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COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper we prove coupled fixed point theorems of multivalued mappings in partially ordered metric spaces by utilizing the combination of multivalued monotone iterative technique and multivalued contraction principle. We also establish some results on the stability of coupled fixed point sets of multivalued mappings in partially ordered metric spaces.

Keywords: coupled fixed point; stability; multivalued mappings; partial order; metric spaces.

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1. Introduction

In 2006, Bhaskar and Lakshmikantham [2] introduced the notions of a mixed monotone mapping and a coupled fixed point, and proved some coupled fixed point theorems for single valued mixed monotone mappings. They further applied their results to the existence of a unique solution to a periodic boundary value problem associated with a first order ordinary differential equation. Afterwards, some coupled fixed point theorems in partially ordered metric spaces

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were developed by Lakshmikantham and Ciric [3], Luong and Thuan [5], Samet [8], among others.

In this paper, we obtain sufficient conditions on the existence of coupled fixed point of multivalued mappings in partially ordered metric spaces by using the combination of multivalued monotone iterative technique and multivalued contraction principle. We also provide sufficient conditions where the components of the coupled fixed point are equal. Results on the stability of coupled fixed point sets of multivalued mappings in partially ordered metric spaces are also proven.

2. Preliminaries

We recall some definitions and important results found in the literature.

Definition 2.1 A partially ordered set is a system $(X,\preceq)$ where X is a non-empty set and $\preceq$ is a binary relation on X satisfying, for all $x,y,z\in X$,

(1) $x\preceq x$ (reflexivity)
(2) if $x\preceq y$ and $y\preceq x$, then $x=y$ (antisymmetry)
(3) if $x\preceq y$ and $y\preceq z$, then $x\preceq z$ (transitivity)

Definition 2.2 A non-empty set X together with a metric $d: X\times X\to\mathbb{R}^+\cup\{0\}$ is called a metric space if the following conditions are satisfied by any $x,y,z\in X$:

a. $d(x,y)\geq 0$ and $d(x,y)=0$ if and only if $x=y$
b. $d(x,y)=d(y,x)$
c. $d(x,z)\leq d(x,y)+d(y,z)$

Definition 2.3 Let (X,d) be a metric space. A sequence $\{x_n\}\in X$ is a Cauchy sequence if it has the property that given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ whenever $n,m\geq N$. The metric (X,d) is complete if every Cauchy sequence in X is convergent.

Definition 2.4 $(X,d,\preceq)$ is a partially ordered complete metric space if (X,d) is a complete metric space and $(X,\preceq)$ is a partially ordered set.

Definition 2.5 The distance between any point a of X and any nonempty subset B of X is defined as:

$$d(a,B) := \inf_{b\in B} d(a,b).$$

For $A,B \in CB(X)$ (set of non-empty, closed, and bounded subsets of X), let

$$D(A,B) := \max\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\}.$$

D is called the Hausdorff metric induced by d.

Lemma 2.6 [7] Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} D(A_n,A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n,x) = 0$, then $x \in A$.

Definition 2.7 Let (X,d) be a complete metric space and $(X,\preceq)$ be a partially ordered set. Define the product metric on $X \times X$ as:

$$d((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + d(y_1,y_2)$$

for all $x_i, y_i \in X (i = 1,2)$.

For notational convenience, we use the same symbol d for the metric on X as well as for the product metric on $X \times X$.

Definition 2.8 Define partial order on the product space $X \times X$ as: for $(u,v),(x,y) \in X \times X$

$$(u,v) \preceq (x,y) \Leftrightarrow u \preceq x \text{ and } y \preceq v.$$

Definition 2.9 [2] Let $F : X \times X \to CB(X)$ ($f : X \times X \to X$) be a multivalued (single valued) mapping. A point (x,y) is said to be a coupled fixed point of the multivalued (single valued) mapping F (f) if $x \in F(x,y)$ ($x = f(x,y)$) and $y \in F(y,x)$ ($y = f(y,x)$).

The main result of Bhaskar and Lakshmikantham in [2] is the following coupled fixed point theorem.

