Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 4, 367-383
https://doi.org/10.28919/afpt/3780
ISSN: 1927-6303

UNIQUE COMMON FIXED POINTS FOR TWO GENERALIZED EXPANSIVE MAPPINGS ON NON-NORMAL CONE METRIC SPACES WITH BANACH ALGEBRAS

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Abstract. By using some elementary results concerning cone metric spaces over Banach algebras and the related ones about c-sequence on cone metric spaces, some new coincidence point and common fixed point theorems for two generalized expansive mappings were discussed and obtained on cone metric spaces over Banach algebras without the assumption of normality and some unique fixed point theorems were given. Also, One of the main results is supported with a relevant example.

Keywords: Cone metric spaces with Banach algebras; Coincidence point; Common fixed point; expansive condition.

2010 AMS Subject Classification: 47H10, 47H25, 54E40, 55M20.

1. Introduction

In 2007, cone metric spaces were reviewed by Huang and Zhang, as a generalization of metric spaces (see [1]). The distance $d(x,y)$ of two elements x and y in a cone metric space X is defined

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Received June 25, 2018

to be a vector in an ordered Banach space E, quite different from that which is defined a non-negative real numbers in general metric space. In 2011, I. Beg, A. Azam and M. Arshad([2]) introduced the concept of topological vector space-valued cone metric spaces, where the ordered Banach space in the definition of cone metric spaces is replaced by a topological vector space.

Recently, some authors investigated the problems of whether cone metric spaces are equivalent to metric spaces in terms of the existence of fixed points of the mappings and successfully established the equivalence between some fixed point results in metric spaces and in (topological vector space-valued) cone metric spaces, see [3-6]. Actually, they showed that any cone metric space (X,d) is equivalent to a usual metric space (X,d^*), where the real-metric function d^* is defined by a nonlinear scalarization function ξ_e(see [4]) or by a Minkowski function q_e(see[5]). After that, some other interesting generalizations were developed, see. for instance, [7].

In 2013, Liu and Xu [8] introduced the concept of cone metric spaces over Banach algebras, replacing a Banach space E by a Banach algebra $\mathscr{A}$ as the underlying spaces of cone metric spaces. And the authors in [8-11] discussed and obtained Banach fixed point theorem, Kannan type fixed point theorem, Chatterjea type fixed point theorem and ćirić type fixed point theorem in cone metric spaces over Banach algebras. Especially, the authors in [10] gave an example to show that fixed point results of mappings in this new space are indeed more different than the standard results of cone metric spaces presented in literature.

In this paper, we use the elementary results of the c-sequences and the basic properties of cone metric spaces over Banach algebras to obtain some new unique common fixed point theorems for two generalized expansive mappings on cone metric spaces over Banach algebras without the assumption of normality and give some unique fixed point theorems. Finally, we give an example to support the main result.

2. Preliminaries

Let $\mathscr{A}$ always be a Banach algebra. That is, $\mathscr{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties(for all $x,y,z \in \mathscr{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;

2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

4. $\| xy \| \leq \| x \| \| y \|$.

In this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathscr{A}$. An element $x \in \mathscr{A}$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more detail, we refer to [12].

We say that $\{x_1, x_2, \cdots, x_n\} \subset \mathscr{A}$ commute if $x_i x_j = x_j x_i$ for all $i, j \in \{1, 2, \cdots, n\}$.

Proposition 2.1.[12] *Let $\mathscr{A}$ be a Banach algebra with a unit e, and $x \in \mathscr{A}$. If the spectral radius $r(x)$ of x is less than* 1*, i.e.,*

$$r(x) = \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = \inf_{n\to\infty} \| x^n \|^{\frac{1}{n}} < 1.$$

Then $(e-x)$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{+\infty} x^i.$$

Remark 2.1. 1) $r(x) \leq \| x \|$ for any $x \in \mathscr{A}$(see [12]).

2) In Proposition 2.1, if the condition $r(x) < 1$ is replaced by the condition $\| x \| < 1$, then the conclusion remains true.

A subset P of a Banach algebra $\mathscr{A}$ is called a cone if

1. P is nonempty closed and $\{0, e\} \subset P$, where 0 denotes the null of the Banach algebra $\mathscr{A}$,

2. $\alpha P + \beta P \subset P$ for all non-negative real numbers $\alpha . \beta$;

3. $P^2 = PP \subset P$;

4. $P \cap (-P) = \{0\}$.

