COUPLED FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTIONS INVOLVING GENERALIZED ALTERING DISTANCE FUNCTION IN METRIC SPACES

R. A. RASHWAN*, S. I. MOUSTAFA

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

Copyright © 2018 R. A. Rashwan and S. I. Moustafa. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we study some unique coupled coincidence points for rational type contractions involving generalized altering distance functions in metric spaces. Our results unify and generalize various known comparable results from the current literature, Gupta et. al. [15], Nashine and Aydi [21], Rashwan and Saleh [25] and many known results. An example is also given to support our main results.

Keywords: coupled coincidence point; coupled fixed point; generalized altering distance function; rational contractions; mixed monotone property; metric and partially ordered metric spaces.

2010 AMS Subject Classification: 54H25, 47H10.

1. Introduction

In 1922, The Banach contraction mapping theorem [5] was introduced and it remains a powerful tool in nonlinear analysis. Generalizations of this theorem have been obtained in several branches of mathematics. In 1984 Khan et al. [16] initiated the use of a control function that
alters distance between two points in a metric space. Such mappings are called an altering distances.

In 2006, Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point and proved some coupled fixed point results under certain conditions, in ordered metric spaces and applied their results to prove the existence and uniqueness of a solution for a periodic boundary value problem. Many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces [1, 2, 3, 4, 10, 11, 18, 26].

2. Preliminaries

At first we state the following definitions and results.

**Definition 1.1.** [27] An altering distance function is a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

(i): $\phi$ is continuous and non-decreasing,

(ii): $\phi(t) = 0 \iff t = 0$.

Altering distances have been generalized to a two-variable function by Choudhury and Dutta [9] and to a three-variable function by Choudhury [8] and was applied for obtaining fixed point results in metric spaces. In [23], Rao et al. introduced the generalized altering distance function in five variables as a generalization of three variables.

**Definition 1.2.** [9] Let $\Psi_2$ be the set of all functions $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ that satisfy,

(i): $\psi$ is continuous and non-decreasing in its two variables,

(ii): $\psi(x, y) = 0 \iff x = y = 0$.

**Definition 1.3.** [23] Let $\Psi_5$ denote the set of all functions $\psi : [0, +\infty)^5 \to [0, +\infty)$. Then $\psi$ is said to be a generalized altering distance function iff,

(i): $\psi$ is continuous and non-decreasing in all five variables,

(ii): $\psi(t_1, t_2, t_3, t_4, t_5) = 0 \iff t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

**Definition 1.4.** [17] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say that $F$ has the mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in its
first argument and monotone g-non-increasing in its second argument, that is, for any \( x, y \in X \),

\[ x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \text{ implies } F(x_1, y) \preceq F(x_2, y) \]

and

\[ y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \text{ implies } F(x, y_1) \succeq F(x, y_2). \]

**Definition 1.5.** [17] An element \((x, y) \in X \times X\) is called a coupled coincidence point of the two mappings \( F : X \times X \to X \) and \( g : X \to X \) if

\[ F(x, y) = g(x), F(y, x) = g(y). \]

If \( g \) is the identity mapping, then \((x, y)\) is called a coupled fixed point of the mapping \( F \).

The aim of this paper is to establish some unique coupled fixed point results for two self-mappings \( F : X \times X \to X \) and \( g : X \to X \) which satisfy rational type contractive condition involving a generalized altering distance function. In order to validate our established theorems and corollaries, an example has been given.

### 3. Main results

Let \( \Phi \) denote all functions \( \phi : [0, \infty) \to [0, \infty) \) which satisfy

(i): \( \phi \) is continuous and non-decreasing,

(ii): \( \phi(t) = 0 \) if and only if \( t = 0 \),

(iii): \( \phi(t + s) \leq \phi(t) + \phi(s), \ \forall t, s \in [0, \infty), \)

(iv): \( \phi \left( \frac{t}{2} \right) \leq \frac{1}{2} \phi(t), \ \forall t \in [0, \infty). \)

**Theorem 3.1.** Let \((X, d)\) be a metric space and \( F : X \times X \to X \), \( g : X \to X \) be two self-mappings such that \( F(X, X) \subseteq g(X) \) and \( g(X) \) is complete subspace of \( X \). Suppose that there exist \( \phi \in \Phi \) and \( \psi_1, \psi_2 \in \Psi_2 \) for which \( \psi_1(x, x) \leq \phi(x) \) and

\[
\phi(d(F(x, y), F(u, v))) \leq \psi_1 \left( \frac{K(x, u) + K(y, v)}{2} \right) - \psi_2 \left( \frac{K(x, u) + K(y, v)}{2} \right),
\]

where

\[
K(x, u) = \left( \frac{d(gu, F(u, v))[1 + d(gx, F(x, y))] + d(gx, gu)}{1 + d(gx, gu)} \right).
\]
and

\[ K(y, v) = \left( \frac{d(gv, F(v, u))[1 + d(gy, F(y, x))]}{1 + d(gy, gv)}, d(gy, gv) \right), \]

for all \( x, y, u, v \in X \). Then there exist \( x, y \in X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \). That is, \( F \) and \( g \) have a coupled coincidence point.