Theorem 2.10 (Bhaskar and Lakshmikantham, 2006) [2] Let $f : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $\alpha \in [0,1)$ with

$$d(f(x,y), f(u,v)) \leq \frac{\alpha}{2}[d(x,u) + d(y,v)], for\ all\ u \preceq x\ and\ y \preceq v$$

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq f(x_0,y_0)\ and\ y_0 \geq f(y_0,x_0).$$

Then there exists $x, y \in X$ such that

$$x = f(x,y) \text{ and } y = f(y,x).$$

3. Main results

This section is divided into two parts: first part provides coupled fixed point theorems of multivalued and single valued in partially ordered metric spaces; second part gives sufficient conditions on the stability of coupled fixed point sets.

Coupled Fixed Point Theorems

Theorem 3.1 Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F : X \times X \to CB(X)$ be a multivalued mapping such that the following conditions are satisfied:

i. There exists $x_0, y_0 \in X$ and some $x_1 \in F(x_0, y_0), y_1 \in F(y_0, x_0)$ such that $(x_0, y_0) \preceq (x_1, y_1)$

ii. If $x_1, x_2, y_1, y_2 \in X$ and $(x_1, y_1) \preceq (x_2, y_2)$, then

   a. for all $u_1 \in F(x_1, y_1)$ there exists $u_2 \in F(x_2, y_2)$ such that $u_1 \preceq u_2$ and $d(u_1, u_2) \leq D(F(x_1, y_1), F(x_2, y_2))$

   b. for all $v_1 \in F(y_1, x_1)$ there exists $v_2 \in F(y_2, x_2)$ such that $v_2 \preceq v_1$ and $d(v_1, v_2) \leq D(F(y_1, x_1), F(y_2, x_2))$

iii. If $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$ for all n, and if $y_n \to y$ is a nonincreasing sequence in X, then $y \preceq y_n$ for all n

iv. There exists $\alpha \in (0,1)$ such that $D(F(x,y), F(u,v)) \leq \frac{\alpha}{2} d((x,y),(u,v))$ for all $(x,y) \preceq (u,v)$

Then F has a coupled fixed point.

Proof. Let $x_0, y_0 \in X$ such that there exists $x_1 \in F(x_0, y_0)$ and $y_1 \in F(y_0, x_0)$ satisfying $x_0 \preceq x_1$ and $y_1 \preceq y_0$, or that is, $(x_0, y_0) \preceq (x_1, y_1)$. If $x_0 = x_1$ and $y_0 = y_1$, then we are done. Suppose that is not the case, then from condition iv

$$D(F(x_0, y_0), F(x_1, y_1)) \leq \frac{\alpha}{2} d((x_0, y_0), (x_1, y_1))$$

Now from condition ii, since $(x_0,y_0) \preceq (x_1,y_1)$, then

- for all $x_1 \in F(x_0,y_0)$ there exists $x_2 \in F(x_1,y_1)$ such that $x_1 \preceq x_2$ and

$$\begin{aligned} d(x_1,x_2) &\leq D(F(x_0,y_0),F(x_1,y_1)) \\ &\leq \frac{\alpha}{2} d((x_0,y_0),(x_1,y_1)) \end{aligned}$$

- for all $y_1 \in F(y_0,x_0)$ there exists $y_2 \in F(y_1,x_1)$ such that $y_2 \preceq y_1$ and

$$\begin{aligned} d(y_1,y_2) &\leq D(F(y_0,x_0),F(y_1,x_1)) \\ &= D(F(y_1,x_1),F(y_0,x_0)) \\ &\leq \frac{\alpha}{2} d((y_1,x_1),(y_0,x_0)) \\ &= \frac{\alpha}{2} d((y_0,x_0),(y_1,x_1)) \\ &= \frac{\alpha}{2} d((x_0,y_0),(x_1,y_1)) \end{aligned}$$

Consider,

$$\begin{aligned} d((x_1,y_1),(x_2,y_2)) &= d(x_1,x_2) + d(y_1,y_2) \\ &\leq \frac{\alpha}{2} d((x_0,y_0),(x_1,y_1)) + \frac{\alpha}{2} d((x_0,y_0),(x_1,y_1)) \\ &= \alpha d((x_0,y_0),(x_1,y_1)) \end{aligned}$$

Similarly, it can be shown that $d((y_1,x_1),(y_2,x_2)) \leq \alpha d((x_0,y_0),(x_1,y_1))$.

