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COMMON FIXED POINT RESULTS FOR WEAKLY COMPATIBLE MAPPINGS UNDER RATIONAL CONTRACTIONS WITH APPLICATION IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we establish some common fixed point results for weakly compatible mappings satisfied generalized contraction under rational expressions in complex valued metric spaces. Our results generalize and extend some of the known results in the literature. Finally, we use our results to obtain the unique common solution of Ursohn integral equation.

Keywords: weakly compatible mappings; complex valued metric spaces; common fixed point theorems.

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1. Introduction

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle gives the existence and uniqueness of a solution of an operator equation and considered as the most widely used fixed point theorem in all analysis. The principle is constructive in nature and is one of the

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most useful tools in the study of nonlinear equations. There are many generalizations of the Banach's contraction mapping principle in the literature. These extension were made either by using contractive conditions on an ambient space. There are been a number of generalizations of metric space such as rectangular metric spaces, pseudo metric spaces, probabilistic metric spaces, D-metric spaces, fuzzy metric spaces, cone metric spaces, 2-metric spaces and G-metric spaces, etc (see [1,12,13]).

Recently, Azam et al. [2] introduced the concept of complex valued metric spaces which is more general than ordinary metric spaces and obtained fixed point theorems of contractive type in the context of complex valued metric spaces (see [3,4,5,7,9,10,14,15,16,17,18,19 ,20]).

2. Preliminaries

In this section, we recall some notations and definitions due to Azam and et al. [2], that will be used in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ iff } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C_1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(C_2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(C_3) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(C_4) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C_1), (C_2) and (C_3) is satisfied and we write $z_1 \prec z_2$ if only (C_3) is satisfied.

Definition 2.1 [2] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions are satisfied:

(CM_1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$,

(CM_2) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(CM_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x,y,z \in X$.

In this case, we say that (X,d) is called a complex valued metric space.

Example 2.1 [15] Let $X = \mathbb{C}$ be a set of complex numbers. Define $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i\,|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is called a complex valued metric space.

Example 2.2 [11] Let $X = \mathbb{R}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = \log z\,|x - y| \quad \forall\ x, y \in \mathbb{R},$$

where z is a fixed complex number such that $0 < \arg(z) < \dfrac{\pi}{2}$ and $|z| > 1$ (Here logarithm takes only the principle value). Then (X, d) is called a complex valued metric space.

Example 2.3 [4] Let $X = \mathbb{C}$. Define a mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(z_1, z_2) = e^{ik}\,|z_1 - z_2| \quad \forall\ z_1, z_2 \in \mathbb{C},$$

where $k \in [0, \pi/2]$. Then (X, d) is called a complex valued metric space.

Definition 2.2 [3] Let X be non-empty set and (S, T) be a pair of self-mappings on X. Then (S, T) is said to be weakly compatible if

$$Sx = Tx \ \Rightarrow\ STx = TSx \quad \forall\ x \in X.$$

Definition 2.3 [2] Let $\{x_r\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then

(i) x is called the limit of $\{x_r\}$ if for every $\varepsilon > 0$ there exist $r_0 \in \mathbb{N}$ such that $d(x_r, x) \prec \varepsilon$ for all $r > r_0$ and we can write $\lim_{r\to\infty} x_r = x$.

(ii) $\{x_r\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exist $r_0 \in \mathbb{N}$ such that $d(x_r, x_{r+s}) \prec \varepsilon$ for all $r > r_0$, where $s \in \mathbb{N}$.

(iii) (X,d) is said to be a complete complex valued metric space if every Cauchy sequence is convergent in (X,d).

Lemma 2.1 [2] Let (X,d) be a complex valued metric space and $\{x_r\}$ be a sequence in X. Then $\{x_r\}$ converges to x if and only if $|d(x_r,x)| \to 0$ as $r \to \infty$.

Lemma 2.2 [2] Let (X,d) be a complex valued metric space. Then a sequence $\{x_r\}$ in X is a Cauchy sequence if and only if $|d(x_r,x_{r+s})| \to 0$ as $r \to \infty$, where $s \in \mathbb{N}$.

The aim of this paper is to obtain some common fixed point theorems for four weakly compatible mappings satisfying rational type contractive conditions in the framework of complex valued metric space. The obtained results are generations of recent results proved by S. U. Khan [8], A. K. Dubey [6] and Azam et al. [2]. Finally, we use our results to obtain the unique common solution of Ursohn integral equation

$$j(t) = f_i(t) + \int_a^b K_i(t,s,j(s))\,ds.$$

3. Main Results

We start to this section with the following theorem.

Theorem 3.1 Let (X,d) be a complete complex valued metric space and S,T,P, $Q : X \to X$ are four mappings satisfy:

$$d(Sj,Tk) \precsim a_1\,d(Pj,Qk) + a_2\,\frac{d(Pj,Sj)\,d(Qk,Tk)}{1+d(Pj,Qk)} + a_3\,\frac{d(Pj,Tk)\,d(Qk,Sj)}{1+d(Pj,Qk)} + a_4\,[\,d(Pj,Sj)+d(Qk,Tk)\,], \tag{1}$$

for all $j,k \in X$, where a_1,a_2,a_3 and a_4 are non-negative reals with $0 \le a_1 + a_2 + a_3 + 2a_4 < 1$. If $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, then S,P,T and Q have a coincidence point. Moreover, if the pairs (S,P) and (T,Q) are weakly compatible, then there exists a unique common fixed point of the four mappings.

Proof. Let j_0 be arbitrary point in X. Since $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, we can construct the sequence $\{j_n\}$ such that,

$$\begin{cases} j_{2n+1} = Sj_{2n} = Qj_{2n+1} \\ j_{2n+2} = Tj_{2n+1} = Pj_{2n+2}, \end{cases} \tag{2}$$

for all $n \in \mathbb{N}$. From (1) and (2), we have

$$\begin{aligned} d(j_{2n+1}, j_{2n+2}) &= d(Sj_{2n}, Tj_{2n+1}) \\ &\precsim a_1\, d(Pj_{2n}, Qj_{2n+1}) + a_2 \frac{d(Pj_{2n}, Sj_{2n})\, d(Qj_{2n+1}, Tj_{2n+1})}{1 + d(Pj_{2n}, Qj_{2n+1})} \\ &\quad + a_3 \frac{d(Pj_{2n}, Tj_{2n+1})\, d(Qj_{2n+1}, Sj_{2n})}{1 + d(Pj_{2n}, Qj_{2n+1})} \\ &\quad + a_4 [d(Pj_{2n}, Sj_{2n}) + d(Qj_{2n+1}, Tj_{2n+1})] \end{aligned}$$

$$\begin{aligned} &= a_1\, d(j_{2n}, j_{2n+1}) + a_2 \frac{d(j_{2n}, j_{2n+1})\, d(j_{2n+1}, j_{2n+2})}{1 + d(j_{2n}, j_{2n+1})} \\ &\quad + a_3 \frac{d(j_{2n}, j_{2n+2})\, d(j_{2n+1}, j_{2n+1})}{1 + d(j_{2n}, j_{2n+1})} \\ &\quad + a_4 [d(j_{2n}, j_{2n+1}) + d(j_{2n+1}, j_{2n+2})]. \end{aligned}$$

