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CONVERGENCE THEOREMS TO COMMON FIXED POINTS OF MULTI-VALUED ρ-QUASI-NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

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Abstract. We constructed a Mann-type iteration process for a finite family of multivalued mappings in modular function spaces and the sequence of the algorithm is proved to be a common ρ-approximate fixed point sequence. Using this algorithm we also proved the ρ-converge theorems to common fixed points of finite family of mappings. By doing so we extended very recent results of Zegeye et al.

Keywords: common fixed point; ρ-quasi-nonexapansive mappings; modular function spaces; Ishikawa Iterates.

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1. Introduction

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The fixed point theory in metric and Banach spaces attracted the attention of well known analysts for the last ten decades. Recently, many mathematicians put their effort in the study of fixed point theory of multivalued mappings in Banach spaces (see eg. [18, 19, 20, 21] and the references therein). Along with the study of fixed point theory on the metric and Banach spaces, many mathematicians are interested in working to this theory on modular spaces, spaces that generalize some classes of Banach spaces.

The idea of modular function spaces first initiated by Nakano [17] which are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces such as Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces,(see eg.[16] and the references therein) and many others.

The importance for applications of modular function spaces consists in the richness of structure of modular function spaces, that besides being Banach spaces (or F-spaces in a more general settings)are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural and modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces. Khamsi et al. [10] gave an example of a mapping which is $\rho-$nonexpansive but it is not norm nonexpansive. They demonstrated that for a mapping T to be norm nonexpansive in a modular function space L_ρ, a stronger than $\rho-$nonexpansiveness assumption is needed:

$$\rho(\lambda(Tx-Ty)) \leq \rho(\lambda(x-y)) \text{ for any } \lambda \geq 0.$$

Khamsi et al. [10] were the pioneer of the study of fixed point theory in the context of modular function. Kozlowski [12] has contributed a lot towards the study of modular function spaces both on his own and with his collaborators and also Kuman [13] obtained some fixed point theorems for nonexpansive mappings in modular function spaces. In 2006, Dhompongsa et.al in [4] proved the existence of fixed points of multivalued $\rho-contarction$ and ρ- nonexpansive mappings in modular function spaces.

However, it has been also shown in [2] that there are fixed point free nonexpansive mppings on, even weakly compact, subsets of uniformly convex Banach spaces. But in modular function spaces the existence of fixed points of multivalued, more difficult than single valued mappings, nonexpansive mappings is guaranteed(see eg. [4], Corollary 3.5)

In some cases the existence of fixed points can be guaranteed by inspection. In such cases studying about the approximation technique is very important than the existence. Therefore, in the fixed point theory, approximating fixed points of nonlinear mappings is an important isue as the existence. Now a days, the approximation processes of fixed points of nonexpansive mappings in modular function spaces is one of the flourishing areas of nonlinear analysis . In 2012, Dehaish and Kozlowski [3] initiated the approximation of fixed points in modular function spaces by Mann iterative process for asymptotically pointwise nonexpansive mappings.

In 2014, Abdou et al. [1] have proved the convergence theorem on common fixed point of two ρ-nonexpansive, single valued, mappings in modular function spaces.

In 2014, Khan and Abbas [7] generalized the results of Dehaish and Kozlowski [3] to the multivalued mapping setting to approximate the fixed points of a ρ-nonexpansive mapping in modular function space by using the Mann iteration process [15]. Zegeye et.al [22] extended the results of [7] to the common fixed points of finite family of ρ- nonexpavsive multivalued mappings.

In 2016, Zegeye et al. [22] proved the convergence theorems of Mann-type iterative algorithm to common fixed points of finite family of a multivalued ρ-nonexpansive mappings in modular function spaces. The have defined Mann-type iteration process as follows and proved the following theorems:

Let $C \subset L_\rho$ be nonempty convex set and $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$f_{n+1} = \alpha_{n,0} f_n + \alpha_{n,1} g_{n,1} + \alpha_{n,2} g_{n,2} + ... + \alpha_{n,m} g_{n,m}, \tag{1.1}$$

where $g_{n,i} \in P_\rho^{T_i}(f_n)$ and $\{\alpha_{n,i}\} \subset (0,1)$ is bounded away from 0 and 1 such that $\sum_{i=0}^{m} \alpha_{n,i} = 1$.