For a given cone $P \subset \mathscr{A}$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ stand for $x \leq y$ and $x \neq y$. While $x \ll y$ sill stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called solid if $\operatorname{int} P \neq \emptyset$.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in \mathscr{A}$.

$$0 \leq x \leq y \implies \| x \| \leq M \ \| y \| .$$

The least positive number satisfying the above is called the normal constant of P.

Here, we always assume that P is a solid and $\leq$ is the partial ordering with respect to P.

Definition 2.1.[1, 9-10] Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathscr{A}$ satisfies

1. $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x,y \in X$;
3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space(over a Banach algebra $\mathscr{A}$).

Remark 2.2. The examples of cone metric spaces(over a Banach algebra $\mathscr{A}$) can be found in [8-10].

Definition 2.2.[1, 8] Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then:

1. $\{x_n\}$ converges to x whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in \mathscr{A}$ with $0 \ll c$ there is a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$.

3. (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.3.[13-14] Let P is a solid cone in a Banach space $\mathscr{A}$. A sequence $\{u_n\} \subset P$ is called a c-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Proposition 2.2.[13] *Let P is a solid cone in a Banach space $\mathscr{A}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences in P. If $\{x_n\}$ and $\{y_n\}$ are c-sequences and $\alpha,\beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.*

Proposition 2.3.[13] *Let P is a solid cone in a Banach algebra $\mathscr{A}$ and $\{x_n\}$ a sequence in P. Then the following conditions are equivalent:*

(1) $\{x_n\}$ is a c-sequence;

(2) for each $c \gg 0$there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for all $n \geq n_0$;

(3) for each $c \gg 0$there exists $n_1 \in \mathbb{N}$ such that $x_n \leq c$ for all $n \geq n_1$.

Proposition 2.4.[10] *Let P is a solid cone in a Banach algebra $\mathscr{A}$ and $\{u_n\}$ a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Proposition 2.5.[10] *Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. The following assertions hold true:*

(i) For any $x,y \in \mathscr{A}$, $a \in P$ with $x \leq y$, $ax \leq ay$;

(ii) For any sequences $\{x_n\},\{y_n\} \subset \mathscr{A}$ with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, where $x,y \in \mathscr{A}$, we have $x_n y_n \to xy$ as $n \to \infty$.

Proposition 2.6.[10] *Let $\mathscr{A}$ be a Banach algebra with a unit e, P a cone in $\mathscr{A}$ and $\leq$ be the semi-order generated by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true:*

(i) Suppose that x is invertible and that $x^{-1} > 0$ implies $x > 0$, then for any integer $n \geq 1$, we have $\lambda^n \leq \lambda \leq e$.

(ii) For any $u > 0$, we have $u \nleq \lambda u$, i.e., $\lambda u - u \notin P$.

(iii) If $\lambda \geq 0$, then $(e-\lambda)^{-1} \geq 0$.

Proposition 2.7.[10] *Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$ and P a solid cone in Banach algebra A. If a sequence $\{x_n\}$ in X converges to $x \in X$, then*

(i) $\{d(x_n,x)\}$ is a c-sequence.

(ii) For any $p \in \mathbb{N}$, $\{d(x_n,x_{n+p})\}$ is a c-sequence.

Lemma 2.1.[15] *If E is a real Banach space with a cone P and if $a \leq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = 0$.*

Lemma 2.2.[16] *If E is a real Banach space with a cone P and if $0 \leq u \ll c$ for all $0 \ll c$, then $u = 0$.*

Lemma 2.3.[16] *If E is a real Banach space with a solid cone P and if $\| x_n \| \to 0$ as $n \to \infty$, then for any $0 \ll c$, there exists $N \in \mathbb{N}$ such that $x_n \ll c$ for any $n > N$.*

Lemma 2.4.[10] *If $\mathscr{A}$ is a Banach algebra and $k \in \mathscr{A}$ with $r(k) < 1$, then $\| k^n \| \to 0$ as $n \to \infty$.*

Lemma 2.5.[10] Let $\mathscr{A}$ be a Banach algebra and $x, y \in \mathscr{A}$. If x and y commute, then the following hold:

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x+y) \leq r(x) + r(y)$;

(iii) $| r(x) - r(y) | \leq r(x-y)$.