For all $n \in \mathbb{N}$, we find

$$\begin{aligned} |d(j_{2n+1}, j_{2n+2})| \le{}& a_1 |d(j_{2n}, j_{2n+1})| + a_2 \frac{|d(j_{2n+1}, j_{2n+2})|\, |d(j_{2n}, j_{2n+1})|}{|1 + d(j_{2n}, j_{2n+1})|} \\ &+ a_4 [|d(j_{2n}, j_{2n+1})| + |d(j_{2n+1}, j_{2n+2})|], \end{aligned}$$

since $|d(j_{2n}, j_{2n+1})| \le |1 + d(j_{2n}, j_{2n+1})|$, we have

$$\begin{aligned} |d(j_{2n+1}, j_{2n+2})| \le{}& a_1 |d(j_{2n}, j_{2n+1})| + a_2 |d(j_{2n+1}, j_{2n+2})| + a_4 [|d(j_{2n}, j_{2n+1})| \\ &+ |d(j_{2n+1}, j_{2n+2})|]. \end{aligned}$$

This implies that

$$(1 - a_2 - a_4)\, |d(j_{2n+1}, j_{2n+2})| \le (a_1 + a_4)\, |d(j_{2n}, j_{2n+1})|,$$

or

$$|d(j_{2n+1}, j_{2n+2})| \le \left(\frac{a_1 + a_4}{1 - a_2 - a_4}\right) |d(j_{2n}, j_{2n+1})|,$$

that is,

$$|d(j_{2n+1}, j_{2n+2})| \le \lambda\ |d(j_{2n}, j_{2n+1})|,$$

where $\lambda = \left(\frac{a_1+a_4}{1-a_2-a_4}\right)$. Similarly,

$$\begin{aligned} d(j_{2n}, j_{2n+1}) &= d(Tj_{2n-1}, Sj_{2n}) \\ &\precsim a_1\, d(Pj_{2n}, Qj_{2n-1}) + a_2\, \frac{d(Pj_{2n}, Sj_{2n})\, d(Qj_{2n-1}, Tj_{2n-1})}{1 + d(Pj_{2n}, Qj_{2n-1})} \\ &\quad + a_3\, \frac{d(Pj_{2n}, Tj_{2n-1})\, d(Qj_{2n-1}, Sj_{2n})}{1 + d(Pj_{2n}, Qj_{2n-1})} \\ &\quad + a_4\, [\, d(Pj_{2n}, Sj_{2n}) + d(Qj_{2n-1}, Tj_{2n-1})\,] \\ &= a_1\, d(j_{2n}, j_{2n-1}) + a_2\, \frac{d(j_{2n}, j_{2n+1})\, d(j_{2n-1}, j_{2n})}{1 + d(j_{2n}, j_{2n-1})} \\ &\quad + a_3\, \frac{d(j_{2n}, j_{2n})\, d(j_{2n-1}, j_{2n+1})}{1 + d(j_{2n}, j_{2n-1})} \\ &\quad + a_4\, [\, d(j_{2n}, j_{2n+1}) + d(j_{2n-1}, j_{2n})\,]. \end{aligned}$$

For all $n \in \mathbb{N}$, we find

$$\begin{aligned} |d(j_{2n}, j_{2n+1})| &\le a_1\, |d(j_{2n}, j_{2n-1})| + a_2\, \frac{|d(j_{2n}, j_{2n+1})|\ |d(j_{2n}, j_{2n-1})|}{|1 + d(j_{2n}, j_{2n-1})|} \\ &\quad + a_4\, [\, |d(j_{2n}, j_{2n+1})| + |d(j_{2n}, j_{2n-1})|\,], \end{aligned}$$

since $|d(j_{2n}, j_{2n-1})| \le |1 + d(j_{2n}, j_{2n-1})|$, we have

$$\begin{aligned} |d(j_{2n}, j_{2n+1})| \le a_1\ |d(j_{2n}, j_{2n-1})| + a_2\ |d(j_{2n}, j_{2n+1})| + a_4\, [\, |d(j_{2n}, j_{2n+1})| \\ + |d(j_{2n}, j_{2n-1})|\,]. \end{aligned}$$

This implies that

$$(1 - a_2 - a_4)\, |d(j_{2n}, j_{2n+1})| \le (a_1 + a_4)\, |d(j_{2n}, j_{2n-1})|,$$

or

$$|d(j_{2n}, j_{2n+1})| \leq \left(\frac{a_1 + a_4}{1 - a_2 - a_4}\right) |d(j_{2n}, j_{2n-1})|,$$

that is,

$$|d(j_{2n}, j_{2n+1})| \leq \lambda \, |d(j_{2n}, j_{2n-1})|.$$

Therefore, for all $n \in \mathbb{N}$,

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda^2 \, |d(j_{2n}, j_{2n-1})|,$$

on continuing this process, we have

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda^{2n+1} \, |d(j_0, j_1)|. \tag{3}$$

Also, for any $n > m$, we get

$$\begin{aligned}|d(j_n, j_m)| &\leq |d(j_n, j_{n-1})| + |d(j_{n-1}, j_{n-2})| + \ldots + |d(j_{m+1}, j_m)| \\ &\leq (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m) \, |d(j_0, j_1)| \\ &\leq \frac{\lambda^m}{1 - \lambda} |d(j_0, j_1)| \longrightarrow 0, \quad as \ n, m \longrightarrow \infty.\end{aligned}$$

This show that $\{j_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, then there exists $u \in X$ such that $j_n \longrightarrow u$ as $n \longrightarrow \infty$. Then from (2), we can write

$$\lim_{n \longrightarrow \infty} S j_{2n} = \lim_{n \longrightarrow \infty} T j_{2n+1} = u$$

$$\text{and} \lim_{n \longrightarrow \infty} P j_{2n} = \lim_{n \longrightarrow \infty} Q j_{2n+1} = u,$$

