INEQUALITIES FOR FIXED POINTS OF THE SUBCLASS \( P(j, \lambda, \alpha, n) \) OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

HUKMI KIZILTUNC\(^1\), HUSEYIN BABA\(^2\)*

\(^1\)Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.
\(^2\)Department of Mathematics, Hakkari Vocational School, Hakkari University, Hakkari, 30000, Turkey.

Abstract. We consider the subclass \( P(j, \lambda, \alpha, n) \) of starlike functions with negative coefficients by using the differential \( D^n \) operator and functions of the form \( f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \) which are analytic in the open unit disk. We examine the subclass \( P(j, \lambda, \alpha, n, z_0) \) for which \( f(z_0) = z_0 \) or \( f'(z_0) = 1 \), \( z_0 \) real. We determine coefficient inequalities for functions belonging to the class \( P(j, \lambda, \alpha, n, z_0) \). As special cases, the results of our paper reduce to Silverman [1].

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1. Introduction and Preliminaries

Let \( \mathcal{A} \) denote the family of functions \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

that are analytic in the open unit disk \( \mathcal{U} := \{ z : z \in \mathbb{C} and \ |z| < 1 \} \). A function \( f \in \mathcal{A} \) is said to be starlike of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathcal{U}).
\]

*Corresponding author

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We denote by $S^*(\alpha)$, the class of all such functions. On the other hand, a function $f \in A$ is said to be convex of order $\alpha$ ($0 \leq \alpha < 1$) if and only if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in U).
\]

Let $C(\alpha)$ denote the class of all those functions which are convex of order $\alpha$ in $U$.

Note that $S^*(0) = S^*$ and $C(0) = C$ are, respectively, the classes of starlike and convex functions in $U$.

Let $A(j)$ denote the class of functions of the form:
\[
f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} := \{1, 2, 3, \ldots\})
\]
which are analytic in the open unit disk $U$.

Sălăgean [3] has introduced the following operator called the Sălăgean operator for a function $f(z)$ in $A(j)$

\[
D^0 f(z) = f(z)
\]
\[
D^1 f(z) = D f(z) = z f'(z) = z + \sum_{k=j+1}^{\infty} k a_k z^k
\]
\[
D^2 f(z) = D (D f(z)) = z f'(z) + z^2 f''(z) = z + \sum_{k=j+1}^{\infty} k^2 a_k z^k
\]
\[\vdots\]
\[
D^n f(z) = D \left( D^{n-1} f(z) \right) = z + \sum_{k=j+1}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}).
\]

With the help of the differential operator $D^n$, we say that a function $f(z)$ belonging to $A(j)$ is in the class $Q(j, \lambda, \alpha, n)$ if and only if
\[
\Re \left\{ \frac{(1 - \lambda) z (D^n f(z))' + \lambda z (D^{n+1} f(z))'}{(1 - \lambda) D^n f(z) + \lambda D^{n+1} f(z)} \right\} > \alpha
\]
for some $\alpha$ ($0 \leq \alpha < 1$) and $\lambda$ ($0 \leq \lambda \leq 1$), and for all $z \in U$.

In [2], M.K.Aouf and H.M.Srivastava, $T(j)$ denoted the subclass of $A(j)$ consisting of functions of the form:
\[
f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \ j \in \mathbb{N}).
\]
Further, M.K. Aouf and H.M. Srivastava defined the class \( P(j, \lambda, \alpha, n) \) by

\[
P(j, \lambda, \alpha, n) = Q(j, \lambda, \alpha, n) \cap T(j).
\]

In [2], for a function \( f(z) \) in \( P(j, \lambda, \alpha, n) \), M.K. Aouf and H.M. Srivastava defined

\[
D^0 f(z) = f(z)
\]

\[
D^1 f(z) = Df(z) = zf'(z) = z - \sum_{k=j+1}^{\infty} k a_k z^k
\]

\[
D^2 f(z) = D(Df(z)) = zf'(z) + z^2 f''(z) = z - \sum_{k=j+1}^{\infty} k^2 a_k z^k
\]

\[
D^n f(z) = D(D^{n-1} f(z)) = z - \sum_{k=j+1}^{\infty} k^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).
\]

In [2], M.K. Aouf and H.M. Srivastava obtained coefficients inequalities, distortion theorems, closure theorems, and some properties involving the modified Hadamard products of several functions belonging to the class \( P(j, \lambda, \alpha, n) \). They also determined the radii of close-to-convexity for, and considered integral operators associated with, functions belonging to the class \( P(j, \lambda, \alpha, n) \). Finally, they extended some of the aforementioned distortion theorems to hold true for certain operators of fractional calculus (that is, fractional integral and fractional derivative).

In order that prove our theorem, the following lemma is needed.

**Lemma 1.1.** [2] Let the function \( f(z) \) be defined by (11). Then \( f(z) \in P(j, \lambda, \alpha, n) \) if only if

\[
\sum_{k=j+1}^{\infty} k^n (k - \alpha) \{1 + (k-1) \lambda\} a_k \leq 1 - \alpha
\]

\((a_k \geq 0; \ j \in \mathbb{N}; \ n \in \mathbb{N}_0; \ 0 \leq \alpha < 1; \ z \in \mathcal{U}; \ 0 \leq \lambda < 1\).

