fixed points. For example the mapping \(T : [0, \infty) \to [0, \infty)\) defined by \(Tx = 1 + x\), has no fixed points, since \(x\) is never equal to \(x + 1\) for any \(x \in [0, \infty)\). If the fixed-point equation \(Tx = x\) does not possesses a solution, it is contemplated to resolve a problem finding an element

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such that \( x \) is in proximity to \( Tx \) in some sense. Best proximity theorems analyze the conditions under which the optimization problem, namely \( \min_{x \in A} d(x, Tx) \) has a solution \([9]\).

Kirk et al. \([7]\) obtained the following interesting fixed point theorem for cyclic mappings.

**Theorem 1.1.** Let \( A \) and \( B \) be nonempty closed subsets of a complete metric space \( X \) and \( T : A \cup B \to A \cup B \) be a cyclic mapping. Assume that there exists \( \lambda \in (0, 1) \) such that

\[
    d(Tx, Ty) \leq \lambda d(x, y)
\]

(1.1)

for all \( x \in A \) and \( y \in B \). Then \( T \) has a unique fixed point in \( A \cap B \).

The condition (1.1) entails \( A \cap B \) being nonempty. Eldred and Veeramani \([4]\) modified the condition (1.1) for the case \( A \cap B = \emptyset \) as follows:

\[
    d(Tx, Ty) \leq \lambda d(x, y) + (1 - \lambda) d(A, B)
\]

(1.2)

for all \( x \in A \) and \( y \in B \), where \( \lambda \in (0, 1) \). The mapping \( T \) satisfying condition (1.2) is called a cyclic contraction. Eldred and Veeramani \([4, \text{Th. 3.10}]\) obtained a unique best proximity point for the mapping \( T \) in a uniformly convex Banach space setting. Subsequently, a number of extensions and generalizations of their results appeared in \([1, 2, 5, 10]\) and many others.

Recently, Al-Tagafi and Shahzad \([1]\) introduced the notion of cyclic \( \varphi \)-contractions and obtained some existence results for this new class of mappings. In this paper we, extend cyclic \( \varphi \)-contractions and introduce the notion of cyclic \( (\psi, \varphi) \)-weakly contractions. Subsequently, this notion is utilized to obtain some fixed and best proximity point theorems which generalize certain results of \([1, 4]\) and \([7]\).

**2. Cyclic \((\psi, \varphi)\)-weakly contractions**

Throughout this section \( \Phi \) denotes the class of the functions \( \varphi : [0, \infty) \to [0, \infty) \) satisfying:

(a) \( \varphi \) is continuous and monotone nondecreasing,

(b) \( \varphi(t) = 0 \iff t = 0 \).
The function $\varphi \in \Phi$ is also known as altering distance function (see, for instance, [6]).

Now we introduce the following notion of a cyclic $(\psi, \varphi)$-weakly contraction mapping.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic mapping. The mapping $T$ will be called a cyclic $(\psi, \varphi)$-weakly contraction if, $\psi, \varphi \in \Phi$ and

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) + \varphi(d(A, B)), \quad (2.1)$$

for all $x \in A$ and $y \in B$ (see also, [3, 8]).

**Remark 2.2.** We remark that:

1. A cyclic $\varphi$-contraction is cyclic $(\psi, \varphi)$-weakly contraction with $\psi(t) = t$ for $t \geq 0$.

2. A cyclic contraction is cyclic $(\psi, \varphi)$-weakly contraction with $\psi(t) = t$, $\varphi(t) = (1 - \lambda)t$ for $t \geq 0$ and $\lambda \in (0, 1)$.

Recall that, a Banach space $X$ is said to be:

(a) uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:

$$\begin{align*}
\|x - p\| \leq R \\
\|y - p\| \leq R \\
\|x - y\| \geq r
\end{align*} \Rightarrow \left\| \frac{x + y}{2} - p \right\| \leq \left(1 - \delta \left(\frac{r}{2^R}\right)\right) R;
$$

(b) strictly convex if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\begin{align*}
\|x - p\| \leq R \\
\|y - p\| \leq R \\
x \neq y
\end{align*} \Rightarrow \left\| \frac{x + y}{2} - p \right\| < R.$$

We begin with the following lemma.