Therefore,

$$\lim_{n \longrightarrow \infty} S j_{2n} = \lim_{n \longrightarrow \infty} T j_{2n+1} = \lim_{n \longrightarrow \infty} P j_{2n} = \lim_{n \longrightarrow \infty} Q j_{2n+1} = u. \tag{4}$$

Since $S(X) \subseteq Q(X)$, there exists $v \in X$ such that

$$Qv = u. \tag{5}$$

We will show that $T v = Q v$, therefore from (1) we obtain

$$
\begin{aligned}
d(u,Tv) &\precsim d(u,Sj_{2n})+d(Sj_{2n},Tv)\\
&\precsim d(u,Sj_{2n})+a_1\,d(Pj_{2n},Qv)+a_2\,\frac{d(Pj_{2n},Sj_{2n})\,d(Qv,Tv)}{1+d(Pj_{2n},Qv)}\\
&\quad+a_3\,\frac{d(Pj_{2n},Tv)\,d(Qv,Sj_{2n})}{1+d(Pj_{2n},Qv)}+a_4\,[\,d(Pj_{2n},Sj_{2n})+d(Qv,Tv)\,]\\
&\precsim d(u,j_{2n+1})+a_1\,d(j_{2n},u)+a_2\,\frac{d(j_{2n},j_{2n+1})\,d(u,Tv)}{1+d(j_{2n},u)}\\
&\quad+a_3\,\frac{d(j_{2n},Tv)\,d(u,j_{2n+1})}{1+d(j_{2n},u)}+a_4\,[\,d(j_{2n},j_{2n+1})+d(u,Tv)\,].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|d(u,Tv)| \le\; & |d(u,j_{2n+1})|+a_1\,|d(j_{2n},u)|+a_2\,\frac{|d(j_{2n},j_{2n+1})|\,|d(u,Tv)|}{|1+d(j_{2n},u)|}\\
&+a_3\,\frac{|d(j_{2n},Tv)|\,|d(u,j_{2n+1})|}{|1+d(j_{2n},u)|}+a_4\,[\,|d(j_{2n},j_{2n+1})|+|d(u,Tv)|\,]. \qquad (6)
\end{aligned}
$$

Taking the limit as $n \longrightarrow \infty$ in (6) and using (4) and (5), we have

$$
(1-a_4)\,|d(u,Tv)| \le 0,
$$

hence $|d(u,Tv)|=0$, thus $Tv=u$ and

$$
u=Tv=Qv. \qquad (7)
$$

Hence v is a coincidence point of T and Q.

By a similar way, since $T(X)\subseteq P(X)$, we can show that

$$
Sw=Pw=u, \qquad (8)
$$

for all $w\in X$. Then, w is a coincidence point of S and P.

Since the pairs (T,Q) and (S,P) are weakly compatible, then

$$
TQv=QTv \text{ and } SPw=PSw. \qquad (9)
$$

Applying (7) and (8) in (9), we can write

$$Tu = Qu \text{ and } Su = Pu, \tag{10}$$

with meaning $u \in X$ is a coincidence point for the four mappings.

Next, we show that u is a common fixed point of T, Q, S and P. We have from (1) that

$$d(Su,Tv) \precsim a_1\, d(Pu,Qv) + a_2 \frac{d(Pu,Su)\, d(Qv,Tv)}{1+d(Pu,Qv)}$$
$$+ a_3 \frac{d(Pu,Tv)\, d(Qv,Su)}{1+d(Pu,Qv)} + a_4 \left[d(Pu,Su) + d(Qv,Tv) \right].$$

Using (7), (8) and (10), we deduce

$$d(Su,u) \precsim a_1\, d(Su,u) + a_2 \frac{d(Su,Su)\, d(u,u)}{1+d(Su,u)}$$
$$+ a_3 \frac{d(Su,u)\, d(u,Su)}{1+d(Su,u)} + a_4 \left[d(Su,Su) + d(u,u) \right].$$

Consequently,

$$|d(Su,u)| \leq a_1\, |d(Su,u)| + a_3 \frac{|d(Su,u)|\, |d(Su,u)|}{|1+d(Su,u)|}.$$

Since $|d(Su,u)| \leq |1+d(Su,u)|$, then we get $(1-a_1-a_3)\,|d(Su,u)| \leq 0$, hence $|d(Su,u)| = 0$ i.e., $Su = u$, then according to (10), we obtain that

$$u = Su = Pu. \tag{11}$$

By a similar way and using (11), we can prove that

$$u = Tu = Qu. \tag{12}$$

i.e., the equations (11) and (12) show that u is a common fixed point for our mappings.

To prove the uniqueness: Suppose that $u^* \neq u$ be another common fixed point of the four mappings, then from (1), one can write

$$d(u,u^*) = d(Su,Tu^*)$$
$$\precsim a_1\, d(Pu,Qu^*) + a_2 \frac{d(Pu,Su)\, d(Qu^*,Tu^*)}{1+d(Pu,Qu^*)}$$

$$+ a_3 \frac{d(Pu,Tu^*)\, d(Qu^*,Su)}{1+d(Pu,Qu^*)} + a_4 [\, d(Pu,Su) + d(Qu^*,Tu^*)\,]$$

$$\precsim a_1\, d(u,u^*) + a_2 \frac{d(u,u)\, d(u^*,u^*)}{1+d(u,u^*)}$$
$$+ a_3 \frac{d(u,u^*)\, d(u^*,u)}{1+d(u,u^*)} + a_4 [\, d(u,u) + d(u^*,u^*)\,].$$

Consequently,

$$|d(u,u^*)| \leq a_1\, |d(u,u^*)| + a_3 \frac{|d(u,u^*)|\, |d(u,u^*)|}{|1+d(u,u^*)|},$$

hence

$$(1-a_1-a_3)\, |d(u,u^*)| \leq 0.$$

Therefore $|d(u,u^*)| = 0$. i.e., $u = u^*$ and so u is a unique common fixed point of S,T,P and Q. Consequently, the proof is completed.

If we take $a_3 = a_4 = 0$ in Theorem 3.1, we obtain the following result:

Corollary 3.1 Let (X,d) be a complete complex valued metric space and S,T,P, $Q : X \to X$ be four mappings satisfy:

$$d(Sj,Tk) \precsim a_1\, d(Pj,Qk) + a_2 \frac{d(Pj,Sj)\, d(Qk,Tk)}{1+d(Pj,Qk)},$$

for all $j,k \in X$, where a_1 and a_2 are non-negative reals with $0 \leq a_1 + a_2 < 1$. If $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, then S,P,T and Q have a coincidence point. Moreover, if the pairs (S,P) and (T,Q) are weakly compatible, then there exists a unique common fixed point of S,T,P and Q.