Theorem 1.1. [22] *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1,2,...,m$. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in equation (1.1). Then,*

(1) $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$;
(2) $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$, for all $i = 1,2,...,m$.

Theorem 1.2. [22] *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1,2,...,m$. Let $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $\{f_n\}$ be as defined in equation (1.1). Then, $\{f_n\}$ ρ-converges to a point in F if and only if $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$.*

Theorem 1.3. [22] *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings satisfying Condition (II) such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1,2,...,m$. Assumethat $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in equation (1.1). Then $\{f_n\}$ ρ-converges to a point in $F := \bigcap_{i=1}^{m} F_\rho(T_i)$.*

It is our purpose in this paper to construct a Mann-type algorithm and show that the sequence is ρ-approximate common fixed pint sequence for a finite family of ρ-quasi-nonexpansive multivalued mappings and under certain mild conditions it ρ-converges to a common fixed point in modular function spaces.

2. Preliminaries

Now, we recall some basic notions and facts about modular spaces as formulated by Kozlowski [11]. For more details the reader may consult [6, 12] and the references therein.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathscr{P}$ be a nontrivial δ-ring of subsets of Ω such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $\Omega = \cup_{n=1} K_n$. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from $\mathscr{P}$. By $\mathscr{M}_\infty$ we denote the space of all extended measurable functions, that is, all functions $f : \Omega \longrightarrow [-\infty,\infty]$ such that there exists a sequence $\{g_n\} \subset \mathscr{E}$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f(w)$ for all $w \in \Omega$. By χ_A we denote the characteristic function of the set A.

Definition 2.1. Let $\rho : \mathscr{M}_\infty \longrightarrow [0,\infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudo-modular if it satisfies the following:

a) $\rho(0) = 0$;
b) ρ is monotone; that is, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$ where $f,g \in \mathscr{M}_\infty$;
c) ρ is orthogonally subadditive; that is, $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$ for any $A,B \in \Sigma$ such that $A \cap B \neq \emptyset$, where $f \in \mathscr{M}_\infty$;
d) ρ has Fatou property; that is, $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies that $\rho(f_n) \uparrow \rho(f)$ where $f \in \mathscr{M}_\infty$ and
e) ρ is order continuous in $\mathscr{E}$; that is, $g_n \in \mathscr{E}$ and $|g_n| \downarrow 0$ implies that $\rho(g_n) \downarrow 0$.

We say that a set $A \in \Sigma$ is ρ-null if $\rho(g\chi_A) = 0$ for every $g \in \mathscr{E}$. We say that a property p holds ρ-almost every where if the exceptional set $\{w \in \Omega : p(w)\ \textit{does not hold}\}$ is ρ-null. As usual we identify any pair of measurable functions f *and* g differing only on ρ-null set by $f = g$ ρ-a.e. With this in mind we define

$$\mathscr{M} = \{f \in \mathscr{M}_\infty : |f(w)| < \infty\ \rho - a.e\},$$

where $f \in \mathscr{M}$ is actually an equivalence class of functions equal ρ-a.e rather than an individual function.

Definition 2.2. Let ρ be a regular convex function pseudo-modular.

a) We say that ρ is a regular convex function semi-modular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies that $f = 0$ ρ-a.e.
b) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies that $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\mathscr{R}$.

Remark 2.1. Let us denote $\rho(f,E) = \rho(f\chi_E)$ for $f \in \mathcal{M}, E \in \Sigma$. Also by convention for $\alpha > 0$ we will write $\rho(\alpha,E)$ instead of $\rho(\alpha\chi_E)$. We will use these notations when convenient. It is easy to prove that $\rho(f,E)$ is a convex function modular in the sense of Definition 2.1.