**Lemma 2.3.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A \cup B$, define $x_{n+1} :=Tx_n$ for each $n \geq 0$. Then for all $x \in A$ and $y \in B$,

(i) $\varphi(d(A, B)) \leq \varphi(d(x, y))$;
(ii) $d(Tx, Ty) \leq d(x, y)$; and

(iii) $d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n) \leq d(x_{n+1}, x_n)$ for each $n \geq 0$.

**Proof.** (i) Since $d(A, B) = d(x, y)$ for all $x \in A$ and $y \in B$ and $\varphi \in \Phi$, we have $\varphi(d(A, B)) \leq \varphi(d(x, y))$.

(ii) Since $T$ is a cyclic $(\psi, \varphi)$-weakly contraction, we have

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$.

From (i) $\varphi(d(A, B)) \leq \varphi(d(x, y))$, hence

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)).$$

Since $\varphi \in \Phi$, it follows that $d(Tx, Ty) \leq d(x, y)$.

(iii) Since $T$ is a cyclic $(\psi, \varphi)$-weakly contraction, we have

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(Tx_{n+1}, Tx_n)) \leq \psi(d(x_{n+1}, x_n)) - \varphi(d(x_{n+1}, x_n)) + \varphi(d(A, B))$$

for all $n \geq 0$. Using (i) and (ii), we get

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(Tx_{n+1}, Tx_n)) \leq \psi(d(x_{n+1}, x_n)).$$

Now since $\psi \in \Phi$, it follows that

$$d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n) \leq d(x_{n+1}, x_n).$$

**Theorem 2.4.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $T : A \cup B \to A \cup B$ a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A \cup B$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $\lim_{n \to \infty} d(x_n, Tx_n) = d(A, B)$.

**Proof.** It follows from Lemma 2.3 (iii) that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Thus $\lim_{n \to \infty} d(x_n, x_{n+1}) = r_0$ for some $r_0 \geq d(A, B)$. If $d(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \geq 1$ then we
are done. Assume that \( d(x_n, x_{n+1}) > 0 \) for each \( n \geq 1 \). Since \( T \) is a cyclic \((\psi, \varphi)\)-weakly contraction, we have
\[
\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_{n+1}, x_{n+2})) + \varphi(d(A, B)) \tag{2.2}
\]
for each \( n \geq 1 \).

Now by Lemma 2.3 (i) and (2.2), we have
\[
\varphi(d(A, B)) \leq \varphi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \psi(d(x_{n+1}, x_{n+2})) + (A, B). \tag{2.3}
\]
Since \( \psi, \varphi \in \Phi \) and \( d(x_n, x_{n+1}) \geq r_0 \geq d(A, B) \), it follows from (2.3) that
\[
\lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) = \varphi(r_0) = \varphi(d(A, B))
\]
for each \( n \geq 1 \). Since \( \varphi \in \Phi \), \( r_0 = d(A, B) \).

In view of Remark 2.2 (1) and (2), Proposition 3.1 of [4] and Theorem 3 of [1] are special cases of Theorem 2.4.

**Theorem 2.5.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \) and \( T : A \cup B \to A \cup B \) a cyclic \((\psi, \varphi)\)-weakly contraction mapping. For \( x_0 \in A \), define \( x_{n+1} := Tx_n \) for each \( n \geq 0 \). If \( \{x_{2n}\} \) has a convergent subsequence in \( A \), then there exists a point \( z \in A \) such that \( d(z, Tz) = d(A, B) \).

**Proof.** Let \( \{x_{2n_k}\} \) be a subsequence of \( \{x_{2n}\} \) such that \( \lim_{k \to \infty} x_{2n_k} = z \). Since
\[
d(A, B) \leq d(z, x_{2n_k-1}) \leq d(z, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})
\]
for each \( k \geq 1 \), it follows from Theorem 2.4 that \( \lim_{k \to \infty} d(x_{2n_k}, x_{2n_k-1}) = d(A, B) \). Since
\[
d(A, B) \leq d(x_{2n_k}, Tz) = d(x_{2n_k-1}, z)
\]
for each \( k \geq 1 \), it follows that \( d(z, Tz) = d(A, B) \).

In view of Remark 2.2 (2), Proposition 3.2 of [4] is a special case of Theorem 2.5.