Taking $P = Q = I$, where I is the identity mapping in Corollary 3.1, we get the following corollary:

Corollary 3.2 [[3],Theorem 4] Let (X,d) be a complete complex valued metric space and $S,T:X\to X$ be two mappings. If S and T satisfy:

$$d(Sj,Tk)\precsim a_1\,d(j,k)+a_2\,\frac{d(j,Sj)\,d(k,Tk)}{1+d(j,k)},$$

for all $j,k\in X$, where a_1 and a_2 are non-negative reals with $0\le a_1+a_2<1$. Then S and T have a unique common fixed point.

By taking $S=T$ in Corollary 3.2, we have the following result:

Corollary 3.3 [[3],Corollary 5] Let (X,d) be a complete complex valued metric space and the mappings $T:X\to X$ satisfy:

$$d(Tj,Tk)\precsim a_1\,d(j,k)+a_2\,\frac{d(j,Tj)\,d(k,Tk)}{1+d(j,k)},$$

for all $j,k\in X$, where a_1 and a_2 are non-negative reals with $0\le a_1+a_2<1$. Then T has a unique common fixed point.

The following theorem is a new version of Theorem 3.1 with various contractive condition.

Theorem 3.2 Let (X,d) be a complete complex valued metric space and $S,T,P,$ $Q:X\to X$ are four mappings satisfy:

$$d(Sj,Tk)\precsim a_1\,d(Pj,Qk)+a_2\,\frac{d(Pj,Sj)\,d(Qk,Tk)}{1+d(Pj,Qk)} +a_3\,\frac{d^2(Pj,Tk)+d^2(Qk,Sj)}{d(Pj,Tk)+d(Qk,Sj)}, \quad (13)$$

for all $j,k\in X$, where a_1,a_2,a_3 are non-negative reals with $0\le a_1+a_2+a_3<1$. If $S(X)\subseteq Q(X)$ and $T(X)\subseteq P(X)$, then S,P,T and Q have a coincidence point. Moreover, if the pairs (S,P) and (T,Q) are weakly compatible, then there exists a unique common fixed point of the four mappings.

Proof. Let j_0 be arbitrary point in X. Since $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, we can construct the sequence $\{j_n\}$ as (2). From (13) and (2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(j_{2n+1}, j_{2n+2}) &= d(Sj_{2n}, Tj_{2n+1}) \\ &\precsim a_1 \, d(Pj_{2n}, Qj_{2n+1}) + a_2 \frac{d(Pj_{2n}, Sj_{2n}) \, d(Qj_{2n+1}, Tj_{2n+1})}{1 + d(Pj_{2n}, Qj_{2n+1})} \\ &\quad + a_3 \frac{d^2(Pj_{2n}, Tj_{2n+1}) + d^2(Qj_{2n+1}, Sj_{2n})}{d(Pj_{2n}, Tj_{2n+1}) + d(Qj_{2n+1}, Sj_{2n})} \\ &= a_1 \, d(j_{2n}, j_{2n+1}) + a_2 \frac{d(j_{2n}, j_{2n+1}) \, d(j_{2n+1}, j_{2n+2})}{1 + d(j_{2n}, j_{2n+1})} \\ &\quad + a_3 \frac{d^2(j_{2n}, j_{2n+2}) + d^2(j_{2n+1}, j_{2n+1})}{d(j_{2n}, j_{2n+2}) + d(j_{2n+1}, j_{2n+1})}. \end{aligned}$$

For all $n \in \mathbb{N}$, we find

$$\begin{aligned} |d(j_{2n+1}, j_{2n+2})| &\leq a_1 |d(j_{2n}, j_{2n+1})| + a_2 \frac{|d(j_{2n}, j_{2n+1})| \, |d(j_{2n+1}, j_{2n+2})|}{|1 + d(j_{2n}, j_{2n+1})|} \\ &\quad + a_3 [\, |d(j_{2n}, j_{2n+1})| + |d(j_{2n+1}, j_{2n+2})| \,] \\ &\leq a_1 \, |d(j_{2n}, j_{2n+1})| + a_2 \, |d(j_{2n+1}, j_{2n+2})| + a_3 [\, |d(j_{2n}, j_{2n+1})| \\ &\quad + |d(j_{2n+1}, j_{2n+2})| \,]. \end{aligned}$$

This implies that

$$|d(j_{2n+1}, j_{2n+2})| \leq \left(\frac{a_1 + a_3}{1 - a_2 - a_3} \right) |d(j_{2n}, j_{2n+1})|,$$

that is,

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda \, |d(j_{2n}, j_{2n+1})|.$$

where $\lambda = \left(\frac{a_1 + a_3}{1 - a_2 - a_3} \right)$. Therefore, for all $n \in \mathbb{N}$,

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda^2 \, |d(j_{2n}, j_{2n-1})|,$$

on continuing this process, we have (3).

Also, for any $n > m$, we get

$$\begin{aligned}|d(j_n,j_m)| &\leq |d(j_n,j_{n-1})|+|d(j_{n-1},j_{n-2})|+...+|d(j_{m+1},j_m)|\\ &\leq (\lambda^{n-1}+\lambda^{n-2}+...+\lambda^m)\,|d(j_0,j_1)|\\ &\leq \frac{\lambda^m}{1-\lambda}\,|d(j_0,j_1)| \longrightarrow 0, \quad as\ n,m\longrightarrow\infty.\end{aligned}$$

This shows that $\{j_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, then there exists $u\in X$ such that $j_n\longrightarrow u$ as $n\longrightarrow\infty$. Then from (2), we can write

$$\lim_{n\longrightarrow\infty} Sj_{2n} = \lim_{n\longrightarrow\infty} Tj_{2n+1} = u$$

$$\text{and}\ \lim_{n\longrightarrow\infty} Pj_{2n} = \lim_{n\longrightarrow\infty} Qj_{2n+1} = u.$$

Therefore, we obtain (4).

Since $S(X)\subseteq Q(X)$, there exists $v\in X$ such that (5) is satisfied.

