Available online at http://scik.org
Adv. Fixed Point Theory, 8 (2018), No. 4, 384-400
https://doi.org/10.28919/afpt/3920
ISSN: 1927-6303

C- CLASS FUNCTION ON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS OF INTEGRAL TYPE IN $n-$ BANACH SPACES

RASHWAN A. RASHWAN[1], D. DHAMODHARAN[2,*], HASANEN A. HAMMAD[3] AND R. KRISHNAKUMAR[4]

[1]Department of Mathematics, Faculty of Science, Assuit University, Assuit 71516, Egypt
[2]Department of Mathematics, Jamal Mohamed College (Autonomous),Tiruchirapplli-620020, India
[3]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[4]Department of Mathematics,Urumu Dhanalakshmi College, Tiruchirappalli-620019, India

Abstract. In this paper, we present some common fixed point theorems for class of mappings satisfying contractive condition of integral type in $n-$Banach spaces via C-Class function. Our results are version of some known results.

Keywords: $n-$Banach space; subadditive integrable function; Lebesgue integrable mapping; common fixed point.

2010 AMS Subject Classification: Primary 47H10; Secondary 54H25.

1. Introduction

In [8, 9] Gähler introduced an attractive theory of $2-$norm and $n-$norm on a linear space. Raymond W. Freese and Y.J. Cho [7] gave as a survey of the latest results on the relations between linear $2-$normed spaces and normed linear spaces and completion of linear $2-$normed

*Corresponding author
E-mail address: dharan_raj28@yahoo.co.in
Received December 7, 2017

spaces. Later on Misiak [18] had also developed the notion of an $n-$norm in1989. The concept on $n-$inner product spaces is also due to Misiak who had studied the same as early as 1980. A systematic development of linear $n-$normed spaces has been extensively made by S.S. Kim and Y.J. Cho [15] and R. Malceski [17], A. Misiak [18] and Hendra Gunawan and Mashadi [13]. For related works of $n-$metric spaces and $n-$inner product spaces see for example [18], [11] and [13]. In 2002, Branciari [5] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality [4]. After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for diferent contractive mappings satisfying some properties, see for example [1, 3, 4, 12, 14, 19, 16]. The aim of this paper is to transform the concept of fixed point in $n-$Banach spaces into fixed point in $n-$Banach spaces of integral type by using known contractive type mapping.We recall some preliminary definitions.

Definition 1.1. [10] let n be a natural number, let X be a real vector space of dimension $d \geq n$ (d may be infinity). A real valued function $\|.,...,.\|$ on X^n satisfying four properties,

(1) $\|x_1,...,x_n\| = 0$ if and only if $x_1,...,x_n$ are linearly dependent in X,

(2) $\|x_1,...,x_n\|$ is invariant under permutation of $x_1,x_2,...,x_n$,

(3) $\|x_1,...,x_{n-1},\alpha x_n\| = |\alpha|\,\|x_1,...,x_{n-1},x_n\|$ for every $\alpha \in R$,

(4) $\|x_1,...,x_{n-1},y+z\| \leq \|x_1,...,x_{n-1},y\| + \|x_1,...,x_{n-1},z\|$ for all y and z in X, is called an $n-$norm over X and the pair $(X,\|.,...,.\|)$ is called a linear $n-$normed spaces.

Example 1.1. [13]. Let $X = R^n$ with the norm $\|.,...,.\|$ on X by

$$\|x_1,...,x_n\| = \left|x_{ij}\right| = \left(\begin{vmatrix} x_{11} & x_{12} & .. & x_{1n} \\ x_{21} & x_{22} & .. & x_{2n} \\ .. & .. & .. & .. \\ x_{n1} & x_{n2} & .. & x_{nn} \end{vmatrix}\right)$$

where $x_i = x_{i1}, x_{i2}, ..., x_{in} \in R^n$ for each $i = 1,2,...,n$. Then $(X,\|.,...,.\|)$ is a linear $n-$normed space.

Definition 1.2. [10]. A sequence x_k in an $n-$normed space $(X,\|.,...,.\|)$ is said to converge to an element $x \in X$ (in the $n-$norm) whenever $\lim_{k\to\infty} \|u_1,...,u_{n-1},x_k - x\| = 0$ for every $u_1,...,u_{n-1} \in X$.