Lemma 2.6.[10] *Let $\mathscr{A}$ be a Banach algebra and $\{x_n\}$ a sequence in $\mathscr{A}$. Suppose that $\{x_n\}$ converge to $x \in \mathscr{A}$ and that x_n and x commute for all n, then $r(x_n) \to r(x)$ as $n \to \infty$.*

Lemma 2.7.[17-18] *Let P be a solid cone in a Banach algebra $\mathscr{A}$ and $\{\alpha, \beta, \gamma\} \subset \mathscr{A}$ with $r(\gamma) < 1$. If $\{\alpha, \beta, \gamma\}$ commute, then*

$$r\Big((e-\gamma)^{-1}(\alpha+\beta)\Big) \leq \frac{r(\alpha+\beta)}{1-r(\gamma)} \leq \frac{r(\alpha)+r(\beta)}{1-r(\gamma)}.$$

Lemma 2.8.[17-18] *(Cauchy Principle) Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$, P a solid cone in $\mathscr{A}$ and $k \in P$ with $r(k) < 1$. If a sequence $\{x_n\} \subset X$ satisfies that*

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \forall n = 0, 1, 2, \cdots.$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.9.[9] *Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$, P a solid cone in $\mathscr{A}$ and $\{x_n\} \subset X$ a sequence. If $\{x_n\}$ is convergent, then the limits of $\{x_n\}$ is unique.*

Definition 2.4.[19] *Two mappings $f, g : X \to X$ are weakly compatible if, for every $x \in X$, $fgx = gfx$ holds whenever $fx = gx$.*

Definition 2.5.[19] *Let $f, g : X \to X$ be two mappings. If $w = fx = gx$ for some $x, w \in X$, then x is called a coincidence point of f and g, and w is a point of coincidence of f and g.*

Lemma 2.10.[19] *If $f, g : X \to X$ be weakly compatible and have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

In 1982, Wang, Li and Gao[20] introduce the following concepts:

Let (X,d) be a real metric space, $f : X \to X$ a mapping. If there exists $a > 1$ such that

$$d(fx, fy) \geq ad(x,y),\ \forall\, x,y \in X.$$

Then f is called I-expansive mapping.

They also proved that any onto I-expansive mapping on complete real metric space has a unique fixed point.

Obviously,

$$d(fx, fy) \geq ad(x,y) \iff bd(fx, fy) \geq d(x,y),$$

where $a > 1$ and $0 < b < 1$ are two constant real numbers.

In this paper, by generalizing the concepts of I-expansive mappings, we will obtain the existence theorems of unique common fixed points for two mappings satisfying generalized expansive conditions on a cone metric space (X,d) over a Banach algebra $\mathscr{A}$ and give some unique fixed point theorems.

3. Unique common fixed points for expansive mappings

Theorem 3.1. *Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$ and $S,T : X \to X$ two mappings satisfying $SX \subset TX$ and P a solid cone in $\mathscr{A}$. Suppose that for each $x,y \in X$ with $x \neq y$,*

$$\alpha d(Tx,Ty) + \beta d(Sx,Ty) + \gamma d(Sy,Tx) \geq d(Sx,Sy), \tag{3.1}$$

where $\{\alpha, \beta, \gamma\} \subset P$ commutes and satisfies $r(\alpha) + r(\beta) + r(\gamma) < 1$. If TX or SX is complete, then S,T have a unique point of coincidence. Furthermore, if S and T are weakly compatible, then S,T have a unique common fixed point.

Proof. Take an $x_0 \in X$. Using $SX \subset TX$, we obtain sequence $\{x_n\}$ and $\{y_n\}$ in X satisfying

$$y_n = Sx_n = Tx_{n+1}, \forall n = 0,1,2,\cdots. \tag{3.2}$$

If there exists n such that $x_n = x_{n+1}$, then $y_n = Sx_n = Tx_n$, hence y_n is the point of coincidence of S and T. So we can assume that $x_n \neq x_{n+1}, \forall n = 0,1,2,\cdots$.

Suppose that $r(\gamma) \leq r(\beta)$, then $r(\alpha) + 2r(\gamma) < 1$.

For any fixed $n = 0, 1, 2, \cdots$, by (3.1),

$$\alpha d(Tx_{n+1}, Tx_{n+2}) + \beta d(Sx_{n+1}, Tx_{n+2}) + \gamma d(Sx_{n+2}, Tx_{n+1}) \geq d(Sx_{n+1}, Sx_{n+2}),$$

using (3.2), we have

$$\alpha d(y_n, y_{n+1}) + \gamma d(y_{n+2}, y_n) \geq d(y_{n+1}, y_{n+2}),$$

hence

$$\alpha d(y_n, y_{n+1}) + \gamma [d(y_{n+2}, y_{n+1}) + d(y_{n+1}, y_n)] \geq d(y_{n+1}, y_{n+2}).$$