Remark 2.2. Note that if ρ is a regular convex function modular, then to verify that a set E is ρ-null it suffices to prove that there exists $\alpha > 0$ such that $\rho(\alpha,E) = 0$.

Definition 2.3. Let ρ be a convex function modular.

(1) A modular function space is the vector space $L_\rho(\Omega,\Sigma)$ or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ } as \text{ } \lambda \to 0\}.$$

(2) The following formula defines a norm in L_ρ frequently called the *Luxemburg norm*:

$$\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}.$$

Definition 2.4. [12] Let $\rho \in \mathscr{R}$.

a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f(\rho)$ if $\rho(f_n - f) \to 0$.

b) A sequence $\{f_n\}$ in L_ρ is called a ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $n,m \to \infty$.

c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $\{f_n\} \subset B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B.

d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter is finite; the ρ-diameter of B is defined as
$\delta_\rho(B) = \sup\{\rho(f-g) : f \in B, g \in B\}$.

e) A set $B \subset L_\rho$ is called ρ-compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ and an $f \in B$ such that $\rho(f_{n_k} - f) \to 0$.

f) A set $B \subset L_\rho$ is called ρ- a.e closed if if for any $\{f_n\}$ in B,which ρ-a.e converges to some f, we have $f \in B$.

g) A set $B \subset L_\rho$ is called ρ-a.e compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e converges to some $f \in B$.

h) Let $f \in L_\rho$ *and* $B \subset L_\rho$. The ρ-distance between f *and* B is defined as $d_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}$.

Theorem 2.1. [12] Let $\rho \in \mathscr{R}$. L_ρ is complete with respect to ρ-convergence.

The following definition plays very crucial role in modular function space and following this definition we get an important property that characterizes the convergence in function modular by the norm (*Luxemburg norm*)convergence(see the detail in [12]).

Definition 2.5. Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$.

Proposition 2.2. [12] The following statements are equivalent:

(i) ρ satisfies the Δ_2-condition.

(ii) $\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for all $\lambda > 0$ if and only if $\|f_n - f\|_\rho \to 0$.

Definition 2.6. [12] Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - type condition if there exists a constant $0 < k < \infty$ such that for every $f \in L_\rho$, we have $\rho(2f) \leq k\rho(f)$.

Remark 2.3. If ρ satisfies the Δ_2 - type condition, then it satisfies Δ_2 - property, and that the converse is not true (see, e.g.,[12]).

Let $\rho \in \mathscr{R}$ and C be a nonempty subset of the modular space L_ρ. We denote a collection of all nonempty ρ-closed and ρ- bounded subsets of C by $\mathscr{C}_\rho(C)$ and a collection of all nonempty ρ-compact subsets of C by $\mathscr{K}_\rho(C)$.

Definition 2.7. [7] *A set $C \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in C$ such that*

$$\rho(f-g) = d_\rho(f,C) = \inf\{\rho(f-h) : h \in C\}.$$

We denote the family of nonempty ρ-bounded ρ-proximinal subsets of C by $P_\rho(C)$.

Definition 2.8. [7] *We define a Hausdorff distance on $\mathscr{C}_\rho(C)$ by,*

$$H_\rho(A,B) = max\{\sup_{f\in A} d_\rho(f,B),\ \sup_{g\in B} d_\rho(g,A)\},$$

$A, B \in \mathscr{C}_\rho(C)$.

Definition 2.9. [7] *A multivalued mapping $T : C \to \mathscr{C}_\rho(C)$ is called ρ-Lipschitzian if there exists a number $k \geq 0$ such that*

$$H_\rho(T(f), T(g)) \leq k\rho(f-g)\ for\ all\ f,g \in C.$$

If $k \leq 1$ then, T is called ρ-nonexpansive and if $k < 1$, T is called ρ-contractive.

Definition 2.10. [22] A mapping $T : C \to \mathscr{C}_\rho(C)$ is said to be ρ-quasi-nonexpansive if $F_\rho(T) \neq \emptyset$ and $H_\rho(T(f), T(h)) \leq \rho(f-h)$ for all $f \in C$ and $h \in F_\rho(T)$.