**Proof.** Choose \( x_0 \) and \( y_0 \) in \( X \) and set \( gx_1 = F(x_0, y_0) \) and \( gy_1 = F(y_0, x_0) \). Repeating this process, set

\[
\begin{cases}
  gx_{n+1} = F(x_n, y_n), \\
  gy_{n+1} = F(y_n, x_n).
\end{cases}
\]

This can be done because \( F(X, X) \subseteq g(X) \). If for some \( n \in \mathbb{N} \),

\[
gx_n = gx_{n+1} \text{ and } gy_n = gy_{n+1}
\]

then, by (2) \((x_n, y_n)\) is a coupled coincidence point of \( F \) and \( g \). Hence we can assume that at least

\[
gx_n \neq gx_{n+1} \text{ or } gy_n \neq gy_{n+1}.
\]

From (1), we have

\[
\phi(d(gx_n, gx_{n+1})) = \phi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\
\leq \psi_1 \left( \frac{K(x_{n-1}, x_n) + K(y_{n-1}, y_n)}{2} \right) - \psi_2 \left( \frac{K(x_{n-1}, x_n) + K(y_{n-1}, y_n)}{2} \right),
\]

where

\[
K(x_{n-1}, x_n) = \left( \frac{d(gx_n, F(x_n, y_n))[1 + d(gx_{n-1}, F(x_{n-1}, y_{n-1}))]}{1 + d(gx_{n-1}, gx_n)}, d(gx_{n-1}, gx_n) \right)
\]

\[ = \left( \frac{d(gx_n, gx_{n+1})[1 + d(gx_{n-1}, gx_n)]}{1 + d(gx_{n-1}, gx_n)}, d(gx_{n-1}, gx_n) \right)
\]

\[ = \left( d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n) \right)
\]

and

\[
K(y_{n-1}, y_n) = \left( d(gy_n, gy_{n+1}), d(gy_{n-1}, gy_n) \right).
\]
Hence
\[ K(x_{n-1}, x_n) + K(y_{n-1}, y_n) = \left( \frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}, \frac{d(gx_n, gx_n) + d(gy_n, gy_n)}{2} \right) = \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) \]

where \( w_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \). Similarly we get
\[ \phi(d(gy_n, gy_{n+1})) \leq \psi_1 \left( \frac{K(y_{n-1}, y_n) + K(x_{n-1}, x_n)}{2} \right) - \psi_2 \left( \frac{K(y_{n-1}, y_n) + K(x_{n-1}, x_n)}{2} \right) \leq \psi_1 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) - \psi_2 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right). \]

By adding (3) and (4) and using properties (iii) and (iv) of \( \phi \), we get
\[ \phi(w_n) \leq \phi(d(gx_n, gx_{n+1})) + \phi(d(gy_n, gy_{n+1})) \leq 2 \psi_1 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) - 2 \psi_2 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) \]

(5)
\[ \phi\left( \frac{w_n}{2} \right) \leq \psi_1 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) - \psi_2 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right). \]

Under the assumption that at least one of \( gx_n \neq gx_{n+1} \) and \( gy_n \neq gy_{n+1} \) holds for all \( n \), then \( w_n \neq 0, \forall n \) and \( \psi_2 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) > 0 \). Also, if \( w_{n-1} < w_n \) then
\[ \phi\left( \frac{w_n}{2} \right) < \psi_1 \left( \frac{w_n}{2}, \frac{w_{n-1}}{2} \right) \]
\[ \leq \psi_1 \left( \frac{w_n}{2}, \frac{w_n}{2} \right) \]
\[ = \phi\left( \frac{w_n}{2} \right), \]

which is a contradiction. Then \( w_n \leq w_{n-1} \forall n \), that is the sequence \( \{w_n\} \) is non-increasing. Therefore there is some \( w \geq 0 \) such that
\[ \lim_{n \to \infty} w_n = w. \]

We shall show that \( w = 0 \). Suppose the contrary, that \( w > 0 \). Taking the limit as \( n \) tends to \( \infty \) of both sides of (5) and using the continuity of \( \phi, \psi_1 \) and \( \psi_2 \) imply that
\[ \phi\left( \frac{w}{2} \right) \leq \psi_1 \left( \frac{w}{2}, \frac{w}{2} \right) - \psi_2 \left( \frac{w}{2}, \frac{w}{2} \right) \]
\[ < \psi_1 \left( \frac{w}{2}, \frac{w}{2} \right) = \phi\left( \frac{w}{2} \right), \]
a contradiction. Thus \( w = 0 \) and

\[
\lim_{n \to \infty} d(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(gy_n, gy_{n+1}) = 0.
\]

In what follows, we shall prove that \( \{gx_n\} \) and \( \{gy_n\} \) are Cauchy sequences in \( X \). Suppose the contrary, that at least one of them is not Cauchy. Then there exists an \( \varepsilon > 0 \) for which we can find two sequences \( \{m_k\} \) and \( \{n_k\} \) with \( n_k > m_k > k \) such that

\[
d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \geq \varepsilon
\]

and

\[
d(gx_{n_k-1}, gx_{m_k}) + d(gy_{n_k-1}, gy_{m_k}) < \varepsilon.
\]