Now we have $(x_1,y_1) \preceq (x_2,y_2)$, then from condition ii,

- there exists $x_3 \in F(x_2,y_2)$ such that $x_2 \preceq x_3$ and

$$\begin{aligned} d(x_2,x_3) &\leq D(F(x_1,y_1),F(x_2,y_2)) \\ &\leq \frac{\alpha}{2} d((x_1,y_1),(x_2,y_2)) \\ &\leq \frac{\alpha}{2} (\alpha d((x_0,y_0),(x_1,y_1))) \\ &= \frac{\alpha^2}{2} d((x_0,y_0),(x_1,y_1)) \end{aligned}$$

- there exists $y_3 \in F(y_2, x_2)$ such that $y_3 \preceq y_2$ and

$$\begin{aligned} d(y_2, y_3) &\le D(F(y_1, x_1), F(y_2, x_2)) \\ &\le \frac{\alpha}{2} d((y_1, x_1), (y_2, x_2)) \\ &\le \frac{\alpha}{2} (\alpha d((y_0, x_0), (y_1, x_1))) \\ &= \frac{\alpha^2}{2} d((x_0, y_0), (x_1, y_1)) \end{aligned}$$

Thus,

$$\begin{aligned} d((x_2, y_2), (x_3, y_3)) &= d(x_2, x_3) + d(y_2, y_3) \\ &\le \frac{\alpha^2}{2} d((x_0, y_0), (x_1, y_1)) + \frac{\alpha^2}{2} d((x_0, y_0), (x_1, y_1)) \\ &= \alpha^2 d((x_0, y_0), (x_1, y_1)) \end{aligned}$$

Continuing this process, we obtain a nondecreasing sequence $\{x_n\}$ and a nonincreasing sequence $\{y_n\}$ such that $x_{n+1} \in F(x_n, y_n)$ and $y_{n+1} \in F(y_n, x_n)$ with $x_n \preceq x_{n+1}$ and $y_{n+1} \preceq y_n$. Thus we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\le \frac{\alpha^n}{2} d((x_0, y_0), (x_1, y_1)) \\ d(y_n, y_{n+1}) &\le \frac{\alpha^n}{2} d((x_0, y_0), (x_1, y_1)) \end{aligned}$$

Hence

$$d((x_n, y_n), (x_{n+1}, y_{n+1})) \le \alpha^n d((x_0, y_0), (x_1, y_1))$$

Next, to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Let $N \in \mathbb{N}$ and $m, n > N$ such that $m > n$, then

$$
\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{m-1},x_m)\\
&\leq \frac{\alpha^n}{2}d((x_0,y_0),(x_1,y_1))+\frac{\alpha^{n+1}}{2}d((x_0,y_0),(x_1,y_1))+\ldots\\
&\quad +\frac{\alpha^{m-1}}{2}d((x_0,y_0),(x_1,y_1))\\
&= \frac{\alpha^n}{2}d((x_0,y_0),(x_1,y_1))(1+\alpha+\ldots+\alpha^{m-1-n})\\
&\leq \frac{\alpha^n}{2}d((x_0,y_0),(x_1,y_1))\frac{1}{1-\alpha}
\end{aligned}
$$

Therefore $d(x_n,x_m)\to 0$ as $n\to\infty$, and this implies that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x\in X$ such that $x_n\to x$ as $n\to\infty$. Similarly we can show that $\{y_n\}$ is a Cauchy sequence and there exists $y\in X$ such that $y_n\to y$ as $n\to\infty$.

Finally, we have to show that $x\in F(x,y)$ and $y\in F(y,x)$. Since $\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to x$, and $\{y_n\}$ is a nonincreasing sequence in X such that $y_n\to y$, therefore from condition iii, $x_n\preceq x$ for all n and $y\preceq y_n$ for all n, that is, $(x_n,y_n)\preceq(x,y)$ for all n. Using condition iv and the fact that $d((x_n,y_n),(x,y))\to 0$ it follows that

$$D(F(x_n,y_n),F(x,y))\leq \frac{\alpha}{2}d((x_n,y_n),(x,y))\to 0.$$

Moreover because $x_{n+1}\in F(x_n,y_n)$ and $\lim\limits_{n\to\infty} d(x_{n+1},x)=0$, it follows from Lemma 2.6 that $x\in F(x,y)$.

Similarly using condition iv and the fact that $d((y_n,x_n),(y,x))\to 0$ it follows that

$$D(F(y_n,x_n),F(y,x))\leq \frac{\alpha}{2}d((y_n,x_n),(y,x))\to 0.$$

Furthermore, since $y_{n+1}\in F(y_n,x_n)$ and $\lim\limits_{n\to\infty} d(y_{n+1},y)=0$, it follows from Lemma 2.6 that $y\in F(y,x)$. Therefore, $(x,y)\in X\times X$ is a coupled fixed point of F. □

A corollary of Theorem 3.1 would automatically follow for the existence of coupled fixed point of single valued mappings.