**Corollary 2.6.** [1, Theorem 4]. Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \) and \( T : A \cup B \to A \cup B \) a cyclic \( \varphi \)-weakly contraction mapping. For \( x_0 \in A \), define
$x_{n+1} := Tx_n$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in $A$, then there exists a point $z \in A$ such that $d(z, Tz) = d(A, B)$.

**Proof.** It comes from Theorem 2.5, when $\varphi(t) = t$.

**Lemma 2.7.** Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex. Let $T : A \cup B \to A \cup B$ be a cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then

$$\lim_{n \to \infty} \|x_{2n+2} - x_{2n}\| = 0 \text{ and } \lim_{n \to \infty} \|x_{2n+3} - x_{2n+1}\| = 0.$$ 

**Proof.** Suppose that $\lim_{n \to \infty} \|x_{2n+2} - x_{2n}\| > 0$. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there is an $n_k \geq k$ satisfying

$$\|x_{2n_k+2} - x_{2n_k}\| \geq \varepsilon_0. \quad (2.4)$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > d(A, B)$ and choose $\varepsilon$ such that

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)} \right\}.$$ 

By Theorem 2.4, there exist $N_1$ and $N_2$ such that

$$\|x_{2n_k+2} - x_{2n_k+1}\| \leq d(A, B) + \varepsilon \text{ and } \|x_{2n_k+1} - x_{2n_k}\| \leq d(A, B) + \varepsilon \quad (2.5)$$

for all $n_k \geq N_1, N_2$. Let $N := \max\{N_1, N_2\}$. It follows from $(2.4), (2.5)$ and the uniform convexity of $X$ that

$$\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| \leq \left( 1 - \delta \left( \frac{\varepsilon_0}{d(A, B) + \varepsilon} \right) \right) (d(A, B) + \varepsilon)$$

for all $n_k \geq N$. As $\frac{x_{2n_k+2} + x_{2n_k}}{2} \in A$, the choice of $\varepsilon$ and the fact that $\delta$ is strictly increasing imply that

$$\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| < d(A, B),$$

for all $n_k \geq N$, a contradiction. Therefore $\lim_{n \to \infty} \|x_{2n+2} - x_{2n}\| = 0$. Similarly we can show that $\lim_{n \to \infty} \|x_{2n+3} - x_{2n+1}\| = 0$.

**Theorem 2.8.** Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex. Let $T : A \cup B \to A \cup B$ be a cyclic $(\psi, \varphi)$-weakly contraction mapping.
mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then for each $\varepsilon > 0$, there exists a positive integer $N_0$ such that for all $m > n \geq N_0$
\[ \|x_{2m} - x_{2n+1}\| < d(A, B) + \varepsilon. \]

**Proof.** Suppose the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$, there exist $m_k > n_k \geq k$ satisfying
\[ \|x_{2m_k} - x_{2n_k+1}\| \geq d(A, B) + \varepsilon_0 \quad \text{and} \quad \|x_{2(m_k-1)} - x_{2n_k+1}\| < d(A, B) + \varepsilon_0. \] (2.6)

By the triangle inequality and (2.6), we have
\[
\begin{align*}
d(A, B) + \varepsilon_0 & \leq \|x_{2m_k} - x_{2n_k+1}\| \\
& \leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - x_{2n_k+1}\| \\
& < \|x_{2m_k} - x_{2(m_k-1)}\| + d(A, B) + \varepsilon_0.
\end{align*}
\]

Making $k \to \infty$ and using Lemma 2.7, we get
\[ \lim_{k \to \infty} \|x_{2m_k} - x_{2n_k+1}\| = d(A, B) + \varepsilon_0. \] (2.7)

Since $T$ is a cyclic $(\psi, \varphi)$-weakly contraction, by Lemma 2.3 (i) and (ii), and the triangle inequality, we obtain
\[
\begin{align*}
\psi(\|x_{2m_k} - x_{2n_k+1}\|) & \leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+2} - x_{2m_k+3}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\
& \leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+1} - x_{2m_k+2}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\
& \leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k} - x_{2m_k+1}\|) \\
& \quad - \varphi(\|x_{2m_k} - x_{2m_k+1}\|) + \varphi(d(A, B)) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|) \\
& \leq \psi(\|x_{2m_k} - x_{2m_k+2}\|) + \psi(\|x_{2m_k} - x_{2m_k+1}\|) + \psi(\|x_{2m_k+3} - x_{2n_k+1}\|). \quad \text{(2.8)}
\end{align*}
\]