Definition 1.3. [19]. A sequence x_k in an $n-$normed space $(X,\|.,...,.\|)$ is said to be Cauchy sequence with respect to $n-$norm if $\lim_{k\to\infty}\|u_1,...,u_{n-1},x_k-x\|=0$ for every $u_1,...,u_{n-1}\in X$.

Definition 1.4. [10]. If every Cauchy sequence in X converges to an element, $x\in X$ then X is said to be complete (with respect to the $n-$norm). A complete $n-$normed space is called an $n-$Banach space.

Definition 1.5. [10]. Let X be a $n-$Banach space and T be a self mapping of X. T is said to be continuous at x if for every sequence x_k in X, $x_k\longrightarrow x$ as $k\to\infty$ implies $Tx_k\longrightarrow Tx$ as $k\to\infty$ in X.

Definition 1.6. A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t)=0$ if and only if $t=0$.

Definition 1.7. An ultra altering distance function is a continuous, nondecreasing mapping $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(t)>0$,$t>0$ and $\varphi(0)\geq 0$.

We denote this set with Φ_u

Definition 1.8. [21] A mapping $F:[0,\infty)^2\to[0,\infty)$ is called cone *C-class* function if it is continuous and satisfies following axioms:

(1) $F(s,t)\leq s$;

(2) $F(s,t)=s$ implies that either $s=0$ or $t=0$; for all $s,t\in P$.

We denote cone C-class functions as $\mathscr{C}$.

Example 1.2. [21] The following functions $F:[0,\infty)^2\to[0,\infty)$ are elements of $\mathscr{C}$, for all $s,t\in[0,\infty)$:

(1) $F(s,t)=s-t$,

(2) $F(s,t)=ks$, where $0<k<1$,

(3) $F(s,t)=s\beta(s)$, where $\beta:[0,\infty)\to[0,1)$,

(4) $F(s,t)=\Psi(s)$, where $\Psi:[0,\infty)\to[0,\infty)$,$\Psi(0)=0$,$\Psi(s)>0$ for all $s\in[0,\infty)$ with $s\neq 0$ and $\Psi(S)\leq s$ for all $s\in[0,\infty)$.,

(5) $F(s,t)=s-\varphi(s)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$;

(6) $F(s,t) = s - h(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) = 0 \Leftrightarrow t = 0$ for all $t,s > 0$.

(7) $F(s,t) = \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a upper semi continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Lemma 1.1. Let ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathscr{C}$ and $\{s_n\}$ a decreasing sequence in $[0,\infty)$ such that

$$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n)) \tag{3.1}$$

for all $n \geq 1$. Then $\lim\limits_{n\to\infty} s_n = 0$.

Definition 1.9. [2]. $\zeta : [0,\infty) \longrightarrow [0,\infty)$ is subadditive on each $[a,b] \subset [0,\infty]$ if,

$$\int_0^{a+b} \zeta(t)dt = \int_0^{a} \zeta(t)dt + \int_0^{b} \zeta(t)dt$$

2. Main results

Theorem 2.1. Let X be a n Banach space. Suppose f be a self mapping of X such that

$$\begin{aligned}\psi\Big(\int_0^{\|fx-fy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt\Big) \leq F\Big(\psi\Big(\alpha \int_0^{\|x-y,u_1,\ldots,u_{n-1}\|} \zeta(t)dt + \beta \int_0^{\|x-fx,u_1,\ldots,u_{n-1}\|+\|y-fy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\ + \gamma \int_0^{\|x-fy,u_1,\ldots,u_{n-1}\|+\|y-fx,u_1,\ldots,u_{n-1}\|} \zeta(t)dt\Big), \varphi\Big(\alpha \int_0^{\|x-y,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\ + \beta \int_0^{\|x-fx,u_1,\ldots,u_{n-1}\|+\|y-fy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\ + \gamma \int_0^{\|x-fy,u_1,\ldots,u_{n-1}\|+\|y-fx,u_1,\ldots,u_{n-1}\|} \zeta(t)dt\Big)\Big)\end{aligned} \tag{2.1}$$

for each $x,y,u_1,\ldots,u_{n-1} \in X$ with non negative reals $\alpha + 2\beta + 2\gamma < 1$. ψ and φ are altering distance and ultra altering distance functions respectively ,$F \in \mathscr{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$, where $\zeta : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, subadditive on each $[a,b] \subset [0,\infty)$, non-negative and for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \zeta(t)dt > 0. \tag{2.2}$$

Then f has a unique fixed point in X, with $\lim_{k\to\infty} f^k x_\circ = z$ for each $x_\circ \in X$.