By using triangle inequality, (7) and (8), one gets

\[
\varepsilon \leq d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})
\]

\[
\leq d(gx_{n_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{m_k})
\]

\[
+ d(gy_{n_k}, gy_{n_k-1}) + d(gy_{n_k-1}, gy_{m_k})
\]

\[
< d(gx_{n_k}, gx_{n_k-1}) + d(gy_{n_k}, gy_{n_k-1}) + \varepsilon.
\]

Letting \( k \to \infty \) in (9) and using (6) imply

\[
\lim_{k \to \infty} d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) = \varepsilon.
\]

Again by triangle inequality,

\[
d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) \leq d(gx_{n_k+1}, gx_{n_k}) + d(gx_{n_k}, gx_{m_k}) + d(gx_{m_k}, gx_{m_k+1})
\]

\[
+ d(gy_{n_k+1}, gy_{n_k}) + d(gy_{n_k}, gy_{m_k}) + d(gy_{m_k}, gy_{m_k+1})
\]

\[
\leq w_{n_k} + w_{m_k} + d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k})
\]

and

\[
d(gx_{n_k}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k}) \leq d(gx_{n_k}, gx_{n_k+1}) + d(gx_{n_k+1}, gx_{m_k+1}) + d(gx_{m_k+1}, gx_{m_k})
\]

\[
+ d(gy_{n_k+1}, gy_{n_k}) + d(gy_{n_k}, gy_{m_k}) + d(gy_{m_k}, gy_{m_k+1})
\]

\[
\leq w_{n_k} + w_{m_k} + d(gx_{n_k+1}, gx_{m_k}) + d(gy_{n_k}, gy_{m_k+1}).
\]
Hence

\[
\lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) = \varepsilon
\]

Now we apply inequality (1) and use (10) and (11) to get a contradiction.

(12) 
\[
\phi \left( d(gx_{n_k+1}, gx_{m_k+1}) \right) = \phi \left( d(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k})) \right) 
\]

\[
\leq \psi_1 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \right) - \psi_2 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \right),
\]

where

\[
K(x_{n_k}, x_{m_k}) = \left( \frac{d(gx_{m_k}, gx_{m_k+1})[1 + d(gx_{n_k}, gx_{n_k+1})]}{1 + d(gx_{n_k}, gx_{m_k})}, d(gx_{n_k}, gx_{m_k}) \right)
\]

and

\[
K(y_{n_k}, y_{m_k}) = \left( \frac{d(gy_{m_k}, gy_{m_k+1})[1 + d(gy_{n_k}, gy_{n_k+1})]}{1 + d(gy_{n_k}, gy_{m_k})}, d(gy_{n_k}, gy_{m_k}) \right)
\].

By (6), we have

\[
\frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \to (0, \varepsilon) \text{ as } k \to \infty.
\]

Similarly we get

(13) 
\[
\phi \left( d(gy_{n_k+1}, gy_{m_k+1}) \right) = \phi \left( d(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k})) \right) 
\]

\[
\leq \psi_1 \left( \frac{K(y_{n_k}, y_{m_k}) + K(x_{n_k}, x_{m_k})}{2} \right) - \psi_2 \left( \frac{K(y_{n_k}, y_{m_k}) + K(x_{n_k}, x_{m_k})}{2} \right).
\]

By adding (12) and (13) and from the properties of \( \phi \), we get

\[
\phi \left( d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1}) \right) \leq 2 \psi_1 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}, \frac{w_{n-1}}{2} \right) 
\]

\[
- 2 \psi_2 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2}, \frac{w_{n-1}}{2} \right)
\]

\[
\phi \left( \frac{d(gx_{n_k+1}, gx_{m_k+1}) + d(gy_{n_k+1}, gy_{m_k+1})}{2} \right) \leq \psi_1 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \right) 
\]

\[
- \psi_2 \left( \frac{K(x_{n_k}, x_{m_k}) + K(y_{n_k}, y_{m_k})}{2} \right).
\]

Taking limit as \( k \) tends to infinity implies

\[
\phi \left( \frac{\varepsilon}{2} \right) \leq \psi_1 \left( 0, \frac{\varepsilon}{2} \right) - \psi_2 \left( 0, \frac{\varepsilon}{2} \right),
\]
which is a contradiction. This shows that \{gx_n\} and \{gy_n\} are Cauchy sequences in \(g(X)\). Since 
\(g(X)\) is complete then there exist \(gx, gy \in X\) such that

\[
\lim_{n \to \infty} gx_n = gx \quad \text{and} \quad \lim_{n \to \infty} gy_n = gy.
\]

Now we show that \((x, y)\) is coupled coincidence point for \(F\) and \(g\). For this purpose we shall use (1) with \(u = x_n\) and \(v = y_n\), then take limit on both sides as \(n\) tends to infinity and use Equation (6).

\[
\phi(d(F(x, y), F(x_n, y_n))) \leq \psi_1\left(\frac{K(x, x_n) + K(y, y_n)}{2}\right) - \psi_2\left(\frac{K(x, x_n) + K(y, y_n)}{2}\right),
\]

where

\[
K(x, x_n) = \left(\frac{d(gx_n, F(x_n, y_n))[1 + d(gx, F(x, y))]}{1 + d(gx, gx_n)}\right), d(gx, gx_n)
\]

and

\[
K(y, y_n) = \left(\frac{d(gy_n, F(y_n, x_n))[1 + d(gy, F(y, x))]}{1 + d(gy, gy_n)}\right), d(gy, gy_n).
\]

As \(n \to \infty\), implies that \(gx = F(x, y)\). A similar argument can be derived to show that \(F(y, x) = gy\). This completes the proof and \((x, y)\) is a coupled coincidence point of \(F\) and \(g\).