Now, we will show that $Tv=Qv$, therefore from (13), we obtain

$$\begin{aligned}d(u,Tv) &\precsim d(u,Sj_{2n})+d(Sj_{2n},Tv)\\ &\precsim d(u,Sj_{2n}) + a_1\,d(Pj_{2n},Qv) + a_2\,\frac{d(Pj_{2n},Sj_{2n})\,d(Qv,Tv)}{1+d(Pj_{2n},Qv)}\\ &\quad + a_3\,\frac{d^2(Pj_{2n},Tv) + d^2(Qv,Sj_{2n})}{d(Pj_{2n},Tv) + d(Qv,Sj_{2n})}\\ &\precsim d(u,j_{2n+1}) + a_1\,d(j_{2n},u) + a_2\,\frac{d(j_{2n},j_{2n+1})\,d(u,Tv)}{1+d(j_{2n},u)}\\ &\quad + a_3\,\frac{d^2(j_{2n},Tv)+d^2(u,j_{2n+1})}{d(j_{2n},Tv)+d(u,j_{2n+1})}.\end{aligned}$$

This implies that

$$\begin{aligned}|d(u,Tv)| &\leq |d(u,j_{2n+1})|+a_1\,|d(j_{2n},u)| + a_2\,\frac{|d(j_{2n},j_{2n+1})|\,|d(u,Tv)|}{|1+d(j_{2n},u)|}\\ &\quad + a_3\left|\frac{d^2(j_{2n},Tv) + d^2(u,j_{2n+1})}{d(j_{2n},Tv) + d(u,j_{2n+1})}\right|. \end{aligned} \tag{14}$$

Taking the limit as $n\longrightarrow\infty$ in (14) and using (4) and (5), we have

$$(1-a_3)\ |d(u,Tv)| \leq 0,$$

hence $|d(u,Tv)|=0$, thus $Tv=u$ and (7) is given.

By a similar way, since $T(X)\subseteq P(X)$, we can show the equation (8).

Since the pairs (T,Q) and (S,P) are weakly compatible, then the equations (8) and (10) are satisfied. Therefore, $u\in X$ is a coincidence point for the four mappings.

Next, we show that u is a common fixed point of T,Q,S and P. We have from (13) that

$$d(Su,Tv) \precsim a_1\, d(Pu,Qv)+a_2\,\frac{d(Pu,Su)\, d(Qv,Tv)}{1+d(Pu,Qv)}+a_3\,\frac{d^2(Pu,Tv)\,+\,d^2(Qv,Su)}{d(Pu,Tv)\,+\,d(Qv,Su)}.$$

Using (7), (8) and (10), we deduce that

$$d(Su,u) \precsim a_1\, d(Su,u)\,+\,a_2\,\frac{d(Su,Su)\, d(u,u)}{1+d(Su,u)}\,+\,a_3\,\frac{d^2(Su,u)\,+\,d^2(u,Su)}{d(Su,u)\,+\,d(u,Su)}.$$

Consequently,

$$|d(Su,u)| \leq a_1\ |d(Su,u)|\ +\ a_3\ |d(Su,u)|\,.$$

Therefore, we get $(1-a_1-a_3)\,|d(Su,u)|\leq 0$, hence $|d(Su,u)|=0$. i.e., $Su=u$, then according to (10), we obtain (11). By a similar way and using (11), we can prove that (12) are verified. This shows that u is a common fixed point for our mappings.

For the uniqueness. Suppose that $u^*\neq u$ be another common fixed point of the four mappings, then from (13), one can write

$$\begin{aligned} d(u,u^*) &= d(Su,Tu^*)\\ &\precsim a_1\, d(Pu,Qu^*)+a_2\,\frac{d(Pu,Su)\, d(Qu^*,Tu^*)}{1+d(Pu,Qu^*)}+a_3\,\frac{d^2(Pu,Tu^*)\,+\,d^2(Qu^*,Su)}{d(Pu,Tu^*)\,+\,d(Qu^*,Su)}\\ &\precsim a_1\, d(u,u^*)\,+\,a_2\,\frac{d(u,u)\, d(u^*,u^*)}{1+d(u,u^*)}\,+\,a_3\,\frac{d^2(u,u^*)\,+\,d^2(u^*,u)}{d(u,u^*)\,+\,d(u^*,u)}. \end{aligned}$$

Consequently,

$$|d(u,u^*)| \leq a_1\, |d(u,u^*)| + a_3\, |d(u,u^*)|,$$

it follows that

$$(1-a_1-a_3)\,|d(u,u^*)| \leq 0.$$

Therefore $|d(u,u^*)| = 0$. i.e., $u = u^*$ and so u is a unique common fixed point of S,T,P and Q. This completes the proof.

For another rational expression, we state and prove the following theorem.

Theorem 3.3 Let (X,d) be a complete complex valued metric space and $S,T,P,Q : X \to X$ are four mappings satisfy:

$$d(Sj,Tk) \precsim a_1\, d(Pj,Qk) + a_2 \frac{d(Pj,Sj)\, d(Qk,Tk)}{1+d(Pj,Qk)} + a_3 \frac{d(Pj,Tk)\, d(Qk,Sj)}{1+d(Pj,Qk)} + a_4 \frac{d(Pj,Sj)\, d(Qk,Tk)}{d(Pj,Qk)+d(Pj,Tk)+d(Qk,Sj)}, \tag{15}$$

for all $j,k \in X$, where a_1,a_2,a_3 and a_4 are nonnegative reals with $0 \leq a_1+a_2+a_3+a_4 < 1$. If $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, and then S,P,T and Q have a coincidence point. Moreover, if the pairs (S,P) and (T,Q) are weakly compatible, then there exists a unique common fixed point of the four mappings.

Proof. Let j_0 be arbitrary point in X. Since $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$, we can construct the sequence $\{j_n\}$ as (2). From (15) and (2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(j_{2n+1}, j_{2n+2}) &= d(Sj_{2n}, Tj_{2n+1}) \\ &\precsim a_1\, d(Pj_{2n}, Qj_{2n+1}) + a_2 \frac{d(Pj_{2n}, Sj_{2n})\, d(Qj_{2n+1}, Tj_{2n+1})}{1+d(Pj_{2n}, Qj_{2n+1})} \end{aligned}$$

$$+ a_3 \frac{d(Pj_{2n}, Tj_{2n+1})\, d(Qj_{2n+1}, Sj_{2n})}{1 + d(Pj_{2n}, Qj_{2n+1})}$$

$$+ a_4 \frac{d(Pj_{2n}, Sj_{2n})\, d(Qj_{2n+1}, Tj_{2n+1})}{d(Pj_{2n}, Qj_{2n+1}) + d(Pj_{2n}, Tj_{2n+1}) + d(Qj_{2n+1}, Sj_{2n})},$$

$$= a_1\, d(j_{2n}, j_{2n+1}) + a_2 \frac{d(j_{2n}, j_{2n+1})\, d(j_{2n+1}, j_{2n+2})}{1 + d(j_{2n}, j_{2n+1})}$$

$$+ a_3 \frac{d(j_{2n}, j_{2n+2})\, d(j_{2n+1}, j_{2n+1})}{1 + d(j_{2n}, j_{2n+1})}$$

$$+ a_4 \frac{d(j_{2n}, j_{2n+1})\, d(j_{2n+1}, j_{2n+2})}{d(j_{2n}, j_{2n+1}) + d(j_{2n}, j_{2n+2}) + d(j_{2n+1}, j_{2n+1})}.$$