Corollary 3.2 Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $f : X\times X \to X$ be a single valued mapping such that the following conditions are satisfied:

i. There exists $x_0,y_0 \in X$ such that $x_1 = f(x_0,y_0)$, $y_1 = f(y_0,x_0)$ and $(x_0,y_0) \preceq (x_1,y_1)$
ii. If $x_1,x_2,y_1,y_2 \in X$ and $(x_1,y_1) \preceq (x_2,y_2)$, then $f(x_1,y_1) \preceq f(x_2,y_2)$ and $f(y_2,x_2) \preceq f(y_1,x_1)$.
iii. If $x_n \to x$ is a nondecreasing sequence in X, then $x_n \preceq x$ for all n, and if $y_n \to y$ is a nonincreasing sequence in X, then $y \preceq y_n$ for all n
iv. There exists $\alpha \in (0,1)$ such that $d(f(x,y), f(u,v)) \leq \dfrac{\alpha}{2} d((x,y),(u,v))$ for all $(x,y) \preceq (u,v)$

Then f has a coupled fixed point.

Theorem 3.3 In addition to the hypotheses of Theorem 3.1, suppose that $x_0,y_0 \in X$ are comparable, then $x = y$.

Proof. Suppose $x_0,y_0 \in X$ are comparable, in particular, assume that $x_0 \preceq y_0$. If $x_0 \preceq y_0$ then $(x_0,y_0) \preceq (y_0,x_0)$, and from condition ii.a of Theorem 3.1 for all $x_1 \in F(x_0,y_0)$ there exists $y_1 \in F(y_0,x_0)$ such that $x_1 \preceq y_1$. Now, assume that $x_k \preceq y_k$, then $(x_k,y_k) \preceq (y_k,x_k)$. Hence, by condition ii.a of Theorem 3.1, for all $x_{k+1} \in F(x_k,y_k)$ there exists $y_{k+1} \in F(y_k,x_k)$ such that $x_{k+1} \preceq y_{k+1}$. Therefore for every $x_n \in F(x_{n-1},y_{n-1})$ there exists $y_n \in F(y_{n-1},x_{n-1})$ such that $x_n \preceq y_n$, for all $n \in \mathbb{N}$.

For a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $D(x,F(x_n,y_n)) \leq \dfrac{\varepsilon}{2}$ and $D(y,F(y_n,x_n)) \leq \dfrac{\varepsilon}{2}$ whenever $n \geq N$. Hence we have that for all $n \geq N$,

$$
\begin{aligned}
D(x,y) &\leq D(x,F(x_{n+1},y_{n+1})) + D(F(x_{n+1},y_{n+1}),y) \\
&\leq D(x,F(x_{n+1},y_{n+1})) + D(F(x_{n+1},y_{n+1}),F(y_{n+1},x_{n+1})) \\
&\quad + D(F(y_{n+1},x_{n+1}),y)
\end{aligned}
$$

$$\le \frac{\varepsilon}{2} + D(F(x_{n+1}, y_{n+1}), F(y_{n+1}, x_{n+1})) + \frac{\varepsilon}{2}$$
$$\le \varepsilon + \frac{\alpha}{2} d((x_{n+1}, y_{n+1}), (y_{n+1}, x_{n+1}))$$
$$\le \varepsilon + \frac{\alpha}{2}(d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_{n+1}))$$
$$= \varepsilon + \alpha d(x_{n+1}, y_{n+1})$$
$$\le \varepsilon + \alpha(d(x_{n+1}, x) + d(x, y) + d(y, y_{n+1}))$$

However, $\alpha(d(x_{n+1}, x) + d(x, y) + d(y, y_{n+1})) \to \alpha d(x, y)$ as $n \to \infty$. Thus we have

$$D(x,y) \le \varepsilon + \alpha d(x,y)$$
$$\le \varepsilon + \alpha D(x,y)$$

Therefore, we have $(1-\alpha)D(x,y) \le \varepsilon$, which implies that $D(x,y) = 0$, since $\varepsilon > 0$ is abitrary and, hence we have $x = y$ □

Remark 3.4 Theorem 3.1 generalizes the result of of Bhaskar and Lakshmikantham in [2].