For uniqueness, let \((x, y)\) and \((x^*, y^*)\) be two coupled coincidence points of \(F\) and \(g\) in \(X\). Then by (1) we have

\[
\phi(d(gx, gx^*)) = \phi(d(F(x, y), F(x^*, y^*))) \leq \psi_1\left(\frac{K(x, x^*) + K(y, y^*)}{2}\right) - \psi_2\left(\frac{K(x, x^*) + K(y, y^*)}{2}\right),
\]

where

\[
K(x, x^*) = \left(\frac{d(gx^*, F(x^*, y^*))[1 + d(gx, F(x, y))]}{1 + d(gx, gx^*)}\right), d(gx, gx^*)
\]

\[
= (0, d(gx, gx^*))
\]

and

\[
K(y, y^*) = \left(\frac{d(gy^*, F(y^*, x^*))[1 + d(gy, F(y, x))]}{1 + d(gy, gy^*)}\right), d(gy, gy^*)
\]

\[
= (0, d(gy, gy^*)).
\]

That is

\[
\frac{K(x, x^*) + K(y, y^*)}{2} = \left(0, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right).
\]
By a similar way

$$\phi(d(gy, gy^*)) = \phi(d(F(y, x), F(y^*, x^*))) \leq \psi_1\left(\frac{K(y, y^*) + K(x, x^*)}{2}\right) - \psi_2\left(\frac{K(y, y^*) + K(x, x^*)}{2}\right).$$

Then,

$$\phi\left(\frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right) \leq \psi_1\left(0, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right) - \psi_2\left(0, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right).$$

Since $$\psi_1$$ is increasing and $$\frac{d(gx, gx^*) + d(gy, gy^*)}{2} \geq 0$$ then,

$$\phi\left(\frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right) \leq \psi_1\left(\frac{d(gx, gx^*) + d(gy, gy^*)}{2}, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right) - \psi_2\left(0, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right).$$

Therefore, $$\psi_2\left(\frac{d(gx, gx^*) + d(gy, gy^*)}{2}, \frac{d(gx, gx^*) + d(gy, gy^*)}{2}\right) = 0 \Rightarrow \frac{d(gx, gx^*) + d(gy, gy^*)}{2} = 0$$, i.e., $$gx = gx^*$$ and $$gy = gy^*$$. That is $$F$$ and $$g$$ have unique point of coincidence.

If we take $$g = I$$ (identity mapping) in Theorem, we obtain the following corollary.

**Corollary 3.2.** Let $$(X, d)$$ be a complete metric space and $$F : X \times X \to X$$. Suppose that there exist $$\phi \in \Phi$$ and $$\psi_1, \psi_2 \in \Psi$$ for which $$\psi_1(x, x) \leq \phi(x)$$ and

$$\phi(d(F(x, y), F(u, v))) \leq \psi_1\left(\frac{K(x, u) + K(y, v)}{2}\right) - \psi_2\left(\frac{K(x, u) + K(y, v)}{2}\right),$$

where

$$K(x, u) = \left(\frac{d(u, F(u, v))[1 + d(x, F(x, y))]}{1 + d(x, u)}, d(x, u)\right)$$

and

$$K(y, v) = \left(\frac{d(v, F(v, u))[1 + d(y, F(y, x))]}{1 + d(y, v)}, d(y, v)\right),$$

for all $$x, y, u, v \in X$$. Then $$F$$ has a unique coupled fixed point.

In the case of partially ordered metric space, we obtain the following theorem.
Theorem 3.3. Let \((X, \preceq, d)\) be a partially ordered metric space. Suppose that \(F : X \times X \to X\) and \(g : X \to X\) are two self mappings on \(X\), \(F\) has the \(g\)-mixed monotone property, \(F(X, X) \subseteq g(X)\) and \(g(X)\) is complete subspace of \(X\). Suppose that there exist \(\phi \in \Psi_1\) and \(\psi_1, \psi_2 \in \Psi_5\) such that \(\psi_1 (x, x, x, x, x) \leq \phi (x)\) and

\[
\phi \left( d(F(x, y), F(u, v)) \right) \leq \psi_1 (M(x, u)) - \psi_2 (M(x, u)),
\]

where

\[
M(x, u) = \left( \frac{d(gx, F(x, y))d(gu, F(u, v))}{d(gx, gu)}, d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v)), \right.
\]

\[
\left. \frac{1}{2} \left[ d(gx, F(u, v)) + d(gu, F(x, y)) \right] \right).
\]

for all \(x, y, u, v \in X\) with \(gx \preceq gu\) and \(gy \succeq gv\). Assume that \(X\) has the following properties:

(a): if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),

(b): if a non-increasing sequence \(y_n \to y\), then \(y_n \succeq y\) for all \(n\).