For all $n \in \mathbb{N}$, we find

$$|d(j_{2n+1}, j_{2n+2})| \leq a_1 |d(j_{2n}, j_{2n+1})| + a_2 \frac{|d(j_{2n}, j_{2n+1})|\, |d(j_{2n+1}, j_{2n+2})|}{|1 + d(j_{2n}, j_{2n+1})|}$$

$$+ a_4 \frac{|d(j_{2n}, j_{2n+1})|\, |d(j_{2n+1}, j_{2n+2})|}{|d(j_{2n}, j_{2n+1})| + |d(j_{2n}, j_{2n+2})|}$$

$$\leq a_1 |d(j_{2n}, j_{2n+1})| + a_2 \frac{|d(j_{2n}, j_{2n+1})|\, |d(j_{2n+1}, j_{2n+2})|}{|1 + d(j_{2n}, j_{2n+1})|}$$

$$+ a_4 \frac{|d(j_{2n}, j_{2n+1})|\, |d(j_{2n+1}, j_{2n+2})|}{|d(j_{2n+1}, j_{2n+2})|},$$

$$\leq a_1 |d(j_{2n}, j_{2n+1})| + a_2 |d(j_{2n+1}, j_{2n+2})| + a_4 |d(j_{2n}, j_{2n+1})|.$$

This implies that

$$|d(j_{2n+1}, j_{2n+2})| \leq \left(\frac{a_1 + a_4}{1 - a_2}\right) |d(j_{2n}, j_{2n+1})|,$$

that is,

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda\, |d(j_{2n}, j_{2n+1})|,$$

where $\lambda = \left(\frac{a_1 + a_4}{1 - a_2}\right)$. Therefore, for all $n \in \mathbb{N}$,

$$|d(j_{2n+1}, j_{2n+2})| \leq \lambda^2\, |d(j_{2n}, j_{2n-1})|,$$

on repeating this process, we obtain (3).

Also, for any $n > m$, we get

$$
\begin{aligned}
|d(j_n, j_m)| &\leq |d(j_n, j_{n-1})| + |d(j_{n-1}, j_{n-2})| + \ldots + |d(j_{m+1}, j_m)|, \\
&\leq (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda^m)\, |d(j_0, j_1)|, \\
&\leq \frac{\lambda^m}{1-\lambda}\, |d(j_0, j_1)| \longrightarrow 0, \ \ as\ \ m,n \longrightarrow \infty.
\end{aligned}
$$

This shows that $\{j_n\}$ is a Cauchy sequence in X. Since (X,d) is complete, then there exists $u \in X$ such that $j_n \longrightarrow u$ as $n \longrightarrow \infty$. Then from (2), we can write

$$\lim_{n\longrightarrow\infty} Sj_{2n} = \lim_{n\longrightarrow\infty} Tj_{2n+1} = u$$

$$\text{and } \lim_{n\longrightarrow\infty} Pj_{2n} = \lim_{n\longrightarrow\infty} Qj_{2n+1} = u.$$

Therefore, we have (4). Since $S(X) \subseteq Q(X)$, there exists $v \in X$ such that (5) is verified.

Now, we will show that $Tv = Qv$, therefore from (15) we obtain

$$
\begin{aligned}
d(u,Tv) &\precsim d(u,Sj_{2n}) + d(Sj_{2n},Tv) \\
&\precsim d(u,Sj_{2n}) + a_1\, d(Pj_{2n},Qv) + a_2 \frac{d(Pj_{2n},Sj_{2n})\, d(Qv,Tv)}{1+d(Pj_{2n},Qv)} \\
&\quad + a_3 \frac{d(Pj_{2n},Tv)\, d(Qv,Sj_{2n})}{1+d(Pj_{2n},Qv)} + a_4 \frac{d(Pj_{2n},Sj_{2n})\, d(Qv,Tv)}{d(Pj_{2n},Qv)+d(Pj_{2n},Tv)+d(Qv,Sj_{2n})} \\
&\precsim d(u,j_{2n+1}) + a_1\, d(j_{2n},u) + a_2 \frac{d(j_{2n},j_{2n+1})\, d(u,Tv)}{1+d(j_{2n},u)} \\
&\quad + a_3 \frac{d(j_{2n},Tv)\, d(u,j_{2n+1})}{1+d(j_{2n},u)} + a_4 \frac{d(j_{2n},j_{2n+1})\, d(u,Tv)}{d(j_{2n},u)+d(j_{2n},Tv)+d(u,j_{2n+1})}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|d(u,Tv)| &\leq |d(u,j_{2n+1})| + a_1\, |d(j_{2n},u)| + a_2 \frac{|d(j_{2n},j_{2n+1})|\, |d(u,Tv)|}{|1+d(j_{2n},u)|} \\
&\quad + a_3 \frac{|d(j_{2n},Tv)|\, |d(u,j_{2n+1})|}{|1+d(j_{2n},u)|} + a_4 \frac{|d(j_{2n},j_{2n+1})|\, |d(u,Tv)|}{|d(j_{2n},u)|+|d(j_{2n},Tv)|+|d(u,j_{2n+1})|}.
\end{aligned}
$$

(16)

Taking the limit as $n \longrightarrow \infty$ in (16) and using (4) and (5), we have

$$|d(u,Tv)| \leq 0,$$

hence $|d(u,Tv)| = 0$, thus $Tv = u$ and we get (7).

By a similar way, since $T(X) \subseteq P(X)$, we can show the equation (8). Since the pairs (T,Q) and (S,P) are weakly compatible, then the equations (8) and (10) are satisfied. Then, $u \in X$ is a coincidence point for the four mappings.