Remark 2.4. *If a mapping $T : C \longrightarrow \mathscr{C}_\rho(C)$ is ρ-nonexpansive with $F_\rho(T) \neq \emptyset$, then T is ρ-quasi-nonexpansive.*

We find the following uniform convexity type property definitions of the function modular ρ in [8] and [9].

Definition 2.11. *Let $\rho \in \mathscr{R}$. Let $t \in (0,1)$, $r > 0$, $\varepsilon > 0$. Define,*

$$D_1(r,\varepsilon) = \{(f,g) : f,g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f-g) \geq \varepsilon r\}.$$

Let

$$\delta_1^t(r,\varepsilon) = \inf\{1 - \frac{1}{r}\rho(tf + (1-t)g : (f,g) \in D_1(r,\varepsilon)\},\ if\ D_1(r,\varepsilon) \neq \emptyset$$

and $\delta_1^t(r,\varepsilon) = 1,\ if\ D_1(r,\varepsilon) = \emptyset$.

We will use the following notational convention: $\delta_1 = \delta_1^{\frac{1}{2}}$.

Definition 2.12. *We say that ρ satisfies (UC1) if for every $r > 0,\ \varepsilon > 0,\ \delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. We say that ρ satisfies (UUC1) if for every $s \geq 0,\ \varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending only on s and ε such that*

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0,\ for\ any\ r > s.$$

Definition 2.13. *A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $t_n \geq a$, for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $0 < b < 1$ such that $t_n \leq b$, for every $n \in \mathbb{N}$.*

Lemma 2.3. [3] *Let ρ satisfies (UUC1) and let $\{t_n\} \subset (0,1)$ be bounded away from both 0 and 1. If there exists $R > 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \leq R,\ \limsup_{n\to\infty} \rho(g_n) \leq R$$

and

$$\lim_{n\to\infty} \rho(t_n f_n + (1-t_n)g_n) = R,\ then$$

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

Definition 2.14. [22] A family of mappings $T_i : C \to C_\rho(C)$, $i = 1,2,...,m$ is said to satisfy Condition (II) if there exists a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0,\infty)$ such that

$$d_\rho(f, T_i(f)) \geq \varphi(d_\rho(f, \cap_{i=1}^m F_\rho(T_i)))$$

for some $i = 1,2,...,m$.

3. Main results

In what follows we shall use the following iteration scheme for finite family of mappings in modular function spaces which was first introduced by Zegeye et al. [22]. Let $C \subset L_\rho$ be nonempty convex set and $T_i : C \to \mathscr{C}_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$f_{n+1} = \alpha_{0,n} f_n + \alpha_{1,n} u_{1,n} + \alpha_{2,n} u_{2,n} + ... + \alpha_{m,n} u_{m,n},, \tag{3.1}$$

where $u_{i,n} \in T_i(f_n)$ and $\{\alpha_{i,n}\} \subset (0,1)$ is bounded away from 0 and 1 such that $\sum_{i=0}^m \alpha_{i,n} = 1$.

Now we prove our first theorem.

Lemma 3.1. *Let $\rho \in \mathscr{R}$ and C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Suppose $T_i : C \to \mathscr{C}_\rho(C)$, $i = 1,2,...,m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Assume that $F := \bigcap_{i=1}^m F_\rho(T_i) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$. Then, the common fixed point set F is ρ-closed.*