Moreover, if there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \succeq F(y_0, x_0)\). Then \(F\) and \(g\) have a coupled coincidence point. This coupled coincidence point is unique if the set of all points of coincidence is totally ordered subset of \(X\), that is, if \((x, y)\) and \((x^*, y^*)\) are two coincidence points of \(F\) and \(g\), then \(gx \preceq gx^*\) and \(gy \succeq gy^*\).

Proof.

Let \(x_0, y_0 \in X\) be such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\). Since \(F(X, X) \subseteq g(X)\), we can find \(x_1, y_1 \in X\) with \(gx_1 = F(x_0, y_0)\) and \(gy_1 = F(y_0, x_0)\). Continuing this process, we construct two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n).
\]

By induction, we will prove that

\[
gx_n \preceq gx_{n+1} \quad \text{and} \quad gy_n \succeq gy_{n+1}.
\]

Since

\[
gx_0 \preceq F(x_0, y_0) = gx_1 \quad \text{and} \quad gy_0 \succeq F(y_0, x_0) = gy_1.
\]
Thus (18) is true for \( n = 0 \). We suppose that (18) is true for some \( n > 0 \). Since \( F \) has the \( g \)-mixed monotone property, by (17) we have that
\[
g_{x_{n+1}} = F(x_n, y_n) \preceq F(x_{n+1}, y_{n+1}) = g_{x_{n+2}} \quad \text{and} \quad g_{y_{n+1}} = F(y_n, x_n) \succeq F(y_{n+1}, x_{n+1}) = g_{y_{n+2}},
\]
that is (18) is true for all natural number \( n \). If for some \( n \in \mathbb{N} \),
\[
g_{x_n} = g_{x_{n+1}} \quad \text{and} \quad g_{y_n} = g_{y_{n+1}}.
\]
Then, by (17) \((x_n, y_n)\) is a coupled coincidence point of \( F \) and \( g \). Hence from now on we assume that at least
\[
g_{x_n} \neq g_{x_{n+1}} \quad \text{or} \quad g_{y_n} \neq g_{y_{n+1}}.
\]
From (16) and (18), we get
\[
\phi\left(d\left(g_{x_n}, g_{x_{n+1}}\right)\right) = \phi\left(d\left(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\right)\right) \\
\leq \psi_1\left(M(x_{n-1}, x_n)\right) - \psi_2\left(M(x_{n-1}, x_n)\right) \\
\leq \psi_1\left(M(x_{n-1}, x_n)\right)
\]
and
\[
M(x_{n-1}, x_n) = \left(\frac{d(g_{x_{n-1}}, F(x_{n-1}, y_{n-1})))d(g_{x_n}, F(x_n, y_n))}{d(g_{x_{n-1}}, g_{x_n})}, d(g_{x_{n-1}}, g_{x_n}), d(g_{x_{n-1}}, F(x_{n-1}, y_{n-1})), \right) \\
\quad \cdot d(g_{x_n}, F(x_n, y_n)), \frac{1}{2}\left[d(g_{x_{n-1}}, F(x_{n-1}, y_{n-1}))) + d(g_{x_n}, F(x_{n-1}, y_{n-1})))\right] \\
\quad = \left(d(g_{x_{n-1}}, g_{x_n}), d(g_{x_{n-1}}, g_{x_n}), d(g_{x_{n-1}}, g_{x_n}), d(g_{x_n}, g_{x_{n+1}}), \frac{1}{2}d(g_{x_{n-1}}, g_{x_{n+1}})\right)
\]
By similar way, we have
\[
\phi\left(d\left(g_{y_{n+1}}, g_{y_n}\right)\right) = \phi\left(d\left(F(y_{n-1}, x_n), F(y_{n-1}, x_{n-1})\right)\right) \\
\leq \psi_1\left(M(y_{n-1}, y_{n-1})\right) - \psi_2\left(M(y_{n-1}, y_{n-1})\right) \\
\leq \psi_1\left(M(y_{n-1}, y_{n-1})\right)
\]
and
\[
M(y_{n-1}, y_{n-1}) = \left(d(g_{y_{n-1}}, g_{y_{n+1}}), d(g_{y_{n-1}}, g_{y_{n-1}}), d(g_{y_{n-1}}, g_{y_{n-1}}), d(g_{y_{n-1}}, g_{y_{n-1}}), \frac{1}{2}d(g_{y_{n-1}}, g_{y_{n+1}})\right)
\]
Since \( \psi_1 \) is increasing with respect to the second argument, then we have for all \( n \geq 1 \)
\[
d(g_{x_n}, g_{x_{n+1}}) \leq d(g_{x_{n-1}}, g_{x_n}) \quad \text{and} \quad d(g_{y_n}, g_{y_{n+1}}) \leq d(g_{y_{n-1}}, g_{y_n}).
\]
With respect to the first inequality we conclude that the sequence \( \{d(gx_n, gx_{n+1})\} \) is non-increasing sequence of positive real numbers, then there exist an \( \varepsilon \geq 0 \) such that

\[
\lim_{n \to \infty} d(gx_n, gx_{n+1}) = \varepsilon
\]