Next, we show that u is a common fixed point of T,Q,S and P. We have from (15) that

$$d(Su,Tv) \precsim a_1\, d(Pu,Qv) + a_2 \frac{d(Pu,Su)\, d(Qv,Tv)}{1+d(Pu,Qv)} + a_3 \frac{d(Pu,Tv)\, d(Qv,Su)}{1+d(Pu,Qv)}$$
$$+ a_4 \frac{d(Pu,Su)\, d(Qv,Tv)}{d(Pu,Qv)+d(Pu,Tv)+d(Qv,Su)}.$$

Using (7), (8) and (10), we deduce that

$$d(Su,u) \precsim a_1\, d(Su,u) + a_2 \frac{d(Su,Su)\, d(u,u)}{1+d(Su,u)} + a_3 \frac{d(Su,u)\, d(u,Su)}{1+d(Su,u)}$$
$$+ a_4 \frac{d(Su,Su)\, d(u,u)}{d(Su,u)+d(Su,u)+d(u,Su)}.$$

Consequently,

$$|d(Su,u)| \leq a_1\, |d(Su,u)| + a_2 \frac{|d(Su,Su)|\, |d(u,u)|}{|1+d(Su,u)|} + a_3 \frac{|d(Su,u)|\, |d(Su,u)|}{|1+d(Su,u)|}$$
$$+ a_4 \frac{|d(Su,Su)|\, |d(u,u)|}{|d(Su,u)|+|d(Su,u)|+|d(u,Su)|}.$$

Therefore we get $(1-a_1-a_3)\, |d(Su,u)| \leq 0$, hence $|d(Su,u)| = 0$. i.e., $Su = u$, then according to (10), we obtain (11). Similarly, by using (11), we can prove that (12) are satisfied. This shows that u is a common fixed point for our mappings.

To prove the uniqueness, Suppose that $u^* \neq u$ be another common fixed point of the four mappings, then from (15), one can write

$$\begin{aligned}
d(u,u^*) &= d(Su,Tu^*) \\
&\precsim a_1 d(Pu,Qu^*) + a_2 \frac{d(Pu,Su)\,d(Qu^*,Tu^*)}{1+d(Pu,Qu^*)} + a_3 \frac{d(Pu,Tu^*)\,d(Qu^*,Su)}{1+d(Pu,Qu^*)} \\
&\quad + a_4 \frac{d(Pu,Su)\,d(Qu^*,Tu^*)}{d(Pu,Qu^*)+d(Pu,Tu^*)+d(Qu^*,Su)} \\
&\precsim a_1\, d(u,u^*) + a_2 \frac{d(u,u)\,d(u^*,u^*)}{1+d(u,u^*)} + a_3 \frac{d(u,u^*)\,d(u^*,u)}{1+d(u,u^*)} \\
&\quad + a_4 \frac{d(u,u)\,d(u^*,u^*)}{d(u,u^*)+d(u,u^*)+d(u^*,u)}.
\end{aligned}$$

Consequently,

$$|d(u,u^*)| \leq a_1\, |d(u,u^*)| + a_3 \frac{|d(u,u^*)|\,|d(u,u^*)|}{|1+d(u,u^*)|},$$

it follows that

$$(1-a_1-a_3)\,|d(u,u^*)| \leq 0.$$

Therefore $|d(u,u^*)| = 0$. i.e., $u = u^*$ and so u is a unique common fixed point of S,T,P and Q. Consequently, this completes the proof.

4. Application

This section deals with the applications of result proved in the previous section. Here we will investigate the solution for the following system of Ursohn integral equations using Theorem 3.1.

$$j(t) = f_i(t) + \int_a^b K_i(t,s,j(s))\,ds, \tag{17}$$

where $i = 1,2,3,4$, $a,b \in \mathbb{R}$, $a \leq b$, $t \in [a,b]$, $j, f_i \in \mathbb{C}([a,b],\mathbb{R}^n)$ and $K_i : [a,b] \times [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a given mapping for each $i = 1,2,3,4$.

Throughout this section, for each $i = 1,2,3,4$ and K_i in (17) we shall use the following symbol

$$\delta_i(j(t)) = \int_a^b K_i(t,s,j(s))\,ds.$$

Theorem 4.1 Consider the Ursohn integral equations (17). Suppose the following assumption hold for each $t \in [a,b]$.

$$(C_1)\ f_1(t) + f_4(t) + \delta_1 j(t) - \delta_4\Big(f_1(t) + f_4(t) + \delta_1 j(t)\Big) = 0$$
$$\text{and } f_2(t) + f_3(t) + \delta_2 j(t) - \delta_3\Big(f_2(t) + f_3(t) + \delta_2 j(t)\Big) = 0,$$

$$(C_2)\ f_1(t) + 4f_3(t) + 6\,\delta_3 j(t) + 2\delta_3\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big) + \delta_1\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big) = 9\,j(t)$$
$$\text{and } f_2(t) + 4f_4(t) + 6\,\delta_4 j(t) + 2\delta_4\Big(3j(t) - 2\delta_4 j(t) - f_4(t)\Big) + \delta_2\Big(3j(t) - 2\delta_4 j(t) - f_4(t)\Big) = 9\,j(t).$$

For all $j,k \in X$ and $a \le t \le b$, we have

$$A_{jk}\sqrt{1+a^2}\,e^{i\cot^{-1}a} \precsim a_1 B_{jk}(t) + a_2 C_{jk}(t) + a_3 D_{jk} + a_4 E_{jk}(t),$$

where a_1, a_2, a_3 and a_4 are non-negative reals with $0 \le a_1 + a_2 + a_3 + 2a_4 \le 1$ and

$$A_{jk}(t) = \|f_1(t) + \delta_1 j(t) - f_2(t) - \delta_2 k(t)\|_\infty,$$

$$B_{jk}(t) = \|3j(t) - 2\delta_3 j(t) - f_3(t) - 3k(t) + 2\delta_4 k(t) + f_4(t)\|_\infty \sqrt{1+a^2}\,e^{i\cot^{-1}a},$$

$$C_{jk}(t) = \left(\frac{\|f_1(t) + \delta_1 j(t) - 3j(t) + 2\delta_3 j(t) + f_3(t)\|_\infty}{1 + \max_{a\le t\le b} B_{jk}(t)}\right) \times \|f_2(t) + \delta_2 k(t) - 3k(t) + 2\delta_4 k(t) + f_4(t)\|_\infty \sqrt{1+a^2}\,e^{i\cot^{-1}a},$$

$$D_{jk}(t) = \left(\frac{\|f_1(t)+\delta_1 j(t)-3k(t)+2\delta_4 k(t)+f_4(t)\|_\infty}{1+\max_{a\le t\le b} B_{jk}(t)}\right)$$

$$\times \|f_2(t)+\delta_2 k(t)-3j(t)+2\delta_3 j(t)+f_3(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},$$

$$E_{jk}(t) = \Big(\|f_1(t)+\delta_1 j(t)-3j(t)+2\delta_3 j(t)+f_3(t)\|_\infty$$

$$+ \|f_2(t)+\delta_2 k(t)-3k(t)+2\delta_4 k(t)+f_4(t)\|_\infty \Big) \sqrt{1+a^2}\, e^{i\cot^{-1}a}.$$

Then the system (17) has a unique common solution.