Proof. Let $\{g_n\}$ be a sequence in F such that $\lim_{n\to\infty} \rho(g_n - f) = 0$ for some $f \in C$. We must show that $f \in F$. Since $g_n \in F_\rho(T_i)$ and T_i is ρ-quasi-nonexpansive for $i = 1,2,...,m$,

$$H_\rho(T_i g_n, T_i f) \leq \rho(g_n - f).$$

Now, let $h_i \in T_i f$ for $i = 1,2,...,m$. Then,

$$\rho(h_i - g_n) \leq H_\rho(T_i f, T_i g_n) \leq \rho(g_n - f).$$

Now, by convexity of ρ, we get

$$\begin{aligned}\rho(\frac{h_i-f}{2}) &= \rho(\frac{h_i-g_n}{2}+\frac{g_n-f}{2})\\ &\leq \frac{1}{2}(\rho(h_i-g_n)+\rho(g_n-f))\\ &\leq \rho(g_n-f)\to\ 0\ , as\ n\to\infty.\end{aligned}$$

This implies that, $h_i = f$, that is, $f \in T_i f$ for all $i = 1,2,...,m$. Since $T_i(f)$ is ρ-closed, $f \in F_\rho(T_i)$ for all $i = 1,2,...,m$. Thus, the common fixed point set is ρ-closed.

Theorem 3.2. Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to \mathscr{C}_\rho(C)$, $i = 1,2,...,m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Assume that $F := \bigcap_{i=1}^{m} F_\rho(T_i) \neq \emptyset$ and $T_i(p) = \{p\}$ for all $p \in F$. Let $\{f_n\}$ be as defined in equation (3.1).Then,

(1) $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$;
(2) $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$, for all $i = 1,2,...,m$.

Proof. Let $p \in F$ be arbitrary. From equation (3.1), we have

$$\begin{aligned}\rho(f_{n+1}-p) &= \rho(\alpha_{0,n}f_n+\alpha_{1,n}u_{1,n}+\alpha_{2,n}u_{2,n}+...+\alpha_{m,n}u_{m,n}-p)\\ &= \rho(\alpha_{0,n}(f_n-p)+\alpha_{1,n}(u_{1,n}-p)+\alpha_{2,n}(u_{2,n}-p)+...+\alpha_{m,n}(u_{m,n}-p))\\ &\leq \alpha_{0,n}\rho(f_n-p)+\alpha_{1,n}\rho(u_{1,n}-p)+\alpha_{2,n}\rho(u_{2,n}-p)+...+\alpha_{m,n}\rho(u_{m,n}-p)..\end{aligned} \tag{3.2}$$

Since $u_{i,n} \in T_i(f_n)$, $T_i(p) = \{p\}$ and T_i is ρ-quasi-nonexpansive for $i = 1,2,...,m$, we have

$$\begin{aligned}\rho(u_{i,n}-p) &\leq H_\rho(T_i(f_n),T_i(p))\\ &\leq \rho(f_n-p)\end{aligned} \tag{3.3}$$

for all $i = 1,2,...,m$.
Now, from (3.2), (3.3) and the assumption $\sum_{i=0}^{m} \alpha_{i,n} = 1$, we obtain that

$$\rho(f_{n+1}-p) \leq \rho(f_n-p),\ \text{for all}\ p \in F.$$

Therefore, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.
Let

$$\lim_{n\to\infty} \rho(f_n - p) = r \tag{3.4}$$

for some $r \geq 0$. So from (3.3) and (3.4), we have

$$\limsup_{n\to\infty} \rho(u_{m,n} - p) \leq r. \tag{3.5}$$

Next, observe that

$$\begin{aligned}
\rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n - p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}(u_{i,n}-p)\Big) &\leq \frac{\alpha_{0,n}}{1-\alpha_{m,n}}\rho(f_n-p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}\rho(u_{i,n}-p) \\
&\leq \frac{1}{1-\alpha_{m,n}}\sum_{i=0}^{m-1}\alpha_{i,n}\rho(f_n-p) \\
&= \rho(f_n-p).
\end{aligned}$$

Therefore, from (3.4) we obtain

$$\limsup_{n\to\infty} \rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}(f_n - p) + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}(u_{i,n}-p)\Big) \leq r. \tag{3.6}$$

Thus (3.4), (3.5), (3.6) and Lemma 2.3, give that

$$\lim_{n\to\infty} \rho\Big(\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - u_{m,n}\Big) = 0. \tag{3.7}$$