We shall show that \( \varepsilon = 0 \). Taking the limit as \( n \to \infty \) in (19) and using the continuity of \( \phi, \psi_1 \) and \( \psi_2 \), we get.

\[
\phi(\varepsilon) \leq \psi_1(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) - \psi_2(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) \leq \phi(\varepsilon) - \psi_2(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon),
\]

which holds unless \( \psi_2(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) = 0 \). Thus \( \varepsilon = 0 \) and

(20) \[ \lim_{n \to \infty} d(gx_n, gx_{n+1}) = 0. \]

Using the same idea, we deduce that

(21) \[ \lim_{n \to \infty} d(gy_n, gy_{n+1}) = 0. \]

Now we shall prove that \( \{gx_n\} \) and \( \{gy_n\} \) are Cauchy sequences in \( X \). For \( \{gx_n\} \), suppose that it is not Cauchy. Then there exists an \( \varepsilon > 0 \) for which we can find two sequences \( \{m_k\} \) and \( \{n_k\} \) with \( n_k > m_k > k \) such that

(22) \[ d(gx_{n_k}, gx_{m_k}) \geq \varepsilon \]

and

(23) \[ d(gx_{n_k-1}, gx_{m_k}) < \varepsilon. \]

By using triangle inequality and having in mind (20) to (23), we have

(24) \[ \lim_{k \to \infty} d(gx_{n_k}, gx_{m_k}), \lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k+1}), \lim_{k \to \infty} d(gx_{n_k}, gx_{m_k+1}), \]

\[ \lim_{k \to \infty} d(gx_{n_k+1}, gx_{m_k}) \to \varepsilon \text{ as } k \to \infty. \]

Since \( n_k > m_k \), so from (18), \( gx_{nk} \geq gx_{mk} \) and \( gy_{nk} \leq gy_{mk} \). By (16) we have

\[
\phi(d(gx_{m_k+1}, gx_{n_k+1})) = \phi(d(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k}))) \leq \psi_1(M(x_{m_k}, x_{n_k})) - \psi_2(M(x_{m_k}, x_{n_k})) \leq \psi_1(M(x_{m_k}, x_{n_k}))
\]
\[ M(x_n, y_n) = \left( \frac{d(gx_n, gx_{n+1})d(gx_n, gy_{n+1})}{d(gx_n, gx_{n+1})}, \frac{d(gx_n, gy_{n+1})}{d(gx_n, gx_{n+1})}, \frac{1}{2} [d(gx_n, gx_{n+1}) + d(gx_{n+1}, gy_{n+1})] \right) \]

Taking limit as \( k \) tends to infinity, using (20, 24) and all properties on \( \phi, \psi_1, \psi_2 \) imply

\[
\phi(\varepsilon) \leq \psi_1(0, \varepsilon, 0, 0, \varepsilon) - \psi_2(0, \varepsilon, 0, 0, \varepsilon) \\
\leq \psi_1(\varepsilon, \varepsilon, \varepsilon, \varepsilon) - \psi_2(0, \varepsilon, 0, 0, \varepsilon) \\
\leq \phi(\varepsilon) - \psi_2(0, \varepsilon, 0, 0, \varepsilon) \\
\Rightarrow \psi_2(0, \varepsilon, 0, 0, \varepsilon) = 0 \Rightarrow \varepsilon = 0,
\]

which is a contradiction. This shows that \( \{gx_n\} \) is Cauchy sequence in \( g(X) \). By a similar way one can deduce that also \( \{gy_n\} \) is Cauchy sequence. Since \( g(X) \) is complete then there exist \( gx, gy \in X \) such that

\[ \lim_{n \to \infty} gx_n = gx \text{ and } \lim_{n \to \infty} gy_n = gy. \]

Now we show that \((x, y)\) is coupled coincidence point for \( F \) and \( g \). For this purpose we shall use the properties on \( X \) and apply (16) with \( x = x_n, y = y_n, u = x \) and \( v = y \), then take limit on both sides as \( n \) tends to infinity and use Equation (25).

\[ \phi \left( d(F(x_n, y_n), F(x, y)) \right) \leq \psi_1 \left( M(x_n, x) \right) - \psi_2 \left( M(x_n, x) \right), \]

where

\[ M(x_n, x) = \left( \frac{d(gx_n, gx_{n+1})d(gx, F(x, y))}{d(gx_n, gx)}, \frac{d(gx_n, gx), d(gx_n, gx_{n+1})}{d(gx_n, gx_{n+1})}, \frac{1}{2} [d(gx_n, F(x, y)) + d(gx, gx_{n+1})] \right) \rightarrow (0, 0, 0, d(gx, F(x, y)), \frac{d(gx, F(x, y))}{2}) \]