Therefore

$$(e - \gamma) d(y_{n+1}, y_{n+2}) \leq (\alpha + \gamma) d(y_n, y_{n+1}).$$

Since $(e - \gamma)$ is invertible and $(e - \gamma)^{-1} \geq 0$ by Proposition 2.1 and Proposition 2.6, we obtain

$$d(y_{n+1}, y_{n+2}) \leq (e - \gamma)^{-1} (\alpha + \gamma) d(y_n, y_{n+1}).$$

Since $r((e - \gamma)^{-1}(\alpha + \gamma)) \leq \frac{r(\alpha) + r(\gamma)}{1 - r(\gamma)} < 1$ by Lemma 2.7, $\{y_n\}$ is a Cauchy sequence by Lemma 2.8. Similarly, $\{y_n\}$ is also Cauchy for the case $r(\beta) \leq r(\gamma)$.

Since TX or SX is complete and $y_n \in SX \subset TX, \forall n$, there exist $z, x \in X$ such that $y_n \to z = Tx$ as $n \to \infty$.

For x_{n+1} and x, using (3.1), we obtain

$$\alpha d(Tx_{n+1}, Tx) + \beta d(Sx_{n+1}, Tx) + \gamma d(Sx, Tx_{n+1}) \geq d(Sx_{n+1}, Sx),$$

that is,

$$\alpha d(y_n, Tx) + \beta d(y_{n+1}, Tx) + \gamma d(Sx, y_n) \geq d(y_{n+1}, Sx),$$

hence

$$\alpha d(y_n, Tx) + \beta d(y_{n+1}, Tx) + \gamma [d(Sx, y_{n+1}) + d(y_n, y_{n+1})] \geq d(y_{n+1}, Sx),$$

so

$$d(y_{n+1}, Sx) \leq (e - \gamma)^{-1} [\alpha d(y_n, Tx) + \beta d(y_{n+1}, Tx) + \gamma d(y_n, y_{n+1})].$$

Since $(e - \gamma)^{-1} [\alpha d(y_n, Tx) + \beta d(y_{n+1}, Tx) + \gamma d(y_n, y_{n+1})]$ is a c-sequence by Proposition 2.2 and Proposition 2.4 and Proposition 2.7, so $d(y_{n+1}, Sx)$ is also a c-sequence, hence $\{y_n\} \to Sx$ as $n \to \infty$. Therefore $z = Tx = Sx$ by Lemma 2.9.

If z_1 is also a point of coincidence of S and T, then there exists $x_1 \in X$ such that $z_1 = Sx_1 = Tx_1$. For x and x_1, using (3.1), we have

$$\alpha d(Tx, Tx_1) + \beta d(Sx, Tx_1) + \gamma d(Sx_1, Tx) \geq d(Sx, Sx_1),$$

that is,

$$d(z, z_1) \leq (\alpha + \beta + \gamma) d(z, z_1),$$

hence

$$d(z, z_1) \leq (\alpha + \beta + \gamma)^n d(z, z_1), \forall n.$$

Using $r(\alpha + \beta + \gamma) \leq r(\alpha) + r(\beta) + r(\gamma) < 1$, we have $\| (\alpha + \beta + \gamma)^n \| \to 0$ as $n \to \infty$ by Lemma 2.4, hence $\{(\alpha + \beta + \gamma)^n d(z, z_1)\}$ is a c-sequence, so $d(z, z_1) = 0$, i.e., $z = z_1$. This means that z is the unique point of coincidence of S and T. If S and T are weakly compatible, then z is the unique common fixed point of S and T by Lemma 2.10.

Using Theorem 3.1, we can obtain the following fixed point theorems:

Theorem 3.2. *Let (X, d) be a cone metric space over a Banach algebra $\mathscr{A}$ and $T : X \to X$ a mapping and let P is a solid cone in $\mathscr{A}$. Suppose that for each $x, y \in X, x \neq y$,*

$$\alpha d(Tx, Ty) + \beta d(T^2x, Ty) + \gamma d(T^2y, Tx) \geq d(T^2x, T^2y),$$

where $\{\alpha, \beta, \gamma\} \subset P$ commutes and satisfies $r(\alpha) + r(\beta) + r(\gamma) < 1$. If TX is complete, then T has a unique fixed point.

Proof. Let $T^2 = S$, then S and T are weakly compatible, hence S, T satisfy all conditions of Theorem 3.1, so T has a unique fixed point by Theorem 3.1.