Proof. Let $x_\circ \in X$, and define the iterate sequence $\{x_k\}$ by

$$x_{k+1} = fx_k = f^{k+1}x, \tag{2.3}$$

then by (2.1) and (2.3), we get

$$\begin{aligned}
\psi(\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt) &= \psi(\int_0^{\|fx_{k-1}-fx_k,u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \\
&\leq F(\psi(\alpha \int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\beta \int_0^{\|x_{k-1}-fx_{k-1},u_1,\ldots,u_{n-1}\|+\|x_k-fx_k,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\gamma \int_0^{\|x_{k-1}-fx_k,u_1,\ldots,u_{n-1}\|+\|x_k-fx_{k-1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt), \\
&\quad \varphi(\alpha \int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\beta \int_0^{\|x_{k-1}-fx_{k-1},u_1,\ldots,u_{n-1}\|+\|x_k-fx_k,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\gamma \int_0^{\|x_{k-1}-fx_k,u_1,\ldots,u_{n-1}\|+\|x_k-fx_{k-1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt))
\end{aligned}$$

$$\leq F(\psi((\alpha+\beta+\gamma)\int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$+(\beta+\gamma)\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt),$$

$$\varphi((\alpha+\beta+\gamma)\int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$+(\beta+\gamma)\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt))$$

$$\leq \psi((\alpha+\beta+\gamma)\int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$+(\beta+\gamma)\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt)$$

$$\Rightarrow \int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt \leq q\int_0^{\|x_{k-1}-x_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \tag{2.4}$$

where $q=(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma})$, for each k and for $u_1,\ldots,u_{n-1}\in X$.

implies that the sequence $\{\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt\}$ is monotonic decreasing and continuous. There exists a real number, say $r\geq 0$ such that

$$\lim_{n\to\infty}\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt = r$$

as $n\to\infty$ equation (2.4) $\Rightarrow$

$$\psi(r)\leq F(\psi(r),\varphi(r))$$

so, $\psi(r)=0$ or $\varphi(r)=0$ which is only possible if $r=0$. Thus

$$\lim_{n\to\infty}\int_0^{\|x_k-x_{k-1},u_1,\ldots,u_{n-1}\|}\zeta(t)dt = 0$$

Claim: $\{x_k\}$ is a Cauchy sequence in X. Suppose $\{x_k\}$ is a Cauchy sequence in X.

Then there exist an $\varepsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$\int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt \geq \varepsilon \text{ and } \int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq \varepsilon$$

$$\varepsilon \leq \int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq \int_0^{\|x_{m_i}-x_{n_{i-1}},u_1,\dots,u_{n-1}\|} \zeta(t)dt + \int_0^{\|x_{n_{i-1}}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

therefore $\lim_{i\to\infty} \int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)d = \varepsilon$

now

$$\varepsilon \leq \int_0^{\|x_{m_{i-1}}-x_{n_{i-1}},u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq \int_0^{\|x_{m_{i-1}}-x_{m_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt + \int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} \int_0^{\|x_{m_{i-1}}-x_{n_{i-1}},u_1,\dots,u_{n-1}\|} \zeta(t)dt = \varepsilon$$

from (2.4) and (2.1) we get,

$$\psi(\varepsilon) \leq \psi(\int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt) = \psi(\int_0^{\|fx_{m_{i+1}}-fx_{n_{i+1}},u_1,\dots,u_{n-1}\|} \zeta(t)dt)$$

$$\leq F(\psi(J(x_{m_i}-x_{n_i},u_1,\dots,u_{n-1})),\varphi(J(x_{m_{i-1}}-x_{n_{i-1}},u_1,\dots,u_{n-1})))$$

where implies

$$\psi(\varepsilon) \leq F(\psi(J(x_{m_i}-x_{n_i},u_1,\dots,u_{n-1})),\varphi(J(x_{m_{i-1}}-x_{n_{i-1}},u_1,\dots,u_{n-1}))) \tag{2.5}$$

$$J(x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}) = \alpha \int_0^{\|x_{m_i}-x_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