Taking limit as \( n \) tends to infinity implies that \( gx = F(x, y) \). A similar argument can be derived to show that \( F(y, x) = gy \). This completes the proof and \((x, y)\) is a coupled coincidence point of the mappings \( F \) and \( g \).
For uniqueness, let \((x, y)\) and \((x^*, y^*)\) be two distinct coupled coincidence points of \(F\) and \(g\) in \(X\). Then by \((16)\) we have
\[
\phi \left( d(F(x, y), F(x^*, y^*)) \right) \leq \psi_1(M(x, x^*)) - \psi_2(M(x, x^*)),
\]
where
\[
M(x, x^*) = \left( \frac{d(gx, F(x, y))d(gx, F(x^*, y^*))}{d(gx, gx^*)}, d(gx, gx^*), d(gx, F(x, y)), d(gx, F(x^*, y^*)) \right),
\]
\[
= \left( 0, d(gx, gx^*), 0, d(gx, gx^*) \right)
\]
By a similar way
\[
\phi \left( d(gy, gy^*) \right) = \phi \left( d(F(y, x), F(y^*, x^*)) \right) \leq \psi_1(M(y, y^*)) - \psi_2(M(y, y^*)),
\]
\[
M(y, y^*) = \left( \frac{d(gy, F(y, x))d(gy, F(y^*, x^*))}{d(gy, gy^*)}, d(gy, gy^*), d(gy, F(y, x)), d(gy, F(y^*, x^*)) \right),
\]
\[
= \left( 0, d(gy, gy^*), 0, d(gy, gy^*) \right)
\]
Therefore, \(gx = gx^*\) and \(gy = gy^*\). That is \(F\) and \(g\) have unique point of coincidence.

Here, we derive the following corollaries from our Theorem. Also we illustrate our results by an example.

**Corollary 3.4.** Let \((X, \preceq, d)\) be a complete partially ordered metric space. Suppose that \(F : X \times X \to X\) is a self mapping on \(X\) such that \(F\) has the mixed monotone property. Suppose that there exist \(\alpha, \beta, \gamma, \delta \in [0, 1)\) with \(\alpha + \beta + 2\gamma + 2\delta < 1\) such that
\[
d(F(x, y), F(u, v)) \leq \alpha \left( \frac{d(x, F(x, y))d(u, F(u, v))}{d(x, u)} \right) + \beta d(x, u) + \gamma \left[ d(x, F(x, y)) + d(u, F(u, v)) \right] + \delta \left[ d(x, F(u, v)) + d(u, F(x, y)) \right]
\]
for all \(x, y, u, v \in X\) with \(x \preceq u\) and \(y \succeq v\). Assume that either

1.: \(F\) is continuous or

2.: \(X\) has the following properties:

   (a): if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),
(b): if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all $n$.

Moreover, if there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then $F$ has a coupled fixed point $(x, y) \in X \times X$. [Theorem 2.1 in [7]]

**Proof.** This corollary is a consequence of Theorem by taking

- $g(x) = x, \quad \forall x \in X$
- $\phi(t) = t, \quad \forall t \geq 0$
- $\psi_1(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma [t_3 + t_4] + 2 \delta t_5$
- $\psi_2(t_1, t_2, t_3, t_4, t_5) = 0$.

**Corollary 3.5.** Let $(X, \preceq, d)$ be a complete partially ordered metric space and $F : X \times X \to X$ be a given mapping having the mixed monotone property such that there exists $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq k \max \left\{ \frac{d(gF(x, y), d(gu, F(u, v)))}{d(gx, gu)}, d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v)) \right\},$$

$$\frac{1}{2} \left[ d(gF(x, y)) + d(gu, F(x, y)) \right]$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$. Assume either

1.: $F$ is continuous or

2.: $X$ has the following properties:

- (a): if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all $n$,
- (b): if a non-increasing sequence $y_n \to y$, then $y_n \succeq y$ for all $n$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then $F$ has a coupled fixed point.

**Proof.** It suffices to take

- $g(x) = x, \quad \forall x \in X$
- $\phi(t) = t, \quad \forall t \geq 0$
- $\psi_1(t_1, t_2, t_3, t_4, t_5) = \max \{t_1, t_2, t_3, t_4, t_5\}$
- $\psi_2(t_1, t_2, t_3, t_4, t_5) = (1 - k) \max \{t_1, t_2, t_3, t_4, t_5\}.$
Corollary 3.6. Let \((X, \preceq, d)\) be a complete partially ordered metric space and \(F : X \times X \to X\) be a given mapping having the mixed monotone property such that there exists \(k \in [0, 1)\) with

\[
(28) \quad d(F(x,y), F(u,v)) \leq k \left[ \frac{d(gx, F(x,y))d(gu, F(u,v))}{d(gx, gu)} + d(gx, F(x,y)), d(gu, F(u,v)), \right]
\]

\[
\frac{1}{2} \left[ d(gx, F(u,v)) + d(gu, F(x,y)) \right]
\]

for all \(x, y, u, v \in X\) with \(x \preceq u\) and \(y \succeq v\). Assume either \(F\) is continuous, or \(X\) has the following properties:

(a): if a non-decreasing sequence \(x_n \to x\), then \(x_n \preceq x\) for all \(n\),

(b): if a non-increasing sequence \(y_n \to y\), then \(y_n \succeq y\) for all \(n\).

If there exist \(x_0, y_0 \in X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \succeq F(y_0, x_0)\). Then \(F\) has a coupled fixed point.