Since $\psi \in \Phi$, (2.8) implies that
\[
\begin{align*}
\|x_{2m_k} - x_{2n_k+1}\| & \leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k} - x_{2m_k+1}\| - \varphi(\|x_{2m_k} - x_{2m_k+1}\|) \\
& \quad + \varphi(d(A, B)) + \|x_{2m_k+3} - x_{2n_k+1}\| \\
& \leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k} - x_{2m_k+1}\| + \|x_{2m_k+3} - x_{2n_k+1}\|. 
\end{align*}
\]
Making \( k \to \infty \) and using (2.7) and Lemma 2.7, we get
\[
d(A, B) + \varepsilon_0 \leq d(A, B) + \varepsilon_0 - \lim_{k \to \infty} \varphi\left(\|x_{2m_k} - x_{2m_{k+1}}\|\right) + \varphi(d(A, B)) \\
\leq d(A, B) + \varepsilon_0.
\]
Hence
\[
\lim_{k \to \infty} \varphi\left(\|x_{2m_k} - x_{2m_{k+1}}\|\right) = \varphi(d(A, B)).
\]
(2.9)
Since \( \varphi \in \Phi \), by (2.6) and (2.9)
\[
\varphi(d(A, B) + \varepsilon_0) \leq \lim_{k \to \infty} \varphi\left(\|x_{2m_k} - x_{2m_{k+1}}\|\right) \\
= \varphi(d(A, B)) < \varphi(d(A, B) + \varepsilon_0),
\]
a contradiction and hence the Theorem.

**Theorem 2.9.** Let \( A \) and \( B \) be nonempty subsets of a uniformly convex Banach space \( X \) such that \( A \) is closed. Let \( T : A \cup B \to A \cup B \) be cyclic \((\psi, \varphi)\)-weakly contraction mapping. For \( x_0 \in A \) define \( x_{n+1} := Tx_n \) for each \( n \geq 0 \). If \( d(A, B) = 0 \), then \( T \) has a unique fixed point \( z \in A \cap B \).

**Proof.** Let \( \varepsilon > 0 \) be given. By Theorem 2.4, there exists \( N_1 \) such that
\[
\|x_{2n} - x_{2n+1}\| < \varepsilon
\]
for all \( n \geq N_1 \). By Theorem 2.8, there exists \( N_2 \) such that
\[
\|x_{2m} - x_{2m+1}\| < \varepsilon
\]
for all \( m > n \geq N_2 \). Let \( N := \max\{N_1, N_2\} \). Then
\[
\|x_{2m} - x_{2n}\| \leq \|x_{2m} - x_{2n+1}\| + \|x_{2n+1} - x_{2n}\| < 2\varepsilon
\]
for all \( m > n \geq N \). Thus \( \{x_{2n}\} \) is a Cauchy sequence in \( A \). Since \( X \) is complete and \( A \) is closed, it follows that \( x_{2n} \to z \in A \) as \( n \to \infty \). Now by Theorem 2.5, we have
\[
d(z, Tz) = d(A, B) = 0,
\]
and \( z \) is a fixed point of \( T \). The uniqueness of fixed point follows easily.

**Corollary 2.10.**[1, Theorem 6]. Let \( A \) and \( B \) be nonempty subsets of a uniformly convex Banach space \( X \) such that \( A \) is closed. Let \( T : A \cup B \to A \cup B \) be cyclic \( \varphi \)-weakly
contraction mapping. For \( x_0 \in A \) define \( x_{n+1} := Tx_n \) for each \( n \geq 0 \). If \( d(A, B) = 0 \), then \( T \) has a unique fixed point \( z \in A \cap B \).

**Proof.** It comes from Theorem 2.9, when \( \psi(t) = t \).

**Theorem 2.11.** Let \( A \) and \( B \) be nonempty subsets of a uniformly convex Banach space \( X \) such that \( A \) is closed and convex. Let \( T : A \cup B \to A \cup B \) be cyclic \((\psi, \varphi)\)-weakly contraction mapping. For \( x_0 \in A \) define \( x_{n+1} := Tx_n \) for each \( n \geq 0 \). Then \( \{x_{2n}\} \in A \) and \( \{x_{2n+1}\} \in B \) are Cauchy sequences.