In [4], Schild examined the class of polynomials of the form \( f(z) = z - \sum_{n=2}^{N} a_n z^n \), where \( a_n \geq 0 \) and \( f(z) \) is univalent in the disk \(|z| < 1\). In [5], Pilat studied the class of univalent polynomials of the from \( f(z) = a_1 z - \sum_{n=2}^{N} a_n z^n \), where \( a_n \geq 0 \) and \( f(z) = z_0 > 0 \). In [1], Silverman dealt with functions of the form

\[
f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n
\]
where either

\begin{equation}
\tag{20}
a_n \geq 0, \quad f(z_0) = z_0 \quad (-1 < z_0 < 1; \quad z_0 \neq 0)
\end{equation}

or

\begin{equation}
\tag{21}
a_n \geq 0, \quad f^{'}(z_0) = 1 \quad (-1 < z_0 < 1).
\end{equation}

Given $\alpha$ and $z_0$ fixed, $S^*_0(\alpha, z_0)$ examined the subclass of functions starlike of order $\alpha$ that satisfy (20), and $S^*_1(\alpha, z_0)$ examined the subclass of functions starlike of order $\alpha$ that satisfy (21). Also denoted by $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ the subclasses of functions convex of order $\alpha$ that satisfy, respectively, (20) and (21).

In [1], Silverman determined necessary and sufficient conditions for functions to be in these classes. Silverman found the extreme points for each of these classes. Besides, Silverman gave a necessary and sufficient condition for a subset $B$ of the real interval $(0,1)$ to have the property that $\bigcup_{z_\gamma \in B} S^*_0(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_0(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} S^*_1(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_1(\alpha, z_\gamma)$ each forms a convex family. The extreme points of each of these classes was then determined. Many of the results in [1] reduced to those in [6] in the special case $z_0 = 0$.

In this paper inspired and motivated by this facts, we consider the subclass $P(j, \lambda, \alpha, n)$ of starlike functions with negative coefficients by using the differential $D^n$ operator and functions of the form

\begin{equation}
\tag{22}
f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \quad j \in \mathbb{N})
\end{equation}

which are analytic in the open unit disk. We examine the subclass $P(j, \lambda, \alpha, n, z_0)$ for which $f(z_0) = z_0$ or $f^{'}(z_0) = 1$, $z_0$ real. The purpose of this paper is to determine coefficient inequalities for functions belonging to the class $P(j, \lambda, \alpha, n, z_0)$. As special cases, the results of this paper reduce to Silverman [1].

2. Main results

**Theorem 2.1.** Let the function $f(z)$ be defined by (11). Then $f(z)$ is $P(j, \lambda, \alpha, n, z_0)$ if only if

\begin{equation}
\sum_{k=j+1}^{\infty} \left[ k^n \left( \frac{k-\alpha}{1-\alpha} \right) \{1 + (k-1) \lambda \} - z_0^{k-1} \right] a_k \leq 1.
\end{equation}

($a_k \geq 0; \quad j \in \mathbb{N}; \quad n \in \mathbb{N}_0; \quad 0 \leq \alpha < 1; \quad z \in U; \quad 0 \leq \lambda < 1; \quad z_0 \in \mathbb{R}$ fixed point).

The result is sharp.
Proof. Assume that \( f(z) \) is \( P(j, \lambda, \alpha, n, z_0) \). We find that

\[
\frac{f(z_0)}{z_0} = 1 \Rightarrow \frac{z_0 - \sum_{k=j+1}^{\infty} a_k z_0^k}{z_0} = 1 \Rightarrow 1 - \sum_{k=j+1}^{\infty} a_k z_0^{k-1} = 1
\]

(24)

\[
\sum_{k=j+1}^{\infty} a_k z_0^{k-1} = 0.
\]

Then, by Lemma 1, we have

\[
\sum_{k=j+1}^{\infty} k^n (k - \alpha) \{1 + (k - 1) \lambda\} a_k \leq 1 - \alpha
\]

(26)

Conversely assume that the inequality (25) holds true. Then we find that

\[
\sum_{k=j+1}^{\infty} \left[ k^n \left( \frac{k - \alpha}{1 - \alpha} \right) \{1 + (k - 1) \lambda\} - z_0^{k-1} \right] a_k \leq 1.
\]

(27)

Consequently, \( f(z) \in P(j, \lambda, \alpha, n, z_0) \).

Corollary 2.2. If we set \( j = 1 \), \( \lambda = 0 \) and \( n = 0 \) in Theorem 1, we immediately obtain. Then we find that

\[
\sum_{k=j+1}^{\infty} \left[ k^n \left( \frac{k - \alpha}{1 - \alpha} \right) \{1 + (k - 1) \lambda\} - z_0^{k-1} \right] a_k \leq 1
\]

(33)

and \( P(1, \alpha, z_0) = S_0^\ast(\alpha, z_0) \). In [1] examined \( S_0^\ast(\alpha, z_0) \).
Corollary 2.3. A special case of Theorem 1 when \( j = 1, \lambda = 1 \) and \( n = 0 \) yields. Then we find that

\[
\sum_{k=j+1}^{\infty} \left[ \frac{k(k-\alpha)}{1-\alpha} - z_0^{k-1} \right] a_k \leq 1
\]

and \( P(1, 1, \alpha, z_0) = K_0(\alpha, z_0) \). In [1] examined \( K_0(\alpha, z_0) \).

References