Stability of Coupled Fixed Point Sets

In this part, we are going to prove a lemma and a theorem for the stabilty of coupled fixed point sets of multivalued mappings in partially ordered metric spaces.

For notational convenience, we denote by $\{(u,v)\} <_G F(u,v)$ if and only if there exists $u_1 \in F(u,v)$ and $v_1 \in F(v,u)$ such that $(u,v) \preceq (u_1, v_1)$ where $d(u,u_1) \le D(u, F(u,v))$ and $d(v,v_1) \le D(v, F(v,u))$.

Lemma 3.5 Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $F_i : X \times X \to CB(X), (i = 1,2)$ satisfy all of the conditions of Theorem 3.1. Denote by $C(F_1)$ and $C(F_2)$ the respective coupled fixed point sets of F_1 and F_2, and $C = C(F_1) \cup C(F_2)$. If

a. for all $(x_0, y_0) \in C(F_1)$, $\{(x_0, y_0)\} <_G F_2(x_0, y_0)$
b. for all $(w_0, z_0) \in C(F_2)$, $\{(w_0, z_0)\} <_G F_1(w_0, z_0)$

then

$$D(C(F_1), C(F_2)) \le \sup_{(x,y)\in C} [D(F_1(x,y), F_2(x,y)) + D(F_1(y,x), F_2(y,x))] \frac{1}{1-\alpha}$$

where $\alpha \in (0,1)$.

Proof. Let $(x_0,y_0) \in C(F_1)$, then $\{(x_0,y_0)\} <_G F_2(x_0,y_0)$, that is, there exists $x_1 \in F_2(x_0,y_0)$ and $y_1 \in F_2(y_0,x_0)$ such that $(x_0,y_0) \preceq (x_1,y_1)$ where $d(x_0,x_1) \leq D(x_0,F_2(x_0,y_0))$ and $d(y_0,y_1) \leq D(y_0,F_2(y_0,x_0))$. Since F_2 satisfies all conditions of Theorem 3.1, then we obtain

- a nondecreasing sequence $\{x_n\}$ such that $x_n \in F_2(x_{n-1},y_{n-1})$ and $d(x_n,x_{n+1}) \leq \frac{\alpha^n}{2} d((x_0,y_0),(x_1,y_1))$
- a nonincreasing sequence $\{y_n\}$ such that $y_n \in F_2(y_{n-1},x_{n-1})$ and $d(y_n,y_{n+1}) \leq \frac{\alpha^n}{2} d((x_0,y_0),(x_1,y_1))$

Hence we have a sequence $\{(x_n,y_n)\}$ such that $x_n \in F_2(x_{n-1},y_{n-1})$ and $y_n \in F_2(y_{n-1},x_{n-1})$ and $d((x_n,y_n),(x_{n+1},y_{n+1})) \leq \alpha^n d((x_0,y_0),(x_1,y_1))$.

Let $N \in \mathbb{N}$ and $m,n > N$ such that $m > n$, then

$$\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\
&\leq \frac{\alpha^n}{2} d((x_0,y_0),(x_1,y_1)) + \frac{\alpha^{n+1}}{2} d((x_0,y_0),(x_1,y_1)) + \ldots \\
&\quad + \frac{\alpha^{m-1}}{2} d((x_0,y_0),(x_1,y_1)) \\
&= \frac{\alpha^n}{2} d((x_0,y_0),(x_1,y_1))(1 + \alpha + \ldots + \alpha^{m-1-n}) \\
&\leq \frac{\alpha^n}{2} d((x_0,y_0),(x_1,y_1)) \frac{1}{1-\alpha}
\end{aligned}$$

Therefore $d(x_n,x_m) \to 0$ as $n \to \infty$, thus $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $w \in X$ such that $x_n \to w$ as $n \to \infty$. In similar manner, we can show that $\{y_n\}$ is a Cauchy sequence and there exists $z \in X$ such that $y_n \to z$ where $(w,z) \in C(F_2)$.