$$+\beta \int_0^{\|x_{m_i}-fx_{m_i},u_1,\dots,u_{n-1}\|+\|x_{n_i}-fx_{n_i},u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

$$+\gamma\int_0^{\|x_{m_i}-fx_{n_i},u_1,\ldots,u_{n-1}\|+\|x_{n_i}-fx_{m_i},u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$=\alpha\int_0^{\|x_{m_i}-x_{n_i},u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$+\beta\int_0^{\|x_{m_i}-x_{m_{i-1}},u_1,\ldots,u_{n-1}\|+\|x_{n_i}-x_{n_{i-1}},u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

$$+\gamma\int_0^{\|x_{m_i}-x_{n_{i-1}},u_1,\ldots,u_{n-1}\|+\|x_{n_i}-x_{m_{i-1}},u_1,\ldots,u_{n-1}\|}\zeta(t)dt$$

Taking limit as $i\to\infty$ we get,

$$\lim_{i\to\infty}J(x_{m_i}-x_{n_i},u_1,\ldots,u_{n-1})=\alpha\varepsilon+0+\gamma\varepsilon$$

$$\lim_{i\to\infty}J(x_{m_i}-x_{n_i},u_1,\ldots,u_{n-1})\le\varepsilon$$

Therefore from (2.5) we have, $\psi(\varepsilon)\le F(\psi(\varepsilon),\varphi(\varepsilon))$ so, $\psi(\varepsilon)=0$ or $\varphi(\varepsilon)=0$. That is a contraction because $\varepsilon>0$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Hence $\{x_n\}$ is a Cauchy sequence. There exists a point z in X such that $\lim_{k\to\infty}f^kx_\circ=z\in X$ and $u_1,\ldots,u_{n-1}\in X$

To prove the uniqueness of z, suppose that $(z\neq w)$ be another fixed point of f, then from (2.1), we get

$$\begin{aligned}\psi\left(\int_0^{\|z-w,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\right) &= \psi\left(\int_0^{\|fz-fw,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\right)\\ &\le F\Bigg(\psi\Bigg(\alpha\int_0^{\|z-w,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\\ &\quad+\beta\int_0^{\|z-fz,u_1,\ldots,u_{n-1}\|+\|w-fw,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\\ &\quad+\gamma\int_0^{\|z-fw,u_1,\ldots,u_{n-1}\|+\|w-fz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Bigg)\end{aligned}$$

$$
\begin{aligned}
& \varphi(\alpha \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt + \beta \int_0^{\|z-fz,u_1,\ldots,u_{n-1}\|+\|w-fw,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
& +\gamma \int_0^{\|z-fw,u_1,\ldots,u_{n-1}\|+\|w-fz,u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \\
\leq\ & F(\psi((\alpha+2\gamma) \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt), \varphi((\alpha+2\gamma) \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \\
\leq\ & \psi((\alpha+2\gamma) \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \\
\Rightarrow \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \leq\ & (\alpha+2\gamma) \int_0^{\|z-w,u_1,\ldots,u_{n-1}\|} \zeta(t)dt
\end{aligned}
$$

which is a contradiction.Since $\alpha+2\gamma<1$. Therefore $z=w$. Hence z is a unique fixed point of f.

Remark 2.1.

(1) On setting $\zeta(t)=1$ over R^+, the contractive condition of integral type transform into a general contractive condition not involving integral.

(2) From Condition 2.1 of integral type several contractive mappings of integral type can be obtained. Now, our next theorem is the extension of the Theorem 2.1 for a pair of mappings

Theorem 2.2. Let X be a n Banach space. Let f and g be a self mappings of X

$$
\begin{aligned}
\psi(\int_0^{\|fx-gy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \leq F(\psi(\alpha \int_0^{\|x-y,u_1,\ldots,u_{n-1}\|} \zeta(t)dt + \beta \int_0^{\|x-fx,u_1,\ldots,u_{n-1}\|+\|y-gy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
+\gamma \int_0^{\|x-gy,u_1,\ldots,u_{n-1}\|+\|y-fx,u_1,\ldots,u_{n-1}\|} \zeta(t)dt), \varphi(\alpha \int_0^{\|x-y,u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
+\beta \int_0^{\|x-fx,u_1,\ldots,u_{n-1}\|+\|y-gy,u_1,\ldots,u_{n-1}\|} \zeta(t)dt + \gamma \int_0^{\|x-gy,u_1,\ldots,u_{n-1}\|+\|y-fx,u_1,\ldots,u_{n-1}\|} \zeta(t)dt))
\end{aligned}
\tag{2.6}
$$