Now,

$$\begin{aligned}
\rho(f_{n+1} - u_{m,n}) &= \rho\Big(\alpha_{0,n}f_n + \sum_{i=1}^{m}\alpha_{i,n}u_{i,n} - u_{m,n}\Big) \\
&= \rho\Big(\alpha_{0,n}f_n + \sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - (1-\alpha_{n,m})u_{m,n}\Big) \\
&= \rho\Big((1-\alpha_{m,n})\Big[\frac{\alpha_{0,n}}{1-\alpha_{m,n}}f_n + \frac{1}{1-\alpha_{m,n}}\sum_{i=1}^{m-1}\alpha_{i,n}u_{i,n} - u_{m,n}\Big]\Big).
\end{aligned}$$

From (3.7) by the Δ_2- property of ρ and Proposition 2.2, we get that

$$\lim_{n\to\infty} \rho(f_{n+1} - u_{m,n}) = 0.$$

In the same way, we can show that

$$\lim_{n\to\infty} \rho(f_{n+1} - f_n) = 0$$

and

$$\lim_{n\to\infty}\rho(f_{n+1}-u_{i,n})=0,$$

for all $i=1,2,...,m-1$.

Now by the convexity of ρ, we get

$$\begin{aligned}\rho(\frac{f_n-u_{i,n}}{2}) &= \rho(\frac{f_n-f_{n+1}}{2}+\frac{f_{n+1}-u_{i,n}}{2})\\ &\leq \frac{\rho(f_n-f_{n+1})}{2}+\frac{\rho(f_{n+1}-u_{i,n})}{2}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\rho(\frac{f_n-u_{i,n}}{2})=0.$$

Since ρ satisfies Δ_2- property by Proposition 2.2 we obtain

$$\lim_{n\to\infty}\rho(f_n-u_{i,n})=0, \tag{3.8}$$

for all $i=1,2,...,m$.

Since $u_{i,n}\in T_i(f_n)$, we have

$$d_\rho(f_n,T_i(f_n))\leq\rho(f_n-u_{i,n})$$

for all $i=1,2,...,m$.

Therefore,

$$\lim_{n\to\infty}d_\rho(f_n,T_i(f_n))=0$$

follows immediately from (3.8) for all $i=1,2,...,m$.

Theorem 3.3. *Let $\rho\in\mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C\subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i: C\to\mathscr{C}_\rho(C)$, $i=1,2,...,m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings. Let $F:=\bigcap_{i=1}^{m}F_\rho(T_i)\neq\emptyset$ with $T_i(p)=\{p\}$ for all $p\in F$ and $\{f_n\}$ be as defined in (3.1). Then, $\{f_n\}$ ρ-converges to a point in F if and only if $\liminf_{n\to\infty}d_\rho(f_n,F)=0$.*

Proof. The necessity is straight forward. Now, we prove the other way round. Suppose that $\liminf_{n\to\infty}d_\rho(f_n,F)=0$. By Theorem 3.2, we have

$$\rho(f_{n+1}-p)\leq\rho(f_n-p),\ \text{for all p }\in F.$$

This implies that

$$d_\rho(f_{n+1},F)\le d_\rho(f_n,F).$$

So that $\lim\limits_{n\to\infty} d_\rho(f_n,F)$ exists. But by hypothesis, $\liminf\limits_{n\to\infty} d_\rho(f_n,F)=0$. Therefore, it must be the case that

$$\lim_{n\to\infty} d_\rho(f_n,F)=0.$$

Consider a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a sequence $\{p_k\}$ in F such that

$$\rho(f_{n_k}-p_k)<\frac{1}{3^k}\ and\ \rho(f_{n_{k+j}}-f_{n_k})<\frac{1}{3^{k+j}} \text{ for all } k,\ j\ge 1.$$