Theorem 3.3. *Let (X, d) be a cone metric space over a Banach algebra $\mathscr{A}$ and $T : X \to X$ a mapping satisfying $TX = T^2X$ and let P is a solid cone in $\mathscr{A}$. Suppose that for each $x, y \in X, x \neq y$,*

$$\alpha d(T^2x, T^2y) + \beta d(Tx, T^2y) + \gamma d(Ty, T^2x) \geq d(Tx, Ty),$$

where $\{\alpha, \beta, \gamma\} \subset P$ commutes and satisfies $r(\alpha) + r(\beta) + r(\gamma) < 1$. If TX is complete, then T has a unique fixed point.

Proof. Let $F = T^2$ and $G = T$, then there exists $x \in X$ such that $Fx = Gx$ by Theorem 3.1, i.e., $T(Tx) = Tx$, hence Tx is the fixed point of T. The uniqueness of fixed point of T is obvious.

Theorem 3.4. *Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$ and $T : X \to X$ a onto mapping and let P is a solid cone in $\mathscr{A}$. Suppose that for each $x,y \in X, x \neq y$,*

$$\alpha d(Tx,Ty) + \beta d(x,Ty) + \gamma d(y,Tx) \geq d(x,y),$$

where $\{\alpha,\beta,\gamma\} \subset P$ commutes and satisfies $r(\alpha)+r(\beta)+r(\gamma) < 1$. Then T has a unique fixed point.

Proof. Let $S = 1_X$ in Theorem 3.1, then the conclusion follows from Theorem 3.1.

Theorem 3.5. *Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$ and $S : X \to X$ a mapping and let P is a solid cone in $\mathscr{A}$. Suppose that for each $x,y \in X, x \neq y$,*

$$\alpha d(x,y) + \beta d(Sx,y) + \gamma d(Sy,x) \geq d(Sx,Sy),$$

where $\{\alpha,\beta,\gamma\} \subset P$ commutes and satisfies $r(\alpha)+r(\beta)+r(\gamma) < 1$. If SX is complete, then S has a unique point.

Proof. Let $T = 1_X$ in Theorem 3.1, then the conclusion follows from Theorem 3.1.

Remark 3.1. If $\beta = \gamma = 0$, then Theorem 3.4 is a new version and generalization of a fixed point theorem for a I-expansive mapping in [20] on cone metric space over Banach algebras; If $\alpha = 0, \beta = \gamma$, then Theorem 3.4 is the expansive version of Chatterjea type fixed point theorem. If $\alpha = 0, \beta = \gamma$, then Theorem 3.5 reduce to Theorem 3.2 in [10], i.e., it is the version of Chatterjea type fixed point theorem on cone metric space over Banach algebras. Hence Theorem 3.1-Theorem 3.5 generalize and improve many known fixed point and common fixed point theorems.

Example 3.1. Let $\mathscr{A} = C^1_{\mathbb{R}}[0,1]$ and define a norm on $\mathscr{A}$ by $\| x \| = \| x \|_\infty + \| x' \|_\infty$ for $x \in \mathscr{A}$. Define multiplication in $\mathscr{A}$ as just pointwise multiplication. Then $\mathscr{A}$ is a real Banach algebra with unit $e = 1$. The set $P = \{x \in \mathscr{A} : x \geq 0\}$ is not normal cone(see[10, 21]).

Let $X = \{1,2,3\}$ and define $d : X \times X \to \mathscr{A}$ by

$$d(1,2)(t) = d(2,1)(t) = e^t, d(1,3)(t) = d(3,1)(t) = 3e^t, d(2,3)(t) = d(3,2)(t) = 2e^t, d(x,x)(t) = 0.$$

Then (X,d) is a complete cone metric space over a Banach algebra $\mathscr{A}$ without normality.

Define two mappings $S,T:X\to X$ by

$$S1=S2=2, S3=1, T1=1, T2=2, T3=3.$$

And let $\alpha,\beta,\gamma\in P$ as follows

$$\alpha(t)=\frac{1}{5}t+\frac{1}{5},\ \beta(t)=\frac{1}{10}t+\frac{1}{5},\ \gamma(t)=\frac{1}{20}t+\frac{1}{5},\ \forall\, t\in[0,1].$$