Furthermore, let us consider:

$$\begin{aligned}
d((x_0,y_0),(w,z)) &= d(x_0,w) + d(y_0,z) \\
&\leq \sum_{i=0}^{\infty} d(x_i,x_{i+1}) + \sum_{j=0}^{\infty} d(y_j,y_{j+1})
\end{aligned}$$

$$
\begin{aligned}
&\leq d(x_0,x_1)+\sum_{i=1}^{\infty}\frac{\alpha^i}{2}d((x_0,y_0),(x_1,y_1))+d(y_0,y_1)\\
&\quad+\sum_{j=1}^{\infty}\frac{\alpha^j}{2}d((x_0,y_0),(x_1,y_1))\\
&=\sum_{n=0}^{\infty}\alpha^n d((x_0,y_0,(x_1,y_1))\\
&\leq d((x_0,y_0),(x_1,y_1))\frac{1}{1-\alpha}\\
&=[d(x_0,x_1)+d(y_0,y_1)]\frac{1}{1-\alpha}\\
&\leq [D(x_0,F_2(x_0,y_0))+D(y_0,F_2(y_0,x_0))]\frac{1}{1-\alpha}
\end{aligned}
$$

$$
d((x_0,y_0),(w,z))\leq [D(F_1(x_0,y_0),F_2(x_0,y_0))+D(F_1(y_0,x_0),F_2(y_0,x_0))]\frac{1}{1-\alpha}
$$

Reversing the roles of F_1 and F_2, and repeating the arguments above, leads to the conclusion that for each $(w_0,z_0)\in C(F_2)$, there exists $(w_1,z_1)\in F_1(w_0,z_0)$ such that $(w_0,z_0)\preceq(w_1,z_1)$ and $(x,y)\in C(F_1)$ such that

$$
d((w_0,z_0),(x,y))\leq [D(F_2(w_0,z_0),F_1(w_0,z_0))+D(F_2(z_0,w_0),F_1(z_0,w_0))]\frac{1}{1-\alpha}
$$

Hence it follows that

$$
D(C(F_1),C(F_2))\leq \sup_{(x,y)\in C}[D(F_1(x,y),F_2(x,y))+D(F_1(y,x),F_2(y,x))]\frac{1}{1-\alpha}
$$

where $\alpha\in(0,1)$. □

By using Lemma 3.5, we can now easily prove the theorem on the stability of fixed point sets.

Theorem 3.6 Let $(X,d,\preceq)$ be a partially ordered complete metric space. Let $F_i : X\times X\to CB(X),(i=0,1,2,\ldots)$ be a sequence of multivalued mappings, each satisfying all of the conditions of Theorem 3.1. Denote by $C(F_0)$, $C(F_1)$, $C(F_2),\ldots$ the fixed point sets of F_0,F_1 and $F_2\ldots$, respectively, and $C=\bigcup C(F_i)$. If $\lim_{n\to\infty}D(F_n(x),F_0(x))=0$ for all $x\in C$, and

a. for all $(x_0,y_0)\in C(F_0)$, $\{(x_0,y_0)\}<_G F_i(x_0,y_0)$ for $i=1,2,\ldots$

b. for all $(w_0,z_0)\in C(F_i)$, $\{(w_0,z_0)\}<_G F_0(w_0,z_0)$ for $i=1,2,\ldots$

then

$$\lim_{n\to\infty} D(C(F_n), C(F_0)) = 0$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} D(F_n(x), F_0(x)) = 0$ for all $x \in C$, then we can choose $N \in \mathbb{N}$ such that for $n \geq N$, $\sup_{x\in C} D(F_n(x), F_0(x)) < (1-\alpha)\varepsilon$. By Lemma 3.5, $D(C(F_n), C(F_0)) < \varepsilon$ for all n, and therefore $\lim_{n\to\infty} D(C(F_n), C(F_0)) = 0$.

□

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] I. Beg, A. R. Butt, Coupled fixed points of set valued mappings in partially ordered metric spaces. J. Nonlinear Sci. Appl. 3 (2010), 179-185.

[2] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., Theory Methods Appl. 65 (2006), 1379-1393.

[3] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., Theory Methods Appl. (2009), 4341-4349.

[4] T. C. Lim, On fixed point stability of set-valued contractive mappings with applications to generalized differential equations, J. Math. Anal. Appl. (1985), 436-441.

[5] N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal., Theory Methods Appl. 74 (2011), 983-992.

[6] J. G. Mehta and M. L. Joshi, On coupled fixed point theorem in partially ordered metric space, Int. J. Pure Appl. Sci. Technol. (2010), 87-92.

[7] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-488.

[8] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., Theory Methods Appl. (2010), 4508-4517.