for each $x,y,u_1,\ldots,u_{n-1} \in X$ with non negative reals $\alpha+2\beta+2\gamma<1$. ψ and φ are altering distance and ultra altering distance functions respectively ,$F \in \mathscr{C}$ such that $\psi(t+s) \leq \psi(t)+\psi(s)$, where $\zeta:[0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which is summable, subadditive on each $[a,b] \subset [0,\infty)$, non-negative and for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \zeta(t)dt > 0.$$

Then f and g have a unique common fixed point in X, with $\lim_{k\to\infty} f^k x_\circ = z$ for each $x_\circ \in X$.

Proof. Let $x_\circ \in X$, and define the iterate sequence x_k by

$$x_{2k+1} = fx_k \text{ and } x_{2k+2} = gx_{k+1}$$

Then by (2.6), we have

$$\begin{aligned}
\psi(\int_0^{\|x_{2k+1}-x_{2k+2},u_1,\ldots,u_{n-1}\|} \zeta(t)dt) &= \psi(\int_0^{\|fx_{2k}-gx_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt) \\
&\leq F(\psi(\alpha \int_0^{\|x_{2k}-x_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\beta \int_0^{\|x_{2k}-x_{2k+1},u_1,\ldots,u_{n-1}\|+\|x_{2k+1}-x_{2k+2},u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\gamma \int_0^{\|x_{2k}-x_{2k+2},u_1,\ldots,u_{n-1}\|+\|x_{2k+1}-x_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt), \\
&\quad \varphi(\alpha \int_0^{\|x_{2k}-x_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\beta \int_0^{\|x_{2k}-x_{2k+1},u_1,\ldots,u_{n-1}\|+\|x_{2k+1}-x_{2k+2},u_1,\ldots,u_{n-1}\|} \zeta(t)dt \\
&\quad +\gamma \int_0^{\|x_{2k}-x_{2k+2},u_1,\ldots,u_{n-1}\|+\|x_{2k+1}-x_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt))
\end{aligned}$$

by simple calculation, we get

$$\int_0^{\|x_{2k+1}-x_{2k+2},u_1,\ldots,u_{n-1}\|} \zeta(t)dt \leq (\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}) \int_0^{\|x_{2k}-x_{2k+1},u_1,\ldots,u_{n-1}\|} \zeta(t)dt$$

by exactly the same argument we produce

$$\int_0^{\|x_{2k}-x_{2k+1},u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq q \int_0^{\|x_{2k-1}-x_{2k},u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

for all $u_1,\dots,u_{n-1} \in X$ and for all k, we get

$$\begin{aligned}\int_0^{\|x_k-x_{k+1},u_1,\dots,u_{n-1}\|} \zeta(t)dt &\leq q \int_0^{\|x_{k-1}-x_k,u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq q^2 \int_0^{\|x_{k-2}-x_{k-1},u_1,\dots,u_{n-1}\|} \zeta(t)dt \\ &\leq \dots \leq q^k \int_0^{\|x_\circ-x_1,u_1,\dots,u_{n-1}\|} \zeta(t)dt,\end{aligned}$$

by the same steps of Theorem 2.1 one can reaches to conclude that x_k is a Cauchy in X, and let $\lim_{k\to\infty} x_k = z \in X$.Now, for $u_1,\dots,u_{n-1} \in X$, by (2.6), we have

$$\begin{aligned}\psi(\int_0^{\|x_{2k+1}-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt) &= \psi(\int_0^{\|fx_{2k}-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt) \\ &\leq F(\psi(\alpha \int_0^{\|x_{2k}-z,u_1,\dots,u_{n-1}\|} \zeta(t)dt \\ &\quad +\beta \int_0^{\|x_{2k}-x_{2k+1},u_1,\dots,u_{n-1}\|+\|z-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt \\ &\quad +\gamma \int_0^{\|x_{2k}-gz,u_1,\dots,u_{n-1}\|+\|z-x_{2k+1},u_1,\dots,u_{n-1}\|} \zeta(t)dt), \\ &\quad \varphi(\alpha \int_0^{\|x_{2k}-z,u_1,\dots,u_{n-1}\|} \zeta(t)dt \\ &\quad +\beta \int_0^{\|x_{2k}-x_{2k+1},u_1,\dots,u_{n-1}\|+\|z-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt \\ &\quad +\gamma \int_0^{\|x_{2k}-gz,u_1,\dots,u_{n-1}\|+\|z-x_{2k+1},u_1,\dots,u_{n-1}\|} \zeta(t)dt)).\end{aligned}$$