It is easy to prove that $r(\alpha)=\frac{2}{5},\ r(\beta)=\frac{3}{10},\ r(\gamma)=\frac{5}{20}$, hence

$$r(\alpha+\beta+\gamma)\le r(\alpha)+r(\beta)+r(\gamma)=\frac{19}{20}<1.$$

And for any $t\in[0,1]$,

$$\begin{aligned}
&[\alpha\, d(T1,T3)+\beta\, d(S1,T3)+\gamma d(S3,T1)](t)\\
=\ &[\alpha\, d(1,3)+\beta\, d(2,3)+\gamma d(1,1)](t)\\
=\ &\Big[3\big(\frac{1}{5}t+\frac{1}{5}\big)+2\big(\frac{1}{10}t+\frac{1}{5}\big)\Big]e^t\\
\ge\ &e^t\\
=\ &d(S1,S3)(t)
\end{aligned}$$

and

$$\begin{aligned}
&[\alpha\, d(T2,T3)+\beta\, d(S2,T3)+\gamma d(S3,T2)](t)\\
=\ &[\alpha\, d(2,3)+\beta\, d(2,3)+\gamma d(1,2)](t)\\
=\ &\Big[2\big(\frac{1}{5}t+\frac{1}{5}\big)+2\big(\frac{1}{5}t+\frac{1}{5}\big)+\big(\frac{1}{20}t+\frac{1}{5}\big)\Big]e^t\\
\ge\ &e^t\\
=\ &d(S2,S3)(t).
\end{aligned}$$

Hence S and T have a unique common fixed point 2 by Theorem 3.1.

Next, we give the second unique common fixed point theorem for two mappings satisfying another generalized expansive condition on cone metric spaces over Banach algebras.

Theorem 3.6. *Let (X,d) be a complete cone metric space over a Banach algebra $\mathscr{A}$, P is a solid cone in $\mathscr{A}$ and $S,T : X \to X$ two surjective mappings. Suppose that for any $x,y \in X, x \neq y$,*

$$\alpha d(Sx,Ty) + \beta d(x,Ty) + \gamma d(y,Sx) \geq d(x,y), \tag{3.3}$$

where $\{\alpha,\beta,\gamma\} \subset P$ commutes and satisfies $r(\alpha)+2\max\{r(\beta),r(\gamma)\} < 1$. Then S,T have an unique common fixed point.

Proof. Taking an element $x_0 \in X$ and using the surjective conditions of S and T, we can construct a sequence $\{x_n\}$ satisfying

$$x_{2n} = Sx_{2n+1}, x_{2n+1} = Tx_{2n+2}, n = 0,1,2,\cdots. \tag{3.4}$$

If there is a $n \in \mathbb{N}$ such that $x_{2n} = x_{2n+1}$, then by (3.3),

$$\alpha d(Sx_{2n+1},Tx_{2n+2}) + \beta d(x_{2n+1},Tx_{2n+2}) + \gamma d(x_{2n+2},Sx_{2n+1}) \geq d(x_{2n+1},x_{2n+2}),$$

using (3.4) and $d(x_{2n},x_{2n+1}) = 0$, we obtain

$$\gamma d(x_{2n},x_{2n+2}) \geq d(x_{2n+1},x_{2n+2}),$$

hence

$$\gamma[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \geq d(x_{2n+1},x_{2n+2}),$$

that is,

$$\gamma d(x_{2n+1},x_{2n+2}) \geq d(x_{2n+1},x_{2n+2}),$$

therefore

$$(e-\gamma)d(x_{2n+1},x_{2n+2}) \leq 0.$$

Hence $d(x_{2n+1},x_{2n+2}) = 0$ by Proposition 2.6, i.e., $x_{2n+1} = x_{2n+2}$.

If there is a $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n+2}$, then by (3.3),

$$\alpha d(Sx_{2n+3},Tx_{2n+2}) + \beta d(x_{2n+3},Tx_{2n+2}) + \gamma d(x_{2n+2},Sx_{2n+3}) \geq d(x_{2n+3},x_{2n+2})$$

using (3.4) and $d(x_{2n+1},x_{2n+2}) = 0$, we obtain

$$\beta d(x_{2n+3},x_{2n+1}) \geq d(x_{2n+3},x_{2n+2}),$$

hence

$$\beta[d(x_{2n+1},x_{2n+2})+d(x_{2n+2},x_{2n+3})] \geq d(x_{2n+1},x_{2n+2}),$$

that is ,

$$\beta d(x_{2n+2},x_{2n+3}) \geq d(x_{2n+2},x_{2n+3}),$$

therefore

$$(e-\beta)d(x_{2n+2},x_{2n+3}) \leq 0.$$

Hence $d(x_{2n+2},x_{2n+3}) = 0$ by Proposition 2.6, i.e, $x_{2n+2} = x_{2n+3}$.