**Proof.** If \( d(A, B) = 0 \), the result follows from Theorem 2.9. So assume that \( d(A, B) > 0 \). Suppose that the sequence \( \{x_{2n}\} \) is not Cauchy. Then there exists \( \varepsilon_0 > 0 \) such that for each \( k \geq 1 \), there exist \( m_k > n_k \geq k \) satisfying

\[
\|x_{2m_k} - x_{2n_k}\| \geq \varepsilon_0. \tag{2.10}
\]

Choose \( 0 < \gamma < 1 \) such that \( \frac{\varepsilon_0}{\gamma} > d(A, B) \) and choose \( \varepsilon \) such that

\[
0 < \varepsilon < \min \left\{ \frac{\varepsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)} \right\}.
\]

By Theorem 2.4, there exists \( N_1 \) such that

\[
\|x_{2n_k} - x_{2n_k+1}\| < d(A, B) + \varepsilon. \tag{2.11}
\]

for all \( n_k \geq N_1 \). By Theorem 2.8, there exists \( N_2 \) such that

\[
\|x_{2m_k} - x_{2n_k+1}\| < d(A, B) + \varepsilon. \tag{2.12}
\]

for all \( n_k \geq N_2 \). Let \( N := \max\{N_1, N_2\} \). It follows from (2.11), (2.12) and the uniform convexity of \( X \) that

\[
\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| \leq \left( 1 - \delta \left( \frac{\varepsilon_0}{d(A, B) + \varepsilon} \right) \right) (d(A, B) + \varepsilon)
\]

for all \( n_k \geq N \). The choice of \( \varepsilon \) and the fact that \( \delta \) is strictly increasing imply that

\[
\left\| \frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1} \right\| < d(A, B),
\]

for all \( n_k \geq N \), a contradiction. Thus \( \{x_{2n}\} \) is a Cauchy sequence in \( A \). Similarly, we can show that \( \{x_{2n+1}\} \) is a Cauchy sequence in \( B \).
Theorem 2.12. Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is closed and convex. Let $T : A \cup B \to A \cup B$ be cyclic $(\psi, \varphi)$-weakly contraction mapping. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $z \in A$ such that $x_{2n} \to z$, $T^2z = z$ and $\|z - Tz\| = d(A, B)$.

Proof. By Theorem 2.11, $\{x_{2n}\}$ is a Cauchy sequence in $A$ and hence $x_{2n} \to z \in A$ as $n \to \infty$. By Theorem 2.5, $\|z - Tz\| = d(A, B)$. To show that $z$ is unique we assume that there exists a $y \in A$ such that $\|y - Ty\| = d(A, B)$ with $T^2y = y$. By Lemma 2.3 (i) and (ii), we have

$$
\|Ty - z\| = \|Ty - T^2z\| \leq \|y - Tz\| \quad \text{and} \quad \|Tz - y\| = \|Tz - T^2y\| \leq \|z - Ty\|.
$$

Thus $\|Tz - y\| = \|z - Ty\|$. In fact $\|z - Ty\| = d(A, B)$; otherwise $\|z - Ty\| > d(A, B)$ and since $T$ is cyclic $(\psi, \varphi)$-weakly contraction, it follows that

$$
\psi(\|Tz - y\|) = \psi(\|Tz - T^2y\|)
\leq \psi(\|z - Ty\|) - \varphi(\|z - Ty\|) + \varphi(d(A, B))
< \psi(\|z - Ty\|) - \varphi(A, B) + \varphi(A, B)
= \psi(\|z - Ty\|) = \psi(Tz - y),
$$

a contradiction. Thus $\|z - Ty\| = d(A, B) = \|y - Tz\|$. Now by convexity of $A$ and $X$

$$
0 < \left\| \frac{y + z}{2} - Ty \right\| = \left\| \frac{y - Ty}{2} + \frac{z - Ty}{2} \right\| < d(A, B),
$$

a contradiction. Thus $y = z$.

In view of Remark 2.2 (1), Theorem 8 of [1] is a special case of Theorem 2.12.

References