Proof. Here we take

- \(g(x) = x, \quad \forall x \in X\)
- \(\phi(t) = t, \quad \forall t \geq 0\)
- \(\psi_1(t_1, t_2, t_3, t_4, t_5) = \frac{1}{5} [t_1 + t_2 + t_3 + t_4 + t_5]\)
- \(\psi_2(t_1, t_2, t_3, t_4, t_5) = \frac{1-k}{5} [t_1 + t_2 + t_3 + t_4 + t_5]\).

Finally, we give the following example to verify Theorem as follows.

Example 3.1. Let \(X = [0, \infty)\) be endowed with its Euclidian metric \(d(x,y) = |x - y|\) and its usual ordering \(\preceq\). Take \(F : X \to X\) defined by

\[
F(x,y) = \begin{cases} \frac{x - 3y}{5} & x \geq 3y; \\ 0 & \text{otherwise.} \end{cases}
\]

and \(\phi, \psi_1\) and \(\psi_2\) as in Corollary. Take \(k = \frac{5}{6}\).

We claim that (28) holds for each \(x \preceq u\) and \(y \succeq v\). We divide the proof into the following four cases.
Case I. If \( x \geq 3y \) and \( u \geq 3v \), here we have \( F(x, y) = \frac{x - 3y}{5} \) and \( F(u, v) = \frac{u - 3v}{5} \).

\[
d(F(x, y), F(u, v)) = \left| \frac{x - 3y}{5} - \frac{u - 3v}{5} \right| = \left| \frac{x - u}{5} + \frac{3}{5}(v - y) \right|
\]
\[
\leq \frac{u - x}{6} + \frac{u - x}{30} + \frac{9 + 3 + 6}{30}(y - v)
\]
\[
\leq \frac{u - x}{6} + \frac{u - x + 3u}{30} + \frac{3y + 4x}{30}
\]
\[
\leq \frac{1}{6} \left( (u - x) + \frac{4u + 3v}{30} + \frac{4x + 3y}{30} \right)
\]
\[
\leq \frac{1}{6} \left( d(x, u) + d(u, F(u, v)) + d(x, F(x, y)) \right)
\]
\[
\leq k \left( \frac{d(gx, F(x, y))d(gu, F(u, v))}{d(gx, gu)} \right), d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v)), \frac{1}{2} \left( d(gx, F(u, v)) + d(gu, F(x, y)) \right)
\]

Note that, since \( 3y \leq x \leq u \Rightarrow 9y \leq 3u \).

Case II. If \( x \geq 3y \) and \( u < 3v \), here we have \( F(x, y) = \frac{x - 3y}{5} \) and \( F(u, v) = 0 \),

\[
d(F(x, y), F(u, v)) = \left| \frac{x - 3y}{5} \right| \leq \frac{x}{5}
\]
\[
\leq \frac{x - u}{6} + \frac{u}{6} + \frac{x}{30}
\]
\[
\leq \frac{x - u}{6} + \frac{u}{6} + \frac{4x + 3y}{30}
\]
\[
\leq \frac{1}{6} \left( (x - u) + u + \frac{4x + 3y}{30} \right)
\]
\[
\leq \frac{1}{6} \left( d(x, u) + d(u, F(u, v)) + d(x, F(x, y)) \right)
\]
\[
\leq k \left( \frac{d(gx, F(x, y))d(gu, F(u, v))}{d(gx, gu)} \right), d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v)), \frac{1}{2} \left( d(gx, F(u, v)) + d(gu, F(x, y)) \right)
\]
Case III. If $x < 3y$ and $u \geq 3v$, here we have $F(x, y) = 0$ and $F(u, v) = \frac{u - 3v}{5}$,

\[
d(F(x, y), F(u, v)) = \left| \frac{u - 3v}{5} \right| \leq \frac{u}{5}
\]

\[
\leq \frac{u - x}{6} + \frac{x}{6} + \frac{u}{30}
\]

\[
\leq \frac{u - x}{6} + \frac{x}{6} + \frac{4u + 3v}{30}
\]

\[
\leq \frac{1}{6}\left[(u - x) + x + \frac{4u + 3v}{30}\right]
\]

\[
\leq \frac{1}{6}\left[d(x, u) + d(x, F(x, y)) + d(u, F(u, v))\right]
\]

\[
\leq \frac{k}{5}\left[d(gx, F(x, y))d(gu, F(u, v))d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v))\right]
\]

\[
\frac{1}{2}\left[d(gx, F(u, v)) + d(gu, F(x, y))\right]
\]

Case IV. If $x < 3y$ and $u < 3v$, here we have $F(x, y) = 0$ and $F(u, v) = 0$,

\[
d(F(x, y), F(u, v)) = 0
\]

\[
\leq \frac{k}{5}\left[d(gx, F(x, y))d(gu, F(u, v))d(gx, gu), d(gx, F(x, y)), d(gu, F(u, v))\right]
\]

\[
\frac{1}{2}\left[d(gx, F(u, v)) + d(gu, F(x, y))\right]
\]

Moreover, it is easy to see that all other hypotheses of Corollary are verified. So $F$ has a coupled fixed point $(0, 0) \in X^2$.

**Conflicts of Interest**

The authors declare that there is no conflict of interests.

**References**