Therefore, we have the following fact: If there is a $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_n = x_{n+1}$ for all $n \geq n_0$. In this case, $\{x_n\}$ must be a Cauchy sequence. So from now on, we assume that $x_n \neq x_{n+1}, \forall n = 0,1,2,\cdots$.

For any fixed $n \in \mathbb{N}$, by (3.3),

$$\alpha d(Sx_{2n+1},Tx_{2n+2})+\beta d(x_{2n+1},Tx_{2n+2})+\gamma d(x_{2n+2},Sx_{2n+1}) \geq d(x_{2n+1},x_{2n+2}),$$

that is,

$$\alpha d(x_{2n},x_{2n+1})+\gamma d(x_{2n+2},x_{2n}) \geq d(x_{2n+1},x_{2n+2}),$$

hence

$$\alpha d(x_{2n},x_{2n+1})+\gamma[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})] \geq d(x_{2n+1},x_{2n+2}),$$

that is,

$$(e-\gamma)d(x_{2n+1},x_{2n+2}) \leq (\alpha+\gamma)d(x_{2n},x_{2n+1}).$$

Using $r(\gamma) < 1$, we obtain

$$d(x_{2n+1},x_{2n+2}) \leq (e-\gamma)^{-1}(\alpha+\gamma)d(x_{2n},x_{2n+1}). \tag{3.5}$$

Similarly, we obtain

$$d(x_{2n+2},x_{2n+3}) \leq (e-\beta)^{-1}(\alpha+\beta)d(x_{2n+1},x_{2n+2}). \tag{3.6}$$

Let $K_1 = (e-\gamma)^{-1}(\alpha+\gamma),\ K_2 = (e-\beta)^{-1}(\alpha+\beta)$ and $K = K_1K_2$. Since $\{\alpha,\beta,\gamma\}$ commute and $(e-\beta)^{-1} = \sum_{i=0}^{\infty}\beta^i$ and $(e-\gamma)^{-1} = \sum_{i=0}^{\infty}\gamma^i$, hence $\{\alpha,\beta,\gamma,K_1,K_2\}$ also commute.

Therefore by Lemma 2.5 and Lemma 3.1,

$$r(K) \leq r(K_1)r(K_2) \leq \frac{r(\alpha)+r(\gamma)}{1-r(\gamma)}\,\frac{r(\alpha)+r(\beta)}{1-r(\beta)} < 1.$$

Using mathematical induction and (3.5)-(3.6), we can obtain

$$d(x_{2n+1},x_{2n+2}) \leq K_1\, d(x_{2n},x_{2n+1}) \leq K_1K_2 d(x_{2n-1},x_{2n}) \leq \cdots \leq K^n K_1 d(x_0,x_1) \tag{3.7}$$

and

$$d(x_{2n+2},x_{2n+3}) \leq K_2 d(x_{2n+1},x_{2n+2}) \leq K^{n+1} d(x_0,x_1). \tag{3.8}$$

So for any $p,q \in \mathbb{N}$ with $p<q$,

$$d(x_{2p+1},x_{2q+1}) \leq \sum_{i=2p+1}^{2q} d(x_i,x_{i+1}) \leq \big(K_1 \sum_{i=p}^{q-1} K^i + \sum_{i=p+1}^{q} K^i\big) d(x_0,x_1) \leq (e-K)^{-1}K^p(K_1+K)d(x_0,x_1). \tag{3.9}$$

Similarly,

$$d(x_{2p},x_{2q+1}) \leq \sum_{i=2p}^{2q} d(x_i,x_{i+1}) \leq \big(\sum_{i=p}^{q} K^i + K_1 \sum_{i=p}^{q-1} K^i\big) d(x_0,x_1) \leq (e-K)^{-1}K^p(e+K_1)d(x_0,x_1); \tag{3.10}$$

$$d(x_{2p},x_{2q}) \leq \sum_{i=2p}^{2q-1} d(x_i,x_{i+1}) \leq \big(\sum_{i=p}^{q-1} K^i + K_1 \sum_{i=p}^{q-1} K^i\big) d(x_0,x_1) \leq (e-K)^{-1}K^p(e+K_1)d(x_0,x_1); \tag{3.11}$$

$$d(x_{2p+1},x_{2q}) \leq \sum_{i=2p+1}^{2q-1} d(x_i,x_{i+1}) \leq \big(K_1 \sum_{i=p}^{q-1} K^i + \sum_{i=p+1}^{q-1} K^i\big) d(x_0,x_1) \leq (e-K)^{-1}K^p(K_1+K)d(x_0,x_1). \tag{3.12}$$