We show that $\{p_k\}$ is a ρ-Cauchy sequence in F. Observe that for $j\ge 1$,

$$\begin{aligned}\rho\Big(\frac{p_{k+j}-p_k}{3}\Big)&=\rho\Big(\frac{p_{k+j}-f_{n_{k+j}}+f_{n_{k+j}}-f_{n_k}+f_{n_k}-p_k}{3}\Big)\\ &\le\frac{1}{3}\rho(p_{k+j}-f_{n_{k+j}})+\frac{1}{3}\rho(f_{n_{k+j}}-f_{n_k})+\frac{1}{3}\rho(f_{n_k}-p_k)\\ &<\frac{1}{3^{k+j+1}}+\frac{1}{3^{k+1}}+\frac{1}{3^{k+j+1}}\longrightarrow 0\ as\ k,\ j\to\infty.\end{aligned}$$

Since ρ-satisfies Δ_2-condition, by Proposition 2.2, $\{p_k\}$ is a ρ-Cauchy sequence in F. But, we know that L_ρ is complete with respect to ρ-convergence and F is ρ-closed, by Lemma 3.1, $\{p_k\}$ ρ-converges to a point in F, say p. Next we show that $\{f_n\}$ ρ-converges to p. In fact, by convexity of ρ, we have

$$\begin{aligned}\rho\Big(\frac{f_{n_k}-p}{2}\Big)&=\rho\Big(\frac{f_{n_k}-p_k+p_k-p}{2}\Big)\\ &\le\rho(f_{n_k}-p_k)+\rho(p_k-p)\to 0\ as\ k\to\infty.\end{aligned}$$

By the Δ_2-condition of ρ, we have $\rho(f_{n_k}-p)\to 0$, as k $\to\infty$. Since $\lim\limits_{n\to\infty}\rho(f_n-p)$ exists, the sequence $\{f_n\}$ ρ-converges to p. Which completes the proof.

Theorem 3.4 Let $\rho\in\mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C\subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i: C\to\mathscr{C}_\rho(C)$, $i=1,2,...,m$, is a finite family of ρ-quasi-nonexpansive multi-valued mappings satisfying Condition (II). Assume that $F:=\bigcap\limits_{i=1}^{m}F_\rho(T_i)\neq\emptyset$ and $T_i(p)=\{p\}$ for all $p\in F$. Let $\{f_n\}$ be as defined in (3.1). Then $\{f_n\}$ ρ-converges to a point in $F:=\cap_{i=1}^{m}F_\rho(T_i)$.

Proof. From Theorem 3.2, we have $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$ and $d_\rho(f_{n+1}, F) \le d_\rho(f_n, F)$. Hence, $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. It then follows from the definition of Condition (II) that,

$$0 = \lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) \ge \lim_{n\to\infty} \varphi(d_\rho(f_n, F))$$

for some $i = 1, 2, ..., m$.

Thus, $\lim_{n\to\infty} \varphi(d_\rho(f_n, F)) = 0$. Since φ is nondecreasing, $\varphi(0) = 0$, and $\varphi(t) > 0$ for all $t \in (0, \infty)$, we have

$$\lim_{n\to\infty} d_\rho(f_n, F) = 0.$$

The rest of the proof follows from the proof of Theorem 3.3, as desired.

The following example supports that, the class of ρ-quasi-nonexpansive multivalued mappings is superior than the class of ρ-nonexpansive multi-valued mappings in modular function spaces.

Example 3.1. *Consider $L_\rho = \mathbb{R}$. Consider $\rho(x) = |x|$, the usual absolute value on reals. Then clearly, ρ is convex function modular on $\mathbb{R}$. Let $C \subset L_\rho$ be given by $C = [0, 5]$. Define $T : C \to \mathscr{C}_\rho(C)$ by*

$$T(x) = \begin{cases} [0, \frac{x}{5}], if x \neq 5 \\ \{1\}, if x = 5 \end{cases}$$

then T is ρ-quasi-nonexpansive ; but it is not ρ-nonexpansive.

Remark 3.1. All the results obained in this paper, extede the results of Zegey et al. [22] in that the class of mappings we have used includes the class of mappings used in [22].

Conflict of Interests

The authors declare that there is no conflict of interests.

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