Taking the limit as $k \to \infty$, gives

$$\int_0^{\|z-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq (\beta+\gamma) \int_0^{\|z-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

since $(\beta+\gamma<1)$ we get $\int_0^{\|z-gz,u_1,\dots,u_{n-1}\|} \zeta(t)dt = 0,.$ $gz = z$. Similarly, we can show that, $fz = z$ and hence z is a common fixed point of f and g. Next, suppose that $(z \neq w)$ be another fixed point of f and g then from (2.6), we get

$$\psi(\int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt) = \psi(\int_0^{\|fz-gw,u_1,\dots,u_{n-1}\|} \zeta(t)dt)$$
$$\leq F(\psi(\alpha \int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt + \beta \int_0^{\|z-fz,u_1,\dots,u_{n-1}\|+\|w-gw,u_1,\dots,u_{n-1}\|} \zeta(t)dt$$
$$+\gamma \int_0^{\|z-gw,u_1,\dots,u_{n-1}\|+\|w-fz,u_1,\dots,u_{n-1}\|} \zeta(t)dt), \varphi(\alpha \int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt$$
$$+\beta \int_0^{\|z-fz,u_1,\dots,u_{n-1}\|+\|w-gw,u_1,\dots,u_{n-1}\|} \zeta(t)dt + \gamma \int_0^{\|z-gw,u_1,\dots,u_{n-1}\|+\|w-fz,u_1,\dots,u_{n-1}\|} \zeta(t)dt))$$
$$\psi(\int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt) \leq (\alpha+2\gamma) \int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt,$$

since $\alpha+2\gamma<1$, we get a contradiction, therefore $\int_0^{\|z-w,u_1,\dots,u_{n-1}\|} \zeta(t)dt = 0,$, we obtain that $z = w$. Therefore z is a unique common fixed point of f and g. The proof is complete.Finally, we extend the result for a sequence of mappings.

Theorem 2.3. Let X be $n-$Banach space with $f : X \longrightarrow X$ and $f_k : X \longrightarrow X$, be a sequence of mappings such that

(i) $$\int_0^{\|f_kx-f_ky,u_1,\dots,u_{n-1}\|} \zeta(t)dt \leq F(\psi(\alpha \int_0^{\|x-y,u_1,\dots,u_{n-1}\|} \zeta(t)dt + \beta \int_0^{\|x-f_kx,u_1,\dots,u_{n-1}\|+\|y-f_ky,u_1,\dots,u_{n-1}\|} \zeta(t)dt$$

$$+\gamma \int_0^{\|x-f_ky,u_1,\dots,u_{n-1}\|+\|y-f_kx,u_1,\dots,u_{n-1}\|} \zeta(t)dt), \varphi(\alpha \int_0^{\|x-y,u_1,\dots,u_{n-1}\|} \zeta(t)dt$$
$$+\beta \int_0^{\|x-f_kx,u_1,\dots,u_{n-1}\|+\|y-f_ky,u_1,\dots,u_{n-1}\|} \zeta(t)dt \tag{2.7}$$
$$+\gamma \int_0^{\|x-f_ky,u_1,\dots,u_{n-1}\|+\|y-f_kx,u_1,\dots,u_{n-1}\|} \zeta(t)dt))$$

(ii) $\lim_{k\to\infty} f_k x = fx$ for each $x \in X$,for each $x, y, u_1, ..., u_{n-1} \in X$ with non negative reals $\alpha + 2\beta + 2\gamma < 1$, where $\zeta : [0,\infty] \longrightarrow [0,\infty]$ is a Lebesgue integrable mapping which is summable, subadditive on each $[a,b] \subset [0,\infty]$, non-negative and for each $\varepsilon > 0, \int_0^{\varepsilon} \zeta(t)dt > 0$.