Proof. Suppose $X = \mathbb{C}([a,b],\mathbb{R}^n)$ and the mapping $d : X\times X \longrightarrow \mathbb{C}$ is defined by

$$d(j,k) = \max_{a\le t\le b} \|j(t)-k(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},$$

where (X,d) is a complete valued metric space.

Also, let $S,T,P,Q : X \longrightarrow X$ be four mappings that can be defined as

$Sj(t) = f_1(t)+\delta_1 j(t) = f_1(t)+\int_a^b K_1(t,s,j(s))\,ds,$

$Tj(t) = f_2(t)+\delta_2 j(t) = f_2(t)+\int_a^b K_2(t,s,j(s))\,ds,$

$Pj(t) = 3j(t)-2\delta_3 j(t)-f_3(t) = 3j(t)-f_3(t)-2\int_a^b K_3(t,s,j(s))\,ds,$

$Qj(t) = 3j(t)-2\delta_4 j(t)-f_4(t) = 3j(t)-f_4(t)-2\int_a^b K_4(t,s,j(s))\,ds.$

For all $j,k\in X$, we find that

$$\left\{\begin{array}{l} d(Sj,Tk) = \max_{a\le t\le b}\|f_1(t)+\delta_1 j(t)-f_2(t)-\delta_2 k(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},\\ d(Pj,Qk) = \max_{a\le t\le b}\|3j(t)-2\delta_3 j(t)-f_3(t)-3k(t)+2\delta_4 k(t)+f_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},\\ d(Pj,Sj) = \max_{a\le t\le b}\|f_1(t)+\delta_1 j(t)-3j(t)+2\delta_3 j(t)+f_3(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},\\ d(Qk,Tk) = \max_{a\le t\le b}\|f_2(t)+\delta_2 k(t)-3k(t)+2\delta_4 k(t)+f_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},\\ d(Pj,Tk) = \max_{a\le t\le b}\|f_2(t)+\delta_2 k(t)-3j(t)+2\delta_3 j(t)+f_3(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a},\\ d(Qk,Sj) = \max_{a\le t\le b}\|f_1(t)+\delta_1 j(t)-3k(t)+2\delta_4 k(t)+f_4(t)\|_\infty \sqrt{1+a^2}\, e^{i\cot^{-1}a}. \end{array}\right\} \tag{18}$$

Since

$$A_{jk}\sqrt{1+a^2}\,e^{i\cot^{-1}a} \precsim a_1 B_{jk}(t)+a_2 C_{jk}(t)+a_3 D_{jk}(t)+a_4 E_{jk}(t).$$

This implies that

$$\max_{a\le t\le b}A_{jk}\sqrt{1+a^2}\,e^{i\cot^{-1}a} \precsim a_1 \max_{a\le t\le b}B_{jk}(t)+a_2 \max_{a\le t\le b}C_{jk}(t) \\ +a_3 \max_{a\le t\le b}D_{jk}(t)+a_4 \max_{a\le t\le b}E_{jk}(t).$$

From (18), we obtain that

$$d(Sj,Tk) \precsim a_1\, d(Pj,Qk) + a_2\frac{d(Pj,Sj)\,d(Qk,Tk)}{1+d(Pj,Qk)} + a_3\frac{d(Pj,Tk)\,d(Qk,Sj)}{1+d(Pj,Qk)} \\ + a_4\left[d(Pj,Sj)+d(Qk,Tk)\right].$$

Next, we show that $S(X)\subseteq Q(X)$, therefore we find that

$$\begin{aligned} Q\Big(Sj(t)+f_4(t)\Big) &= 3\Big(Sj(t)+f_4(t)\Big)-2\delta_4\Big(Sj(t)+f_4(t)\Big)-f_4(t) \\ &= Sj(t)+2\Big\{Sj(t)+f_4(t)-\delta_4\Big(Sj(t)+f_4(t)\Big)\Big\} \\ &= Sj(t)+2\Big\{f_1(t)+\delta_1 j(t)+f_4(t)-\delta_4\Big(f_1(t)+\delta_1 j(t)+f_4(t)\Big)\Big\} \\ &= Sj(t)+2\Big\{f_1(t)+f_4(t)+\delta_1 j(t)-\delta_4\Big(f_1(t)+f_4(t)+\delta_1 j(t)\Big)\Big\}. \end{aligned}$$

From (C_1), we obtain that $Q\big(Sj(t)+f_4(t)\big)=Sj(t)$. This implies that $S(X)\subseteq Q(X)$. By a similar way, we can show that $T(X)\subseteq P(X)$.

Also, we show that (S,P) and (T,Q) are weakly compatible. Then, for all $j,k\in X$, we have

$$\begin{aligned} \Big\|PSj(t)-SPj(t)\Big\| &= \Big\|P\Big(f_1(t)+\delta_1 j(t)\Big)-S\Big(3j(t)-2\delta_3 j(t)-f_3(t)\Big)\Big\| \\ &= \Big\|3\Big(f_1(t)+\delta_1 j(t)\Big)-2\delta_3\Big(f_1(t)+\delta_1 j(t)\Big)-f_3(t) \\ &\quad -f_1(t)-\delta_1\Big(3j(t)-2\delta_3 j(t)-f_3(t)\Big)\Big\|. \end{aligned} \tag{19}$$

If $Sj(t) = Pj(t)$, then we deduce that

$$f_1(t) + \delta_1 j(t) = 3j(t) - 2\delta_3 j(t) - f_3(t).$$

Therefore, we can write (19) as

$$\begin{aligned}\Big\|PSj(t) - SPj(t)\Big\| &= \Big\|3\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big) - 2\delta_3\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big) \\ &\qquad - f_3(t) - f_1(t) - \delta_1\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big)\Big\|. \\ &= \Big\|9j(t) - f_1(t) - 4f_3(t) - 6\,\delta_3 j(t) - 2\delta_3\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big) \\ &\qquad - \delta_1\Big(3j(t) - 2\delta_3 j(t) - f_3(t)\Big)\Big\|.\end{aligned}$$

Using (C_2), we get $\|PSj(t) - SPj(t)\| = 0$. So, $PSj(t) = SPj(t)$ whenever $Sj = Pj$. Thus, (S,P) is weakly compatible. By a similar way, we can show that (T,Q) is weakly compatible.

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