Since $r(K)<1$, $\| K^n \| \to 0$ as $n\to\infty$ by Lemma 2.4, hence

$$\| (e-K)^{-1}K^p(K_1+K)d(x_0,x_1) \| \to 0 \text{ as } p\to\infty \tag{3.13}$$

and

$$\| (e-K)^{-1}K^p(e+K_1)d(x_0,x_1) \| \to 0 \ \text{ as } p\to\infty. \tag{3.14}$$

Therefore by Lemma 2.3, for any $0 \ll c$ there exists N such that

$$(e-K)^{-1}K^p(K_1+K)d(x_0,x_1) \ll c,\ \forall p>N \tag{3.15}$$

and

$$(e-K)^{-1}K^p(e+K_1)d(x_0,x_1) \ll c,\ \forall p>N. \tag{3.16}$$

Combining (3.9)-(3.12) and (3.15)-(3.16), we can show that there is a $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $n > m > n_0$. Hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. And since S and T are surjective, there exist $x, y \in X$ such that $z = Tx = Sy$.

For x_{2n+1} and x, we have

$$\alpha d(Sx_{2n+1}, Tx) + \beta d(x_{2n+1}, Tx) + \gamma d(x, Sx_{2n+1}) \geq d(x_{2n+1}, x)$$

that is,

$$\alpha d(x_{2n}, Tx) + \beta d(x_{2n+1}, Tx) + \gamma d(x, x_{2n}) \geq d(x_{2n+1}, x)$$

hence

$$\alpha d(x_{2n}, Tx) + \beta d(x_{2n+1}, Tx) + \gamma[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x)] \geq d(x_{2n+1}, x)$$

which implies that

$$(e - \gamma)d(x_{2n+1}, x) \leq \alpha d(x_{2n}, Tx) + \beta d(x_{2n+1}, Tx) + \gamma d(x_{2n}, x_{2n+1}). \tag{3.17}$$

Since $r(\gamma) < 1$ implies $(e - \gamma)^{-1} \geq 0$, we obtain

$$d(x_{2n+1}, x) \leq (e-\gamma)^{-1}\alpha d(x_{2n}, Tx) + (e-\gamma)^{-1}\beta d(x_{2n+1}, Tx) + (e-\gamma)^{-1}\gamma d(x_{2n}, x_{2n+1}). \tag{3.18}$$

Since $x_n \to Tx$ as $n \to \infty$ and $\{x_n\}$ is Cauchy, the right-hand side of (3.18) is a c-sequence by Proposition 2.2 and Proposition 2.4 and Proposition 2.7, hence for each $c \gg 0$ there exists N such that

$$(e-\gamma)^{-1}\alpha d(x_{2n}, Tx) + (e-\gamma)^{-1}\beta d(x_{2n+1}, Tx) + (e-\gamma)^{-1}\gamma d(x_{2n}, x_{2n+1}) \ll c, \ \forall n > N. \tag{3.19}$$

Combining (3.18), we have for each $c \gg 0$ there exists N such that

$$d(x_{2n+1}, x) \ll c, \ \forall n > N, \tag{3.20}$$

hence $x_{2n+1} \to x$ as $n \to \infty$. But $x_{2n+1} \to z$ as $n \to \infty$, hence $z = x$ by Lemma 2.10. Similarly, we can obtain $z = y$. Therefore $z = Tz = Sz$, that is, z is a common fixed point of S and T.

If z_1 is also a common fixed point of S and T, i.e., $z_1 = Sz_1 = Tz_1$, then by (3.3),

$$\alpha d(Sz, Tz_1) + \beta d(z, Tz_1) + \gamma d(z_1, Sz) \geq d(z, z_1)$$

hence

$$d(z, z_1) \leq (\alpha + \beta + \gamma) d(z, z_1).$$

Since $r(\alpha+\beta+\gamma) \leq r(\alpha)+r(\beta)+r(\gamma) \leq r(\alpha)+2\max\{r(\beta), r(\gamma)\} < 1$, so $z = z_1$ by the proof process in Theorem 3.1. Hence z is the unique common fixed point of S and T.

Remark 3.2. Using Theorem 3.6, we can give many fixed point theorems, but we omit those here.

Conflict of Interests

The authors declare that there is no conflict of interests.

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