Then f has a unique fixed point in X, such that $\lim_{k\to\infty} z_k = z$, z_k being the unique fixed point of f_k, $k = 1, 2, ...$

Proof. If we take the limit in (2.7), we have

$$\begin{aligned}\psi(\int_0^{\|fx-fy,u_1,...,u_{n-1}\|} \zeta(t)dt) \leq\ & F(\psi(\alpha \int_0^{\|x-y,u_1,...,u_{n-1}\|} \zeta(t)dt \\ & +\beta \int_0^{\|x-fx,u_1,...,u_{n-1}\|+\|y-fy,u_1,...,u_{n-1}\|} \zeta(t)dt \\ & +\gamma \int_0^{\|x-fy,u_1,...,u_{n-1}\|+\|y-fx,u_1,...,u_{n-1}\|} \zeta(t)dt), \\ & \varphi(\alpha \int_0^{\|x-y,u_1,...,u_{n-1}\|} \zeta(t)dt \\ & +\beta \int_0^{\|x-fx,u_1,...,u_{n-1}\|+\|y-fy,u_1,...,u_{n-1}\|} \zeta(t)dt \\ & +\gamma \int_0^{\|x-fy,u_1,...,u_{n-1}\|+\|y-fx,u_1,...,u_{n-1}\|} \zeta(t)dt))\end{aligned}$$

for all $x, y, u_1, ..., u_{n-1} \in X$ and hence f satisfies (2.7). Hence by Theorem (2.1), f has a unique fixed point say $z \in X$. Now for all $x, y, u_1, ..., u_{n-1} \in X$

$$\int_0^{\|z-z_k,u_1,...,u_{n-1}\|} \zeta(t)dt = \int_0^{\|fz-f_kz_k,u_1,...,u_{n-1}\|} \zeta(t)dt$$

$$\leq \int_0^{\|fz-f_kz,u_1,...,u_{n-1}\|} \zeta(t)dt + \int_0^{\|f_kz-f_kz_k,u_1,...,u_{n-1}\|} \zeta(t)dt, \tag{2.6}$$

again from (2.7), we obtain

$$
\begin{aligned}
\psi\Big(\int_0^{\|f_kz-f_kz_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big) \;\leq\; & F\Big(\psi\Big(\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
& +\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
& +\gamma\int_0^{\|z-f_kz_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big), \\
& \varphi\Big(\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
& +\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
& +\gamma\int_0^{\|z-f_kz_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big)\Big)
\end{aligned}
$$

then, by the condition (ii) of this theorem and taking the limit as $k \to \infty$, we get

$$
\begin{aligned}
\psi\Big(\int_0^{\|f_kz-f_kz_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big) \leq\; & F\Big(\psi\Big(\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta)(t)dt, \\
& +\gamma\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big) \\
& \varphi\Big(\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
& +\gamma\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt\Big)
\end{aligned}
\tag{2.7}
$$

from (2.7) in (2.6), we have

$$\begin{aligned}
\psi(\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt) &= \psi(\int_0^{\|fz-f_kz_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt) \\
&\leq F(\psi(\int_0^{\|fz-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\gamma\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt), \\
&\quad \varphi(\int_0^{\|fz-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\alpha\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +\beta\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\gamma\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|+\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt)) \\
&\leq F(\psi((1+\beta)\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+(\alpha+\gamma)\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +\gamma\int_0^{\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt),\varphi((1+\beta)\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +(\alpha+\gamma)\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+\gamma\int_0^{\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt)) \\
&\leq \psi((1+\beta)\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt+(\alpha+\gamma)\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +\gamma\int_0^{\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt)
\end{aligned}$$

$$\begin{aligned}
\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt &\leq (\frac{1+\beta}{1-\alpha-\gamma})\int_0^{\|z-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \\
&\quad +(\frac{\gamma}{1-\alpha-\gamma})\int_0^{\|z_k-f_kz,u_1,\ldots,u_{n-1}\|}\zeta(t)dt.
\end{aligned}$$

So by condition (ii) of this theorem we get a contradiction, therefore

$$\int_0^{\|z-z_k,u_1,\ldots,u_{n-1}\|}\zeta(t)dt \leq 0, \text{ as } k\to\infty.$$

we get $\lim_{k\to\infty} z_k = z$ (This complete the proof).

Conflict of Interests

The authors declare that there is no conflict of interests.

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